# Ruin probability*

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**Abstract**

A review on ruin probabilities in the collective risk theory is given. Two-sided bounds of ruin probabilities are proposed for a classical risk model. Under Cramér's condition, these bounds coincide with those obtained by Rossberg and Siegel. In the heavy-tailed case, the bounds are new.

## 1 Introduction

Ruin probability is a convenient characteristic of a collective risk process which can be regarded as surplus dynamics of an insurance company. The surplus varies owing to two reasons: clients' premium incomes and claim expenses. Usually (but not always) models with deterministic premium income process are considered. However, the insurance claim sizes are always stochastic. Therefore, the risk process is random. The policy of an insurance company determines the premium and claim sizes. Obviously, a stochastic nature of risk process should be taken into account in this determination. In practice this is performed via the central limit theorem. Namely, a normal approximation of the accumulated claim sizes is yielded by a sufficiently large number of claims during some time interval and this provides the possibility to calculate an appropriate premium size for any given confidence level. Application of the large deviations theory permits to improve the approximation.

However, for decisions of any kind and especially for determination of the premium size, it is intentional to have a criterion evaluating the quality of these decisions which is sensitive to parameter variations. One of the most popular criteria is the ruin probability, conceivable as the probability that the risk process falls below a prescribed level (for example, zero) during a given time interval (finite or infinite). Usually, ruin probability is regarded as a function of an initial capital of an

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insurance company (*initial surplus*). The mathematical research of ruin probability began with the classical works of H. Cramér [39], [40], and his results can be found in textbooks on probability theory. Recently this problematic enjoys great popularity. The involved questions are very interesting from the mathematical point of view and require elaboration of new research methods. It is sufficient to note that even for the analysis of classical models, such methods as the Wiener–Hopf factorization, Spitzer's intentity, the martingale theory, the Markov process theory, the random walk theory and many others were enrolled. The desire to exanime more realistic models leads to consideration of new factors (inflation, reinsurance, etc.), to the unavoidable enlarging the spectrum of the applicable methods, and also to new qualitative phenomena. For example, the asymptotic behaviour of the ruin probability (as the initial capital of an insurance company infinitely increases) is completely different in the two cases, where a random claim size has an exponential moment or where its distribution has a "heavy tail".

Since it is impossible to find the ruin probability in a closed form for many models of interest, it is necessary to seek its approximations. The famous Cramér-Lundberg formula (see [75]) provides a nice example of such an approximation. This and other similar approximations are discussed in Section 2. Heuristic approximations (like De Vylder's formula; see [98] and [181]) represent another type of approximations. If we consider a diffusion approximation, it can be regarded as a third type of approximations obtained with the help of limit theorems. All these approximations are discussed in Section 2. Their essential defect is the absence of their accuracy estimates. Moreover, these approximations may yield huge relative errors. Because of this, it seems preferable to seek *two-sided bounds* of ruin probabilities. A significant part of this paper is devoted to elaboration of methods that yield such bounds showing at least the area which an unknown function belongs to. Among the variety of approximations, numerical and simulation methods should be mentioned. The effectiveness and correctness of these methods is usually provided by solutions of non-trivial mathematical problems. For example, the standard methods of mathematical statistics do not work for estimation of ruin probabilities by crude simulation of risk processes because the ruin represents a rare event. This leads to the necessity of simulation, for example, of processes transformed from an initial risk processes by appropriate change of the underlying probability measure. However, both the form of such a transfomation and its computer realization are far from trivial. A series of works that examine similar problems, are referred to in Section 2.

The structure of this paper is the following. In Section 2 a short review of the literature on ruin probability is given. The rest of the paper is devoted to the search of bounds for ruin probabilities. In Section 3, additional notation is introduced needed for study of the ruin probability as distribution function of a so-called *geometric sum*, that means a sum of independent identically distributed random variables (i.i.d.r.v.) in a random number which is a geometrically distributed and independent of the summands. Section 4 is devoted to two-sided bounds of the ruin probability in the case of Cramér's condition. These estimates were derived earlier by Rossberg and Siegel [146], but our method is essentially simpler and based on probabilistic arguments only (not analytic). Sections 5 and 6 include correspondingly lower and upper estimates of ruin probability in the case when claim sizes have no finite exponential moment. Numerical calculations illustrating these results are given in Section 7.

# 2 Review and basic notation

Let $x > 0$ denote an initial capital of an insurance company; $S(t)$ is the total premium size, paid by clients during time interval $[0, t]$; $T_i, Z_i, i \geq 1$, are sequences of times at which claims are paid and corresponding claim sizes, $0 \leq T_1 \leq T_2 \leq T_3 \leq \ldots$; $N(t) = \max\{n : T_n \leq t\}$ is the number of claims in the time interval $[0, t]$. We consider (if it is not said an opposite) that the function $S(t)$ is non-negative, continuous and non-decreasing, and the claim sizes $Z_i$ are positive. We define *risk* *process* $X(t)$, $t \geq 0$, by the following balance equation

$$X(t) = x + S(t) - \sum_{i \leq N(t)} Z_i \tag{2.1}$$

and let a *ruin epoch* be given by the relation

$$\tau = \inf\{t : X(t) < 0\}. \tag{2.2}$$

In general, the process $\{S(t)\}$ and sequences $\{T_i\}$ and $\{Z_i\}$ are random. Hence, the risk process $\{X(t)\}$ as well as the ruin epoch $\tau$ are random too. But the r.v. $\tau$ is not proper (in general) in the sense that $\mathbf{P}(\tau < \infty) < 1$. The function

$$\Psi(x) = \mathbf{P}(\tau < \infty \mid X(0) = x) \tag{2.3}$$

is called an *infinite-time ruin probability* given initial capital $x$. The function

$$\Psi(t, x) = \mathbf{P}(\tau \leq t \mid X(0) = x) \tag{2.4}$$

is called a *finite-time ruin probability* over horizon $t$ given an initial capital $x$. Probability (2.3) serves as the main object of our research, although we will touch a finite-time ruin probability also.

We will speak about the *Sparre Andersen risk model* (see [3], [98]) in the cases, where process $S(t)$ is deterministic and $S(t) = ct$ (constant $c > 0$ represents a gross premium rate), sequence $\{Z_i\}$ consists of i.i.d.r.v.'s, and $\{T_i\}$ is *a renewal process*, that is *inter-occurrence times* $\theta_i = T_i - T_{i-1}, i \geq 1$, are i.i.d. In this case, we denote the disribution function (d.f.) of claim sizes by

$$B(u) = \mathbf{P}(Z_1 \leq u) \tag{2.5}$$

and the d.f. of inter-occurrence times by

$$A(u) = \mathbf{P}(\theta_1 \leq u). \tag{2.6}$$

We will speak about a *classical risk model*, if it is a Sparre Andersen model with

$$A(u) = 1 - \exp(-\lambda u). \tag{2.7}$$

Let

$$a_1 = \int_0^\infty u\, dA(u) < \infty, \tag{2.8}$$

$$b_1 = \int_0^\infty u\, dB(u) < \infty, \tag{2.9}$$

$$\rho = \frac{ca_1 - b_1}{b_1} > 0. \tag{2.10}$$

The last relation means particularly that the risk process decreases in the mean, and constant $\rho$ is a *relative safety loading.*

Mathematical aspects, associated with ruin probabilities, are examined in a great number of works, among them several books that bring to light the state of art to the moment of their publication from the point of view of the author's interest. We mention the most important monographies, where further references can be found: [12], [27], [38], [55], [84], [93], [98], [140], [153].

The modern theory of ruin probabilities can be traced back to F. Lundberg's works [126] and [127] where the first estimates of ruin probability (in paricular, the famous *Lundberg inequality*) were obtained. However, as it usually happens, Lundberg's works do not contain rigorous mathematical statements. Because of this, the establishment of the *mathematical collective risk theory* is usually reffered to Cramér's works [39] and [40]. In these works, a classical risk model is examined under the additional assumption (known as *the Cramér condition*) that a unique positive solution $R > 0$ (called *Lundberg's exponent*) of the equation

$$\frac{\lambda}{c}\int_0^\infty e^{Ru}(1-B(u))\,du = 1 \tag{2.11}$$

exists. In this case (see [75], Chapter 11) function $\Psi_R(x) = \Psi(x)e^{Rx}$ satisfies the renewal equation yielding the limiting relation

$$\lim_{x\to\infty}\Psi_R(x) = k_{CL}, \tag{2.12}$$

where

$$k_{CL} = \frac{\int_0^\infty e^{Ru}\int_u^\infty(1-B(v))\,dv\,du}{\int_0^\infty ue^{Ru}(1-B(u))\,du} \tag{2.13}$$

is the so-called *Cramér-Lundberg constant*, and the relation (2.12) can be written in the form of the *Cramér-Lundberg asymptotic formula*

$$\Psi(x) \sim k_{CL}e^{-Rx}. \tag{2.14}$$

Note that in the case of the *Markov risk model*, for which

$$B(u) = 1 - \exp\left(-\frac{u}{b_1}\right), \tag{2.15}$$

the function $\Psi(x)$ can be found in a closed form

$$\Psi(x) = \frac{1}{1+\rho}\exp(-Rx), \tag{2.16}$$

where

$$R = \frac{\rho}{b_1(1+\rho)}. \tag{2.17}$$

A classical risk model is generalized in [59], where the risk process process $X(t)$ is defined by the relation

$$X(t) = x + ct - \sum_{i\le N(t)} Z_i + W(t),$$

where $W(t)$ is the Wiener process, $\mathbf{E}W(t) = 0$. It was shown, in [59] that the ruin probability satisfies a renewal equation for a stopped renewal process and the cases where its solution can be found in a closed form were discussed.

Relation (2.14) holds also for Sparre Andersen models, if one replaces (2.11) by the Cramér condition in the form

$$\mathbf{E}\exp(R(Z_1 - c\theta_1)) = 1 \tag{2.18}$$

and redefines the constant $k_{CL}$. This generalization is discussed in [72] (see also works [23], [98], [167], [168], [170], and [171]). All these works used the Wiener-Hopf theory and relationships between the ruin probability $\Psi(x)$ and the distribution of maximum of a random walk and the corresponding ladder epochs and heights (see [23]):

$$\Psi(x) = \mathbf{P}(M > x), \tag{2.19}$$

where

$$\begin{aligned} M &= \sup_k \sigma_k, & (2.20)\\ \sigma_0 &= 0, \quad \sigma_k = \xi_1 + \cdots + \xi_k, & (2.21)\\ \xi_i &= Z_i - c\theta_i & (2.22)\end{aligned}$$

We note that the waiting time in a single server queue equals (in distribution) to the total maximum of a random walk (see [6], [8], [75], [125], [155]), and this relationship permits to translate the queueing theory results to the risk theory and vice versa. This analogy was examined in details in [11].

In order for the Cramér conditions (2.11) (or (2.18)) to hold, it is necessary, but not sufficient (in general; see [72]) the existence of an exponential moment

$$\int_0^\infty e^{\kappa u}\, dB(u) < \infty \tag{2.23}$$

for some $\kappa > 0$.

In works [167] and [72] the asymptotic behaviour of $\Psi(x)$ in the case where (2.23) holds, but the Cramér condition (2.18) fails, is examined. If

$$r = \sup\{\alpha : \mathbf{E}\exp(\alpha(Z_1 - c\theta_1)) < 1\},$$

but

$$\mathbf{E}\exp(r(Z_1 - c\theta_1)) = \infty,$$

then

$$\Psi(x) = o\left(e^{-rx}\right).$$

It is adopted that relations of the Cramér-Lundberg type approximate $\Psi(x)$ to a good accuracy. But the real accuracy of these approximations remains unknown and can attain tens of percentages (see [95], [96], [98], [101], [181]). In addition, the computation of the Lundberg exponent requires the exact knowledge of d.f.'s $A$ and $B$ that does not hold in practice. A variety of approximations of ruin probability were suggested. They can schematically be divided into two classes: heuristic approximations and approximations resulted from limit theorems of probability theory. It is true that the real accuracy of all approximations in both classes remains unknown. Practical calculations showed that *heuristic De Vylder's approximation* is accurate enough (see [97], [98], [181]). It is applicable to a classical risk model and consists in the following. Let $X^*(t)$ be a classical risk process with exponential distributed claim sizes (Markov model) that can be determined by parameters $\lambda^*, b_1^*$ and $\rho^*$. Let $X(t)$ be the initial (classical) risk process with the mean claim size $b_1$ and parameters $\lambda$ and $\rho$. Let us find the parameters $\lambda^*, b_1^*$ and $\rho^*$ from the following three equations:

$$\mathbf{E}\left(X^*(t)\right)^k \equiv \mathbf{E}\left(X(t)\right)^k, \quad k = 1, 2, 3.$$

It can be proved that these equations define the desired parameters uniquely. De Vylder's approximation $\Psi_{DV}(x)$ of $\Psi(x)$ can be regarded as ruin probability for process $X^*(t)$ and it is defined by formulae the (2.16) and (2.17), in which one must substitute $\lambda^*, b_1^*$ and $\rho^*$ instead of $\lambda, b_1$ and $\rho$. Let us mention that De Vylder's approximation can be used also for estimates of *finite time ruin probability* since the Markov risk model permits this kind of calculations.

Let us mention also the *Beeckman-Bowers approximation* (see [28], [98]), which is applicable to the classical risk models and consists in the replacement of the function $H(x) = 1 - (1+\rho)\Psi(x)$ by a gamma distribution function such that the first two moments of this distribution coincide with the first two moments of $H(x)$.

The diffusion approximation is an example of usage of limit theorems to find ruin probability. For this, a Sparre Andersen risk process is replaced by a Wiener process (see [95], [98], [109]). The correctness of such a replacement (accompanied by standard normalizations of time and state variables) is yielded from the invariance principle (see [109] and [98]). For Wiener's process, the problem is reduced to the first passage time problem which has been already solved. Weak points of this approach are the following: (1) low real accuracy of the estimates, reaching *hundreds of percentages* (see [98]); (2) disagreement between the exponential form of ruin probability resulted from the diffusion approximation and the minimal requirements in concern with the form of the claim size d.f. In reality, under this approximation only existence of the two first moments of the d.f. $B$ is required that does not yield the exponentiality of $\Psi(x)$.

It seems that Gerber's works [82] – [84] opened the gate for martingales to the theory of ruin probability. In order to give an impression of this approach, we examine a classical risk model that satisfy the Cramér condition. Let

$$\mu(t) = \exp(-RX(t)), \tag{2.24}$$

where $R > 0$ is the Lundberg exponent (see (2.11)). Then, for any $u > 0$,

$$\mathbf{E}(\mu(t+u) \mid \mu(t)) = e^{-RX(t)} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-Rcu} \mathbf{E} e^{R(Z_1+\cdots+Z_k)} = \mu(t) \tag{2.25}$$

that is $\mu(t)$ is a martingale. Let $\tau$ be a ruin epoch (see (2.2)). As $X(t) \to \infty$ a.s. when $t \to \infty$, it follows from (2.25) that

$$\mathbf{E}(\mu(\tau);\ \tau < \infty) = \mu(0) = e^{-Rx}.$$

By (2.3), the ruin probability takes the form

$$\Psi(x) = \frac{\exp(-Rx)}{\mathbf{E}(\mu(\tau) \mid \tau < \infty)}. \tag{2.26}$$

Since $\mu(\tau) > 1$, relation (2.26) yields the *Lundberg inequality*

$$\Psi(x) \le e^{-Rx}. \tag{2.27}$$

In fact, the equation (2.26) relates the ruin probability with variable $\mathbf{E}(\mu(\tau) \mid \tau < \infty)$. A series of estimates of this variable can be found in [82], [84]. Two-sided estimates of ruin probability presented in [146] immediately follow from (2.26):

$$\underline{C} e^{-Rx} \le \Psi(x) \le \bar{C} e^{-Rx}, \tag{2.28}$$

where

$$\begin{aligned} \underline{C} &= \inf_{y>0} \left\{ e^{-Ry}(1 - B(y)) \left( \int_y^{\infty} e^{Ru}\, dB(u) \right)^{-1} \right\}, \\ \bar{C} &= \sup_{y>0} \left\{ e^{-Ry}(1 - B(y)) \left( \int_y^{\infty} e^{Ru}\, dB(u) \right)^{-1} \right\}. \end{aligned}$$

It is noteworthy that the estimates (2.28) were found in [146] not for a classical model, but for a more general Sparre Andersen model. The martingale approach and its "relatives" based on the theory of Markov processes and generalized Dynkin's formula turned to be productive and were applied to a variety of models, the majority of which satisfying analogs of the Cramér condition; see [43], [59], [65], [90], [134], [142], [166], [180], [190], and [191]. We mention a recent work [78], where the results of paper [59] are spreaded to the case of Sparre Andersen's and more general models. Using the structure of Markov piecewise-deterministic processes (see [44], [45], [112], [114]) and constructing martingales of an exponential form for them, the authors obtain two-sided estimates of finite- and infinite-time ruin probabilities. Finite-time ruin probability $\Psi(t,x)$ was examined also in [7], [26], [51], [68], [89], [108], and [175]. As a matter of fact, probability $\Psi(t,x)$ is more interesting in collective risk theory. However, it is extremely difficult to find it and this explains the small number of works devoted to this problem.

A series of works is devoted to ruin probability in a *Cox model*, where claim occurrence times $\{T_i\}$ form a *Cox process*, conceivable as a Poisson process with randomly varying intensity. One of the first work in this direction was done by Ammeter (see [2]), and this model was systematically studied in Grandel's book [98] (see also [33], [64], [111], [110], and [143]). The interest to the Cox model can easily be explained by practical needs to account claim size (or rate) fluctuations. A particular case of Cox process is the so-called *Markov modulated process*, when the intensity is governed by a Markov process. A number of works are devoted to this case, see [9], [10], [13], [102] and references therein. Among recent works on this topic, using martingale approach, we mention [68], [150] and [151].

One of the important directions in collective risk theory is the ruin probability theory in the case, where the Cramér condition is not satisfied and particularly, where the claim size distribution has *heavy tail* that means when it has no finite exponential moment. Now the ruin mechanism differs from the corresponding mechanism in Cramér's case. While in the Cramér case the ruin occurs as the result of accumulation of many claims of "modest size", in the heavy tailed case the typical ruin is resulted from one large claim size. Simulation of risk processes shows impressively this difference (see [147]). Of course, if the variance of claim sizes is small, then the ruin mechanism is approximately the same as in the Cramér case, even in the presense of large claims.

Let us consider the Sparre Andersen risk model (see notation (2.5) – (2.10)) and use the ruin probability representation as maximum of a random walk (2.19). We define a *ladder epoch* of the random walk (see [8], [75])

$$L = \inf\{k : \sigma_k > 0\}. \tag{2.29}$$

By condition (2.10), $L$ is an improper random variable. Let

$$q = \mathbf{P}(L = \infty). \tag{2.30}$$

We define the conditional distribution of a *ladder height* $\sigma_L$:

$$F_\chi(u) = \mathbf{P}(\sigma_L \le u \mid L < \infty). \tag{2.31}$$

It is well known (see for example, [8]) that the maximum $M$ of the random walk $\sigma_k$ satisfy the following equality in distribution

$$M \stackrel{d}{=} \chi_1 + \cdots + \chi_{\nu-1}, \tag{2.32}$$

where r.v. $\nu$ does not depend on the sequence of r.v.'s $\chi_1, \chi_2, \ldots$, follows the geometric distribution

$$\mathbf{P}(\nu = k) = q(1-q)^{k-1}, \quad k \ge 1, \tag{2.33}$$

and

$$\mathbf{P}(\chi_1 \le u) = F_\chi(u), \tag{2.34}$$

For the classical risk model, the following relations hold (they may be viewed as a consequence of the Pollacheck-Khinchine formula; see [8]):

$$q = \frac{\rho}{1+\rho}, \tag{2.35}$$

$$F_\chi(u) = \frac{1}{b_1}\int_0^u (1 - B(z))\,dz, \tag{2.36}$$

which facilitate significantly the ruin probability analysis.

If we remove the assumption about the exponentiality of the distribution of r.v.'s $\{\theta_i\}$, then formulae similar to (2.35) and (2.36) can be found for stationary sequences of occurrence times (see [21] and [133]), but in this case we can not use any relation of the form (2.32). Equality (2.19) together with (2.30) – (2.34) renders to the following representation of ruin probability

$$\Psi(x) = \sum_{k=1}^{\infty} q(1-q)^{k-1}\bar{F}_\chi^{(k-1)}(x), \tag{2.37}$$

where $F_\chi^{(k-1)}$ is the $(k-1)$-fold convolution of d.f. $F_\chi$, and $\bar{F}_\chi^{(k-1)}(u) = 1 - F_\chi^{(k-1)}(u)$.

The following class $\mathcal{S}$ of d.f.'s plays an important role in examination of the risk processes when the Cramér condition does not hold: $F_\chi \in \mathcal{S}$, if

$$\lim_{u\to\infty} \frac{1 - F_\chi^2(u)}{1 - F_\chi(u)} = 2. \tag{2.38}$$

It follows from equality (2.38) that $1 - F_\chi^{(n)}(u) \sim 1 - F_\chi^n(u)$. This means that the sum $\chi_1 + \cdots + \chi_n$ behaves asymptotically as $\max(\chi_1, \dots \chi_n)$. Various properties of d.f.'s from the class $\mathcal{S}$ as well as from other related classes are studied in works [31], [66], [67], [72], [79], [92], [112], [144], and [171]. If $F_\chi \in \mathcal{S}$, then, by (2.37),

$$\Psi(x) \sim \frac{1-q}{q}(1 - F_\chi(x)),$$

see [72]. Particularly, if we examine the classical risk model, then

$$\Psi(x) \sim \frac{1}{\rho b_1}\int_x^\infty (1 - B(u)\,du. \tag{2.39}$$

The result (2.39) was found at first in the two special cases: when the claim sizes follow the Pareto distribution (see [24]) and the lognormal distribution (see [176]). A general case is examined in the fundamental work [72], where the case $F_\chi \in \mathcal{S}$ as well as other cases were studied using the Wiener-Hopf theory. Known defect of any asymptotic approximation in general and of (2.39) in particular is the absense of convergence rate estimates. As we will see in Section 7, relation (2.39) can be too optimistic. Among other works devoted to the "heavy tailed" case we mention [1], [15], [22], [73], [116], and [121]. In particular, the asymptotic of ruin epoch and the claim size that yields a ruin is studied in [15].

Similar to the Cramér case, in the heavy tailed case (large claim sizes) there is an interest to generalizations accounting fluctuations in the intensities and claim sizes. A series of results in this direction can be found in [9], [10], [12], [13], and [14].

A few works are devoted to the analysis of the influence of new factors on ruin probability. Thus, it is proved in a recent work [123] that the introduction of an interest rate into the classical risk model in the presense of large claim sizes leads to an asymptotic behaviour of $\Psi(x)$ that differs from (2.39). Namely, if $\delta > 0$ is the interest rate coefficient (this means that the capital $y$ increases up to the value $y\exp(\delta t)$ during time $t$), then

$$\Psi(x) \sim k_\delta(1 - F_\chi(x)), \quad \delta > 0, \tag{2.40}$$

where the coefficient $k_\delta$ depends on $\delta$ (in the case $\delta = 0$ we have formula (2.39)). This topic is considered in [161] as well.

Formula (2.40) profess the thesis that an asymptotic representation does not disclose complete information about the behaviour of ruin probability. It is proved in [185] that if the inflation factor is taken into account in the Sparre Andersen model, then $\Psi(x) \equiv 1$ (see also [183]). Reinsurance models that have a significant applied interest, lead to complicated mathematical problems (see [182], [187]). Works [77], [78], [147], and [148] examine models in the presense of loans, investments, and random character of premium income.

As we have already seen, the analytical calculation of ruin probability is difficult and, for this reason, numerical procedures for calculation of ruin probabilities are of interest (see [132], [139], [157], [158], [172], [177], and [160]). An effective algorithm for calculation of ruin probability for a classical risk model, based on representation (2.37), was proposed in [58] and [94]. A reccurent procedure that works with upper and lower discrete approximations of d.f. $F_\chi$ was constructed (see [29] and [55]). A recursive estimate of finite time ruin probability $\Psi(t,x)$ is described in [118]. Similar problems are considered in [50] and [175]. A matrix approach for calculation of ruin probability is developed in [18]. For this, the distribution $B$ of claim sizes is replaced by a phase type distribution, which has a finite exponential moment. Despite this, the accuracy of the proposed approach remains high, even in the cases where $B$ has heavy tail. Monte-Carlo methods can be regarded as specific numerical methods. Obviously, the estimates that use a relative number of successes (of ruins) do not work as the ruin probability is small. Because of this, special methods for rare event estimates are elaborated and an analogy between ruin probability and the stationary waiting time distribution in a single server queue is studied (see [11], [20], [58], [105]).

Finally, we mention statistical aspects of the risk theory. Here we list works [30], [52], [69], [70] that propose different models to describe the appearance of extremal situations. A series of papers is devoted to estimates of parameters in risk models (see [16], [41], [46], [106], [145], and [152]). Various estimates of ruin probability are proposed in works [42], [74], [76], and [107]. In particular, paper [42] is devoted to the asymptotic behaviour of a nonparametric estimate of ruin probability in the classical risk model. This estimate is based on representation (2.36), in which the real distribution of claim sizes is replaced by the corresponding empirical distribution.

# 3 Additional notations

We return to the representation of ruin probability at a Sparre Andersen model in the form of a geometric sum (2.37). As we have already noted, this is quite convenient from many points of view. In the present paper, we shall start from this representation as well.

Let us introduce a *renewal process*

$$S_0 = 0, \quad S_n = \chi_1 + \cdots + \chi_n \ (n \geq 1) \tag{3.1}$$

and let

$$N(x) = \min\{n : S_n > x\} \tag{3.2}$$

and

$$\epsilon_x = S_{N(x)} - x \tag{3.3}$$

be the *excess* of renewal process (3.1) over level $x \geq 0$. It is not difficult to show (see, for example, [113] or [115]), that

$$\Psi(x) = \mathbf{E}(1-q)^{N(x)}. \tag{3.4}$$

Formula (3.4) serves as starting point for various estimates of $\Psi(x)$.

Let us introduce the following classes of *non-negative monotone increasing functions defined on* $[0,\infty)$:

(1) class $\mathcal{L}$ consists of these functions $\Lambda$ such that $\Lambda(0) = 0$, $\lim_{u\to\infty} \Lambda(u) = \infty$, $\Lambda(u)/u$ does not increase and $\lim_{u\to\infty} \Lambda(u)/u = 0$;

(2) class $\mathcal{G}^s$, $s \geq 0$, consists of these functions $G(u) = \exp(\Lambda(u))$, $\Lambda \in \mathcal{L}$, such that

$$\lim_{u\to\infty} \frac{G(u)}{(1+u)^s} = \infty;$$

(3) class $\mathcal{G}_c$ consists of these differentiable functions $G$ such that $G(u)$ is *convex* for $u \geq 0$ and its derivative $g(u) = dG(u)/du$ is *concave* for $u \geq 0$ and $g(u) \to \infty$ for $u \to \infty$.

Let us note that class $\mathcal{G}_c$ appears in a natural way in the study of the uniformly integrable r.v.'s (see [115], Section 1.1).

The following assertions hold true. We will not prove them since the proof is quite similar to that of Theorem 2.3.1 from [114].

**Lemma 1** *Let $Y$ be a non-negative r.v. If $\mathbf{E}Y^s < \infty$, $s \geq 0$, then*
*(1) there exists a function $G \in \mathcal{G}_c$ such that $\mathbf{E}\left((1+Y)^{s-1}G(Y)\right) < \infty$;*
*(2) there exists a function $G \in \mathcal{G}^s$ such that $\mathbf{E}G(Y) < \infty$.*

Obviously, $\mathcal{G}^s \subset \mathcal{G}^t$ for $s > t$. In addition, if $G \in \mathcal{G}^s$, $s \geq 0$, then the function $G$ is *subexponential* in the sense that

$$\lim_{u\to\infty} e^{-\kappa u} G(u) = 0$$

for any $\kappa > 0$. As for any finite (a.s.) r.v. $Y$ there exists a function $G \in \mathcal{G}^s$ such that $\mathbf{E}G(X) < \infty$, the finiteness of $\mathbf{E}G(X)$ does not mean, in general, that $X \in \mathcal{S}$, where class $\mathcal{S}$ is defined in Section 2 (see also [72] and [165]).

We introduce additional notation

$$\begin{aligned} q' &= -\ln(1-q), \qquad (3.5)\\ \mathbf{1}(u) &= \begin{cases} 0, & \text{if } u < 0,\\ 1, & \text{if } u \geq 0,\end{cases}\\ I(B) &= \begin{cases} 0, & \text{if the event } B \text{ occurs},\\ 1, & \text{otherwise},\end{cases}\\ a \wedge b &= \min(a,b),\\ a \vee b &= \max(a,b),\\ (a)_+ &= a \vee 0, \end{aligned}$$

(3.6)

which we use together with other notation introduced in this section and with notation (2.29) – (2.37) without any comments.

# 4 Two-sided estimates under the Cramér condition

We first examine the case, in which the Cramér condition holds. That is we assume the existence of a positive solution $R > 0$ of equation (2.18). Since sequence $\exp(R\sigma_n)$, $n \geq 0$, is a martingale, standard arguments lead to the equation

$$(1-q)\mathbf{E}\exp(R\chi_1) = 1. \qquad (4.1)$$

Let us consider the Markov sequence

$$\zeta_n = (n, S_n), \quad n \geq 0, \qquad (4.2)$$

and put

$$V(\zeta_n) = (1-q)^n \exp(RS_n), \quad V(\zeta_0) = 1. \tag{4.3}$$

The Cramér condition (4.1) yields that $V(\zeta_n)$ is a martingale and as $N(x)$ is a stoping time ($\mathbf{E}N(x) < \infty$), the following equation holds true

$$\mathbf{E}V(\zeta_{N(x)}) = \mathbf{E}\left((1-q)^{N(x)} \exp(RS_{N(x)})\right) = V(\zeta_0) = 1.$$

We rewrite it in the form (cf. (3.3))

$$\mathbf{E}\left((1-q)^{N(x)} \exp(R\epsilon_x)\right) = \exp(-Rx).$$

Hence,

$$\mathbf{E}\left((1-q)^{N(x)} \mathbf{E}\left(\exp(R\epsilon_x) \mid N(x)\right)\right) = \exp(-Rx), \tag{4.4}$$

It can be proved that

$$\underline{k} \le \mathbf{E}\left(\exp(R\epsilon_x) \mid N(x)\right) \le \bar{k} \tag{4.5}$$

with probability one, where

$$\underline{k} = \inf_{v \ge 0} \frac{\int_v^\infty \exp(Ru)\, dF_\chi(u)}{\exp(Rv)(1 - F_\chi(v))}, \tag{4.6}$$

$$\bar{k} = \sup_{v \ge 0} \frac{\int_v^\infty \exp(Ru)\, dF_\chi(u)}{\exp(Rv)(1 - F_\chi(v))}. \tag{4.7}$$

Constants $\underline{k}$ and $\bar{k}$ are not convenient for calculations because they are expressed in terms of the auxiliary distribution $F_\chi$. However, it can be shown that estimates (4.6) and (4.7) still hold if we put there $B$ instead of $F_\chi$. Two-sided estimate (2.28) follows from (4.4). As it has been already mentioned, this estimate was firstly derived by Rossberg and Siegel in [146] by compicated analytical methods. We also remark that the Lundberg inequality (2.27) follows directly from relation (4.4). The accuracy of estimates (2.28) in concrete examples was examined in [117].

To close this section we give a *heuristic approximation* of ruin probability as a naive consequence of formula (4.4). If we consider $\epsilon_x$ and $N(x)$ as independent r.v.'s (which is not true in general), then it follows from (4.4) that

$$\Psi(x) = \frac{\exp(-Rx)}{\mathbf{E}\exp(R\epsilon_x)}.$$

Further, we put instead of the real distribution of the excess $\epsilon_x$ its limit distribution that has the form (2.36) and exists under well-known non-restrictive conditions on the renewal process (3.1) (see [115]). Then

$$\mathbf{E}\exp(R\epsilon_x) = \frac{1}{b_1}\int_0^\infty e^{Ru}(1 - F_\chi(u))\, du = \frac{q}{(1-q)Rb_1}.$$

As a result, we arrive at the following heuristic approximation

$$\Psi(x) \approx \Psi_H(x) = \frac{(1-q)Rb_1}{q}\, e^{-Rx}. \tag{4.8}$$

The accuracy of the proposed heuristic approximations (4.8) is illustrated in the following two examples, where the exact values of ruin probability are taken from Grandell's book [98]. In both examples, we examine a classical risk model.

**Example 1.** *Mixture of exponential distributions*

In this example the claim size distribution is a mixture of three exponential distributions:

$$1 - B(u) = 0.0039793\, e^{-0.014631u} + 0.1078392\, e^{-0.190206u} + 0.8881815\, e^{-5.514588}.$$

The mean claim size is equal to 1. The results of calculations by formula (4.8) are shown in the following table

| $\rho$ | $x$ | $\Psi(x)$ | $\Psi_H(x)$ |
|---|---|---|---|
| | 10 | 0.8897 | 0.8597 |
| 0.05 | 100 | 0.7144 | 0.7161 |
| | 1000 | 0.1149 | 0.1151 |
| | 10 | 0.7993 | 0.7513 |
| 0.10 | 100 | 0.5393 | 0.5431 |
| | 1000 | 0.0210 | 0.0211 |
| | 10 | 0.7242 | 0.6657 |
| 0.15 | 100 | 0.4247 | 0.4301 |
| | 10 | 0.6611 | 0.5969 |
| 0.20 | 100 | 0.3455 | 0.3522 |
| | 10 | 0.6073 | 0.5406 |
| 0.25 | 100 | 0.2886 | 0.2961 |
| | 10 | 0.5610 | 0.4936 |
| 0.30 | 100 | 0.2461 | 0.2540 |
| | 10 | 0.2634 | 0.2199 |
| 1.00 | 100 | 0.0724 | 0.0787 |

The figures appearring in this table show a pretty high accuracy of the proposed approximation.

**Example 2.** *Gamma distribution*

Now we consider that the claim sizes follow the gamma distribution with mean 1:

$$B(u) = \int_0^u \frac{z^{\beta-1}\beta^{1/\beta}}{\Gamma(\beta)} \exp(-z\beta)\, dz,$$

where $\Gamma(\beta)$ is Euler's gamma-function. The following table contains the results of calculations for $\beta = 0.01$ and $\rho = 0.1$:

| $x$ | 300 | 600 | 900 | 1200 | 1500 | 1800 | 2100 | 2400 | 2700 | 3000 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Psi(x)$ | 0.5211 | 0.3087 | 0.1829 | 0.1083 | 0.0642 | 0.0380 | 0.0225 | 0.0134 | 0.0079 | 0.0047 |
| $\Psi_H(x)$ | 0.5221 | 0.3093 | 0.1832 | 0.1084 | 0.0643 | 0.0381 | 0.0226 | 0.0134 | 0.0079 | 0.0047 |

These figures also claim the high accuracy of the approximation.

# 5 Lower bounds in the presense of large claims

In this and the following sections we shall examine the case, where the Sparre Andersen model does not satisfy the Cramér condition (2.18) and there is no exponential moment of the claim sizes $Z_i$. We say in such a case that d.f. $B$ has *heavy tail*. Specific restrictions, under which the following consideration holds, will be posed when necessary.

In order to avoid cumbersome expressions, we suppose that

$$\mathbf{E}\chi_1 < \infty$$

and examine the *normalized r.v.'s*

$$X_i = \frac{\chi_i}{\mathbf{E}\chi_i}, \quad i \geq 1, \tag{5.1}$$

and the corresponding sum

$$\Psi_0(x) = \sum_{k=1}^{\infty} q(1-q)^{k-1}\bar{F}^{(k-1)}, \tag{5.2}$$

where

$$F(u) = \mathbf{P}(X_1 \leq u). \tag{5.3}$$

By force of (2.34), (2.37), (5.1), and (5.2) the following relations are obvious and can be used for transformation of "normalized characteristics" into those required by the original setup:

$$F_\chi(u) = F\left(\frac{u}{\mathbf{E}\chi_1}\right), \tag{5.4}$$

$$\Psi(x) = \Psi_0\left(\frac{x}{\mathbf{E}\chi_1}\right). \tag{5.5}$$

Let us regard all r.v.'s having *different notations* (in the sense of either different literal notations or different indices) as *independent*. Moreover, variables with different indices only, will be regarded as *indentically distributed*. For example, r.v.'s $X, X_1, X_2$ and $Y$ are considered as independent by default, and $X, X_1$ and $X_2$ are i.i.d. as well. Besides, we shall omit the index of a r.v. when stating any property in terms of the d.f. of a generic r.v. from a set of i.i.d.r.v.'s. For example, the mean value of a r.v. $X_i$ will be denoted as $\mathbf{E}X$ (instead of $\mathbf{E}X_i$).

Let us introduce "truncated" non-negative r.v.'s (the truncation parameter is $b > 0$), defining their d.f.'s on the non-negative half-axis:

$$F^{(b)}(u) = \mathbf{P}(X(b) \leq u) = \begin{cases} F(u), & \text{if } u \leq b, \\ 1, & \text{otherwise,} \end{cases} \tag{5.6}$$

$$T^{(b)}(u) = \mathbf{P}(Y(b) \leq u) = \begin{cases} F(b), & \text{if } u \leq b, \\ F(u), & \text{otherwise.} \end{cases} \tag{5.7}$$

Thus,

$$X(b) \stackrel{d}{=} X \wedge b, \tag{5.8}$$

$$Y(b) \stackrel{d}{=} XI(X > b). \tag{5.9}$$

Let us denote the moments of the variable $X$ (if they exist) by

$$\varphi_i = \mathbf{E}X^i, \quad i \geq 1, \quad (\varphi_1 = 1). \tag{5.10}$$

If $G$ is a non-negative measurable function then we denote

$$\varphi(G) = \mathbf{E}G(X), \tag{5.11}$$

where the case $\varphi(G) = \infty$ is possible. Let now

$$\varphi_i(b) = \mathbf{E}X^i(b), \quad i \geq 1.$$

Obviously, there exist all power and exponential moments for the truncated r.v. $X(b)$. Let

$$N^{(b)}(x) = \min\left\{n : X_1(b) + \cdots + X_n(b) > x\right\}.$$

Renewal function $\mathbf{E}N^{(b)}(x)$ satisfies the following Lorden's inequality (see [115], Section 1.5):

$$\mathbf{E}N^{(b)}(x) \leq \frac{x}{\varphi_1(b)} + \frac{\varphi_2(b)}{\varphi_1^2(b)}. \tag{5.12}$$

**Lemma 2** *Let $\varphi_2 < \infty$. Then*

$$\mathbf{E}N^{(b)}(x) \leq \frac{2bx}{(2b-\varphi_2)_+} + \varphi_2. \tag{5.13}$$

*Proof.* Since $\varphi_1 = 1$ and $\varphi_2 < \infty$,

$$\varphi_1(b) = 1 - \int_b^\infty (1-F(u))\,du \geq 1 - \frac{1}{b}\int_b^\infty u(1-F(u))\,du \geq 1 - \frac{\varphi_2}{2b}.$$

It is easy to see that $\varphi_2(b)/\varphi_1^2(b)$ is a non-decreasing function of $b$. Hence $\varphi_2(b)/\varphi_1^2(b) \leq \varphi_2/\varphi_1^2 = \varphi_2$. Thus, inequality (5.13) follows from (5.12). □

**Lemma 3** *Let $\varphi_2 = \infty$ and $\varphi(G) < \infty$ for a function $G \in \mathcal{G}_c$, $G(0) = 0$. Then*

$$\mathbf{E}N^{(b)}(x) \leq \frac{g(b)}{(g(b)-\varphi(G))_+}\left(x + \frac{2b\varphi(G)}{(g(b)-\varphi(G))_+}\right). \tag{5.14}$$

*Proof.* By the condition, $\varphi_2 = \infty$ and therefore the function $G$ cannot increase faster than the quadratic function. By Lemma 1, there exists a function $G \in \mathcal{G}_c$ such that $\varphi(G) < \infty$. Similarly to Lemma 2,

$$\varphi_1(b) \geq 1 - \frac{\varphi(G)}{g(b)}.$$

We have

$$\varphi_2(b) = 2\int_0^b u(1-F(u))\,du \leq \frac{2b}{g(b)}\int_0^b g(u)(1-F(u))\,du \leq \frac{2b\varphi(G)}{g(b)}.$$

Relation (5.14) follows now from Lorden's inequality (5.12). □

Let us find lower estimates of the function $\Psi_0(x)$.

**Theorem 1** *Let $\varphi_2 < \infty$. Then*

$$\Psi_0(x) \geq E(x;\varphi_2) + K(q,x)\frac{1-F(x)}{q}, \tag{5.15}$$

*where*

$$\begin{aligned}
E(x;\varphi_2) &= \exp(-q'\varphi_2)\exp\left(-\frac{q'(x\vee(2\varphi_2))^2}{(x\vee(2\varphi_2)-\varphi_2)}\right),\\
K(q,x) &= \frac{q^2(1-q)}{(q')^2}\left(1+\exp(-q'y_*) - \frac{2}{q'y_*}(1-\exp(-q'y_*))\right),\\
y_* &= \frac{x}{2} + \frac{\varphi_2-1}{2}\left(1-\sqrt{1+\frac{2x}{\varphi_2-1}}\right).
\end{aligned}$$

*Proof*. By definition of $X_i(b)$ and accounting relation (3.4),

$$\begin{aligned}\Psi_0(x) &= \mathbf{P}\left(\sum_{i\le\nu-1} X_i(b) > x\right) + \mathbf{P}\left(\sum_{i\le\nu-1} X_i > x,\ \sum_{i\le\nu-1} X_i\wedge b \le x\right)\\ &= \mathbf{E}(1-q)^{N^{(b)}(x)} + \sum_{n\ge 1} q(1-q)^{n-1}\mathbf{P}\left(\sum_{i\le n-1} X_i > x,\ \sum_{i\le n-1} X_i\wedge b\le x\right). \quad (5.16)\end{aligned}$$

Put

$$b = \frac{x}{2}. \quad (5.17)$$

Then by Jensen's inequality and (5.13),

$$\mathbf{E}(1-q)^{N^{(b)}(x)} \ge \exp\left(-\frac{q'x^2}{(x-\varphi_2)_+} - q'\varphi_2\right), \quad (5.18)$$

Now, we take into account the fact that the function in the left-hand side of (5.18), does not increases along $x$. At the same time, the function in the right-hand side of (5.18) monotone decreases for $x \ge 2\varphi_2$. Therefore,

$$\mathbf{E}(1-q)^{N^{(b)}(x)} \ge E(x;\varphi_2) \quad (5.19)$$

Denote

$$p_n = \mathbf{P}\left(\sum_{i\le n-1} X_i > x,\ \sum_{i\le n-1} X_i\wedge b\le x\right), \quad n\ge 1. \quad (5.20)$$

Obviously, $p_1 = 0$. The total probability formula, relation (5.17), and the fact that $X_i$ are i.i.d.r.v.'s, render as consequence the inequality, true for any $n \ge 2$:

$$\begin{aligned}p_n &\ge (n-1)\mathbf{P}\left(\sum_{i\le n-1} X_i > x,\ \sum_{i\le n-1} X_i\wedge b\le x,\ X_1\le\frac{x}{2},\dots,X_{n-2}\le\frac{x}{2}, X_{n-1}>\frac{x}{2},\right)\\ &= (n-1)\mathbf{P}\left(X_{n-1} > x - \sum_{i\le n-2} X_i,\ \sum_{i\le n-2} X_i\le\frac{x}{2},\ X_1\le\frac{x}{2},\dots,X_{n-2}\le\frac{x}{2}\right)\\ &\ge (n-1)\mathbf{P}(X_{n-1}>x)\mathbf{P}\left(\sum_{i\le n-2} X_i\le\frac{x}{2},\ X_1\le\frac{x}{2},\dots,X_{n-2}\le\frac{x}{2}\right)\\ &= (n-1)\mathbf{P}(X_{n-1}>x)\mathbf{P}\left(\sum_{i\le n-2} X_i\le\frac{x}{2}\right).\end{aligned}$$

Hence,

$$p_n \ge (n-1)(1-F(x))F^{(n-2)}\left(\frac{x}{2}\right). \quad (5.21)$$

We estimate $F^{(n-2)}(x/2)$. As $\varphi_2 < \infty$, we have by Chebyshev's inequality

$$F^{(k)}(z) \ge \mathbf{1}(z-k)\left(1 - \frac{k(\varphi_2-1)}{(z-k)^2}\right)_+. \quad (5.22)$$

Define

$$d(u) = 1 - \frac{u(\varphi_2 - 1)}{(x/2 - u)^2}, \quad u \le y_* < \frac{x}{2}. \tag{5.23}$$

Obviously, function $d(u)$ is concave on $[0, y_*]$, $d(0) = 1$, and $d(y_*) = 0$. Hence,

$$d(u) \ge 1 - \frac{u}{y_*}, \quad 0 \le u \le y_*,$$

and

$$\begin{aligned}
\sum_{n\ge1} q(1-q)^{n-1}p_n &\ge (1 - F(x)) \sum_{n\ge1} q(1-q)^{n-1}(n-1)F^{(n-2)}\left(\frac{x}{2}\right) \\
&\ge (1 - F(x)) \sum_{n\le y_*+2} q(1-q)^{n-1}(n-1)\left(1 - \frac{n-2}{y_*}\right) \\
&\ge q(1-q)(1-F(x)) \int_0^{y_*} e^{-q'u} u \left(1 - \frac{u}{y_*}\right) du \\
&= K(q,x)\frac{1-F(x)}{q}.
\end{aligned} \tag{5.24}$$

The assertion of the theorem follows from (5.16), (5.19), and (5.24). □

A similar estimate can be obtained in the case, where $\varphi_2 = \infty$ but $\varphi(G) < \infty$ for a function $G \in \mathcal{G}_c$. Let

$$x_* = \min\{x : g\left(\frac{x}{2}\right) \ge \varphi(G)\}, \tag{5.25}$$

where $g$ is the derivative of function $G$. Using properties of the function $g$, one can prove that function $xg^2(x/2)/(g(x/2) - \varphi(G))^2$ is unimodal for $x \ge x_*$. We denote by $x_{**}$ the point where this function takes its maximal value that is

$$x_{**} = \operatorname{argmax}\left\{\frac{xg^2(x/2)}{(g(x/2) - \varphi(G))^2} : x \ge x_*\right\}. \tag{5.26}$$

Let

$$E_1(x;G) = \exp\left(-\frac{q'x'g^2(x'/2)}{(g(x'/2) - \varphi(G))^2}\right), \tag{5.27}$$

where

$$x' = x \vee x_{**}.$$

**Theorem 2** *Let $\varphi_2 = \infty$ and $\varphi(G) < \infty$ for a function $G \in \mathcal{G}_c$, $G(0) = 0$. Then*

$$\Psi_0(x) \ge E_1(x;G) + K_1(q,x)\frac{1-F(x)}{q}, \tag{5.28}$$

*where*

$$K_1(q,x) = \frac{q^2(1-q)}{(q')^2}\left(1 + \exp(-q'y_{**}) - \frac{2}{q'y_{**}}(1 - \exp(-q'y_{**}))\right), \tag{5.29}$$

*and $y_{**}$ is the unique solution of the equation*

$$y_{**}\varphi(G) + G(y_{**}) = G\left(\frac{x}{2}\right). \tag{5.30}$$

*Proof.* Main arguments in the proof of Theorem 1 remain true in this case too. There are only two differences. To estimate variable $\mathbf{E}(1-q)^{N^{(b)}}(x)$ by Jensen's inequality, one should use inequality (5.14), from where the term $E_1(x;G)$ in the right-hand side of estimate (5.28) appears.

Term $F^{(n-2)}(x/2)$ in formula (5.21) can be estimated with the help of Chebyshev's inequality:

$$F^{(k)}(z) \geq \left(1 - \frac{\mathbf{E}G(X_1 + \cdots + X_k)}{G(z)}\right)_+ . \tag{5.31}$$

Using concavity of $g(u)$ and Jensen's inequality, we arrive at the relation

$$\begin{aligned} \mathbf{E}G\left(\sum_{i=1}^{k} X_i\right) &= \mathbf{E}\int_0^{X_1+\cdots+X_k} g(u)\,du \leq \mathbf{E}\left(\sum_{i=1}^{k} G(X_i) + \sum_{i=2}^{k} X_i g(X_1 + \cdots + X_{i-1})\right) \\ &\leq k\varphi(G) + g(1) + \cdots + g(k-1) \leq k\varphi(G) + G(k). \end{aligned}$$

Therefore,

$$F^{(k)}(z) \geq \left(1 - \frac{k\varphi(G) + G(k)}{G(z)}\right)_+ .$$

Define the function

$$d(u) = 1 - \frac{u\varphi(G) + G(u)}{G(x/2)},$$

which is obviously concave on the interval $[0, y_{**}]$. Using the arguments of Theorem 1, we arrive at the desired assertion. □

# 6 Upper bounds in the presense of large claims

We find now upper estimates of probability $\Psi_0(x)$ (see (5.2)). For this, let us introduce truncated r.v.'s $X(b)$ and $Y(b)$ (see (5.8) – (5.7)) with truncation level

$$b = \frac{a}{q}, \tag{6.1}$$

where $a$ is a constant to be appropriately chosen.

The assertion of the following lemma results from the definition of r.v.'s $X(b)$ and $Y(b)$ and equality (6.1) (equality (5.16) was derived similarly).

**Lemma 4** *If variable $b$ is defined by (6.1), then*

$$\Psi_0(x) \leq \begin{cases} \Psi_0^{(a)}(x), & \text{if } 0 \leq x \leq a/q, \\ \Psi_0^{(a)}(x) + r_a(x), & \text{if } x > a/q, \end{cases} \tag{6.2}$$

*where*

$$\begin{aligned} \Psi_0^{(a)}(x) &= \mathbf{P}\left(\sum_{i<\nu} X_i(b) > x\right), \\ r_a(x) &= \mathbf{P}\left(\sum_{i<\nu} (X_i(b) + Y_i(b)) > x;\ \sum_{i<\nu} Y_i(b) \leq x\right). \end{aligned}$$

Summand $\Psi_0^{(a)}$ depends on the truncated r.v. $X_i(b) = X_i(a/q)$ and therefore it can be estimated with the help of Lundberg's inequality. Let

$$M(a) = \frac{2}{a^2}\left(e^a - 1 - a\right). \tag{6.3}$$

**Lemma 5** *Let $\varphi_2 < \infty$ and*

$$R(a) = \frac{1}{\varphi_2 M(a)}\left(-1 + \sqrt{1 + 2q\varphi_2 M(a)}\right). \tag{6.4}$$

*Then for every $a > 0$ and $b$, defined by equation (6.1),*

$$\Psi_0^{(a)}(x) \le \exp(-R(a)x). \tag{6.5}$$

*Proof.* For every $p > 0$

$$\exp\left(pX(b)\right) \le 1 + pX(b) + p^2X^2(b)\,\frac{\exp(pa/q) - 1 - pa/q}{(pa/q)^2}.$$

Let $p = R(a)$. Evidently, $p \le q$. Since $\mathbf{E}X(b) \le 1$ and $\mathbf{E}X^2(b) \le \varphi_2$,

$$\mathbf{E}\exp\left(pX(b)\right) \le 1 + p + \frac{1}{2}p^2\varphi_2 M(a) \equiv 1 + q.$$

As $b = a/q$,

$$(1-q)\,\mathbf{E}\exp\left(R(a)X(b)\right) \le (1+q)(1-q) < 1.$$

It follows from the results of Section 4 that relation (6.5) holds. □

If $\varphi_2 = \infty$, then the following result can be stated.

**Lemma 6** *Let $\varphi_2 = \infty$ and $\varphi(G) < \infty$ for $G \in \mathcal{G}_c$. Let $g(u) = dG(u)/du$, $b$ be defined by (6.1), and*

$$R(a) = q\left(1 - (\varphi(G) - G(0))M(a)\frac{a}{g(a/q)}\right)_+. \tag{6.6}$$

*Then inequality (6.5) holds true.*

*Proof.* For any $p > 0$

$$\exp\left(pX(b)\right) \le 1 + pX(b) + (G(X(b)) - G(0))\frac{p^2X^2(b)}{(G(X(b)) - G(0))}\,\frac{\exp(pX(b)) - 1 - pX(b)}{p^2X^2(b)}.$$

As $G \in \mathcal{G}_c$,

$$G(u) - G(0) \ge \frac{ug(u)}{2},$$

and therefore,

$$\mathbf{E}\exp(pX(b)) \le 1 + p + p^2(\varphi(G) - G(0))M(a)\,\frac{X(b)}{g(X(b))} \le 1 + p + p^2(\varphi(G) - G(0))M(a)\,\frac{a}{qg(a/q)}.$$

Put $p = R(a)$ (see (6.6)). Evidently, $p \le q$ and hence

$$(1-q)\,\mathbf{E}\exp\left(R(a)X(b)\right) \le 1,$$

which yields the assertion of the lemma. □

We estimate now the summand $r_a(x)$ from the right-hand side of relation (6.2):

$$r_a(x) = \sum_{k\geq 1} q(1-q)^k r_a^{(k)}(x), \tag{6.7}$$

where

$$r_a^{(k)}(x) = \mathbf{P}\left(\sum_{i\leq k}(X_i(b)+Y_i(b)) > x;\ \sum_{i\leq k} Y_i(b) \leq x\right). \tag{6.8}$$

**Lemma 7** *For every $a>0$, the uniform estimate*

$$\sup_{x\geq a/q} r_a(x) \leq \frac{1-q}{q}\left(1-F\left(\frac{a}{q}\right)\right) \tag{6.9}$$

*holds.*

*Proof.* As

$$\left\{\sum_{i\leq k}(X_i(b)+Y_i(b)) > x;\ \sum_{i\leq k} Y_i(b) \leq x\right\} \subset \left\{\sum_{i\leq k} Y_i(b) > b\right\} \subset \bigcup_{i\leq k}\left\{Y_i > \frac{a}{q}\right\},$$

we have

$$r_a^{(k)} \leq k\left(1-F\left(\frac{a}{q}\right)\right),$$

from which, using (6.7), we arrive at the assertion of the lemma. □

Let us obtain now a non-uniform estimate of $r_a(x)$, $x > a/q$. For this, we need in d.f.'s $F^{(b)}(u)$ and $T^{(b)}(u)$ (see (5.6) and (5.7)) for $b = a/q$. Denote by $F_k^{(b)}(u)$ and $T_k^{(b)}(u)$ the $k$-fold convolutions of the corresponding d.f.'s. Take an arbitrary number $0<\epsilon\leq 1$. By definition of $r_a^{(k)}(x)$,

$$r_a^{(k)}(x) = \int_0^x \left(1-T_k^{(b)}(x-u)\right)\, dF_k^{(b)}(u).$$

Further,

$$\begin{aligned} 1-T_k^{(b)}(x-u) &= \mathbf{P}\left(\sum_{i\leq k} Y_i(b) > x-u;\ \bigcup_{i\leq k}\{Y_i(b) > \epsilon x\}\right) \\ &+ \mathbf{P}\left(\sum_{i\leq k} Y_i(b) > x-u;\ \bigcap_{i\leq k}\{Y_i(b) \leq \epsilon x\}\right) \equiv A+B. \end{aligned}$$

Obviously, for $x > a/q$,

$$A \leq k\,\mathbf{P}(Y(b) > \epsilon x) \leq k\left(1-F(\epsilon x)\right).$$

We define $Y_i$, $i\geq 1$ as i.i.d.r.v.'s with a common d.f.

$$\mathbf{P}(Y\leq v) = \mathbf{P}(Y(b)\leq v \mid Y(b)\leq \epsilon x) = \frac{T^{(b)}(v)}{T^{(b)}(\epsilon x)}, \quad v\leq \epsilon x.$$

Then for every $\lambda > 0$,

$$B = \left(T^{(b)}(\epsilon x)\right)^k \mathbf{P}\left(\sum_{i\le k} Y_i > x-u\right)$$
$$\le \exp(-\lambda(x-u))\left(\int_0^{\epsilon x}\exp(\lambda v)\,dT^{(b)}(v)\right)^k.$$

Assume that $\varphi(G) < \infty$ for a function $G(u) = \exp(\Lambda(u)) \in \mathcal{G}^s$ $(s \ge 0)$. We choose

$$\lambda = \frac{\Lambda(\epsilon x)}{\epsilon x}.$$

Such a choice yields (by condition $G \in \mathcal{G}^s$)

$$\exp(\lambda v) \le \exp(\Lambda(v)) \equiv G(v), \quad 0 \le v \le \epsilon x,$$

and because of this

$$B \le \frac{\exp(\lambda u)}{G^{1/\epsilon}(\epsilon x)}\left(\int_0^{\infty} G(v)\,dT^{(b)}(v)\right)^k.$$

Substituting the above estimates for $A$ and $B$ in the expression for $r_a^{(k)}(x)$, we arrive at

$$r_a^{(k)}(x) \le k(1-F(\epsilon x)) + G^{-1/\epsilon}(\epsilon x)\,(\mathbf{E}G(Y(b)))^k\left(\mathbf{E}\exp\left(\frac{\Lambda(\epsilon x)}{\epsilon x}X(b)\right)\right)^k. \tag{6.10}$$

This relation, equalities (6.7) and (6.8) and Lemmas 4, 5 and 6 yield the following assertion.

**Lemma 8** *Let, for fixed $q > 0$, $a > 0$, $x > a/q$, $0 < \epsilon \le 1$, and $G \in \mathcal{G}^s$ $(s \ge 0)$, the conditions*

$$\varphi(G) < \infty$$

*and*

$$d \equiv (1-q)\,\mathbf{E}G(Y(b))\,\mathbf{E}\exp\left(\frac{\Lambda(\epsilon x)}{\epsilon x}X(b)\right) < 1 \tag{6.11}$$

*are satisfied. Then*

$$\Psi_0(x) \le \exp(-R(a)x) + \frac{1-q}{q}(1-F(\epsilon x)) + \frac{qd}{1-d}G^{-1/\epsilon}(\epsilon x), \tag{6.12}$$

*where the constant $R(a)$ is defined in Lemma 5 for $\varphi_2 < \infty$ and in Lemma 6 for $G \in \mathcal{G}_c \cap \mathcal{G}_s$.*

In spite of the fact that the assertion of Lemma 8 does not seem attractive, it is convenient as a starting point for numerical calculations, as all the variables involved in the right-hand side of inequality (6.12) can be either computed or estimated.

The form of the estimate, appearing in Lemma 8 as well as of the lower estimate (see (5.15) and (5.28)) is rather intuitive. The first summand in all these estimates, by its origin and meaning, corresponds to the exponential form of ruin probability in Cramér's case.

Moreover the power of the exponent in lower estimates (see Theorems 1 and 2) and in upper estimates (see (6.12)) are asymptotically equivalent to $q$ for $q \to 0$. The same property is inherent to Lundberg's exponent, defined by equation (2.18). The first summand may play an important role for comparatively small sizes of initial capital $x > 0$. But when $x$ increases, the main role is passed on other components. The second summand in the lower and upper bounds corresponds to asymptotic formula (2.39), although there is the difference between (2.39) and these summands: lower bounds (5.15) and (5.28) contain coefficient $K(q,x)$ such that

$$\lim_{x\to\infty} K(q,x) = \frac{1-q}{1+q} + o(q).$$

In upper estimate (6.12), the second summand includes an arbitrary constant $\epsilon$ $(0 < \epsilon \leq 1)$. For $\epsilon = 1$, this summand equals exactly to the asymptotic approximation (2.39). However, the variable $1/G(x)$, appearing in the third summand in (6.12), decreases slower than the second summand when $x \to \infty$. Perhaps, this is resulted from the fact that estimate (6.12) is valid for a class of d.f.'s $F$ which is wider than class $\mathcal{S}$. In order for the second term to dominate in the right-hand side of (6.12), a constant $\epsilon$ was introduced. The following lemma clears up partly the role of $\epsilon$. It is concerned with the class of d.f.'s $F$ that plays an important role in the risk theory (see [72] and [165]). This class is characterized by the property that function $\Lambda_F(u) = -\ln(1-F(u))$ is *finally concave*. This means that there exists $u_0 \geq 0$ such that $\Lambda_F(u)$ is concave for $u \geq u_0$. For example, the lognormal distribution, the Pareto distribution and integrated tails of these distributions have this property.

**Lemma 9** *Let the function* $-\ln(1-F(u))$ *be finally concave. Then there exists a fuction* $G \in \mathcal{G}^s$ *(for some* $s \geq 0$*) such that*

$$G^{-1/\epsilon}(\epsilon x) = o(1-F(\epsilon x)) \tag{6.13}$$

*for every* $\epsilon < 1$ *which is close enough to* 1.

*Proof.* By the conditions of the lemma, there exists a number $u_* \geq 0$ such that $\Lambda_F(u)/u$ decreases for $u \geq u_*$. Fix a positive number $\kappa < 1$ and choose

$$\Lambda(u) = \begin{cases} \kappa u \Lambda_F(u_*)/u_*, & \text{for } u \leq u_*, \\ \kappa \Lambda_F(u) & \text{for } u > u_*. \end{cases}$$

Function $G(u) = \exp(\Lambda(u))$ belongs to a class $\mathcal{G}^s$ (at least, $G \in \mathcal{G}^{(0)}$) and $\varphi(G) = \mathbf{E}G(X) < \infty$. In addition,

$$G^{-1/\epsilon}(\epsilon x) = (1-F(\epsilon x))^{\kappa/\epsilon}$$

for $u \geq u_*$. Hence the assertion of the lemma is true for $\epsilon > \kappa$. □

Let us note that the asymptotic behaviour of the summands give information that is true only for $x \to \infty$. In concrete calculations when it is necessary to estimate the ruin probability for *finite* values of the initial capital $x$, the behaviour of the summands can be cardinally different from the limiting behaviour. We note also that the quality of the upper estimate depends on the choice of the function $G$. Such a situation is typical in mathematics. For example, the second Lyapunov's method from the theory of stability and its various generalizations uses the idea of the choice of a test function in order to obtain the required property of the underlying system (see for example [112]).

Let us examine now the components of upper estimate (6.12) in details.

We start with variable $\mathbf{E}\exp(\Lambda(\epsilon x)X(b)/\epsilon x)$. Fix $a > 0$ and $0 < \epsilon \leq 1$ and denote

$$p(x) = \frac{\Lambda(\epsilon x)}{\epsilon x}.$$

Obviously, $p(x) \to 0$ for $x \to \infty$.

Assume first that $\varphi_2 < \infty$. Then the proof of Lemma 5 yields that

$$\mathbf{E}\exp(\Lambda(\epsilon x)X(b)/\epsilon x) \leq 1 + p(x) + \frac{p^2(x)\varphi_2 M(a)}{2} \equiv 1 + r(x). \tag{6.14}$$

In the case $\varphi_2 = \infty$ and $G \in \mathcal{G}_c$, we have, with the help of Lemma 6,

$$\mathbf{E}\exp\left(\Lambda(\epsilon x)X(b)/\epsilon x\right) \le 1 + p(x) + \frac{qa(\varphi(G) - G(0))M(a)}{g(a/q)} \equiv 1 + r(x). \tag{6.15}$$

Function $r(x)$ (its different forms are presented in equations (6.14) and (6.15)) has the property that $r(x) \to 0$ along $x \to \infty$.

It follows from the proof of Lemma 8 that for every *fixed* $q > 0$ there exists a number $a > 0$ such that

$$r_a(x) \le \Psi_0^{(1)}(x) \sim \frac{(1-q)(1-F(\epsilon x))}{q} + \frac{qd}{1-d} G^{-1/\epsilon}(\epsilon x), \tag{6.16}$$

for every $\epsilon > 0$, where the number $d$ is define in the (6.11) and $\Psi_0^{(1)}$ in the relation after the (6.20).

Although $\mathbf{E}G(Y(a/q)) \to 1$ for $a/q \to \infty$, this does not yield (in general) the uniformity (with respect to $q$) of this limit for $a \to \infty$.

The results above can be summed up in the form of theorem. For this, we denote

$$\psi(G, b) \equiv 1 + \mathbf{E}(G(X)I(X > b)) \tag{6.17}$$

Obviously (see (5.9)),

$$\mathbf{E}G(Y(b)) \le \psi(G, b).$$

**Theorem 3** *Let $G \in \mathcal{G}^s$ $(s \ge 0)$, the function $\psi$ defined in the (6.17), and the variables $a$, $\epsilon$ and $x$ satisfy the relation*

$$d_1 \equiv (1-q)(1 + r(x))\psi(G, a/q) < 1. \tag{6.18}$$

*Then*

$$\Psi_0(x) \le \exp(-R(a)x) + 1\left(x - \frac{a}{q}\right)\left(\frac{(1-q)(1-F(\epsilon x))}{q} + \frac{qd_1}{1-d_1} G^{-1/\epsilon}(\epsilon x)\right), \tag{6.19}$$

*where $R(a)$ is given in Lemma 4 and $r(x)$ is defined by equality (6.14) in the case $\varphi_2 < \infty$, and $R(a)$ is given in Lemma 5 and $r(x)$ is defined by equality (6.15) in the case $G \in \mathcal{G}_c \cap \mathcal{G}^s$.*

Inequality (6.19) can be improved by accounting the following two features. First, let us recall that a uniform upper bound of $r_a(x)$ was found in Lemma 7 (see (6.9)). Hence,

$$\Psi_0(x) \le \exp(-R(a)x) + 1\left(x - \frac{a}{q}\right)\Psi_0^{(1)}(x) \equiv \Psi_1(x), \tag{6.20}$$

where

$$\Psi_0^{(1)}(x) = \min\left(\frac{1-q}{q}\left(1 - F\left(\frac{a}{q}\right)\right), \frac{(1-q)(1-F(\epsilon x))}{q} + \frac{qd_1}{1-d_1} G^{-1/\epsilon}(\epsilon x)\right),$$

Second, $\Psi_0(x)$ is a monotone non-increasing function. This yields the validity of the estimate

$$\Psi_0(x) \le \min_{u \le x} \Psi_1(u). \tag{6.21}$$

# 7 Numerical examples

We examine a classical risk model, determined by parameters $\lambda$, $c$ and d.f. $B(u)$. We denote by $b_k$ ($k \geq 1$) the $k$th moment of d.f. $B$. In order to use the results of Sections 5 and 6, we introduce probability $q$ by formlula (2.35) and d.f. $F$ of the normalized ladder height $X$ by formula

$$F(u) = \int_0^{b_2 u/2} (1 - B(z))\, dz. \tag{7.1}$$

The power moments of r.v. $X$ (see (5.10)) are equal to

$$\varphi_k = \mathbf{E}X^k = \frac{2^k b_{k+1}}{(k+1) b_2^k}.$$

In this section we shall restrict ourselves to the case $\varphi_2 < \infty$ and find upper and lower bounds of ruin probability $\Psi(x)$, deriving them from the corresponding bounds of probability $\Psi_0(x)$ (see formula (5.5)). Lower bounds are taken from formlula (5.15) while upper bounds are taken from the assertion of Theorem 3. The main part of the calculations can be written in the form of the following algorithm.

**Calculation of upper bounds of ruin probability**

1. Fix parameters $q$, $a$, $\epsilon$, $x$, and function $G \in \mathcal{G}^s$.
2. Find $M(a)$ from formula (6.3) and $R(a)$ from formlula (6.4).
3. Calculate (or estimate from above) $\psi(G, a/q)$ (see (6.17)).
4. Find $r(x)$ from formula (6.14).
5. Define $d_1$ from formula (6.18).
6. If $d_1 \geq 1$, then an upper bound is not defined for given values of parameters. In the opposite case, take the right-hand side of formula (6.19) as an upper bound.

In the calculation process, parameters $q$ and $x$ are given. Other parameters ($a$, $\epsilon$) and function $G$ are chosen so that they minimize the value of the upper estimate. The possibility to get $d_1 < 1$, permiting the calculation of an upper estimate in point 6 of the algroritihm, was proved in Section 6. It is natural that the choice of function $G$ is realized as the choice of some parameters defining the function.

We provide two examples where two-sided bounds of ruin probability in the presense of large claims are found. We do not display routine calculations (like calculations of lower and upper bounds of probability $\Psi_0(x)$ by formulae (5.15) and (6.21) and transition from them to lower $\underline{\Psi}(x)$ and upper $\bar{\Psi}(x)$ estimates of ruin probability $\Psi(x)$ (see (5.5)), the Embrechts-Veraverbeke asymptotic approximation (2.39), etc.), but we display the main formulae only.

**Example 3.** *Lognormal distribution of claim sizes*

Let $B(u)$ be a lognormal distribution with density

$$b(u) = \frac{1}{\sqrt{2\pi}\, u\sigma} \exp\left(-\frac{(\ln u + \sigma^2/2)^2}{2\sigma^2}\right).$$

In this case, $b_k = \exp(k(k-1)\sigma^2/2)$ and, in particular, $b_1 = 1$. Function $F$, defined by equality (7.1), has an asymptotic representation (see [176])

$$1 - F(u) \sim \frac{\sigma^2 b_2}{2\sqrt{2\pi}\,(\ln^2(b_2 u/2) - \sigma^4/4)} \exp\left(-\frac{(\ln(b_2 u/2) - \sigma^2/2)^2}{2\sigma^2}\right).$$

Let $G(u) = \exp(\Lambda(u))$, where

$$\Lambda(u) = \begin{cases} \Lambda_1(u), & \text{if } u > x^*, \\ \Lambda_1(x^*)u/x^*, & \text{if } u \le x^*, \end{cases}$$

$$\Lambda_1(u) = \frac{1}{2\sigma^2}\left(\ln\left(\frac{b_2 u}{2}\right) + \frac{\sigma^2}{2}\right)^2 - t\ln\left(\frac{b_2 u}{2}\right) + \kappa,$$

Variable $x^*$ will be defined later on, and both $\kappa \in (-\infty, \infty)$ and $t > 0$ serve as parameters of function $G$ to be chosen.

It is easy to see that $\varphi(G) < \infty$ for $t > 1$. In calculations, we consider that $t > 1$. If

$$\kappa \ge \frac{1 + t(t-1)\sigma^4}{2\sigma^2},$$

the function $\Lambda_1(u)/u$ decreases while $u \ge 0$ increases. Thus, $\Lambda_1(u)$ takes its minimal value at the point

$$x^* = 2\exp\left(\left(t - \frac{3}{2}\right)\sigma^2\right).$$

Evidently, function $G$ thus constructed belongs to any class $\mathcal{G}^s$ (for any $s \ge 0$).

The following tables contain the results of calculations. In them, $a$, $t$, $\kappa$, $\epsilon$ are the values of the parameters, for which $\bar{\Psi}(x)$ takes the minimal value at point $x$, and the values of the asymptotic approximation $\Psi_{EV}(x)$ computed by formula (2.39).

The first table contains numerical data corresponding to the value $\sigma = 1.8$ ($b_2 = 25.53$).

| $\rho$ | $x$ | $\bar{\Psi}(x)$ | $a$ | $t$ | $\kappa$ | $\epsilon$ | $\underline{\Psi}(x)$ | $\Psi_{EV}(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 10000 | $5.9 \cdot 10^{-3}$ | 1.1 | 1.5 | 1.41 | 0.9 | $3.0 \cdot 10^{-5}$ | $3.5 \cdot 10^{-5}$ |
| | 50000 | $2.8 \cdot 10^{-4}$ | 2.7 | 1.6 | 1.72 | 1.0 | $3.6 \cdot 10^{-7}$ | $4.0 \cdot 10^{-7}$ |
| 0.2 | 1000 | $6.7 \cdot 10^{-2}$ | 3.3 | 1.4 | 1.07 | 0.93 | $9.8 \cdot 10^{-4}$ | $2.9 \cdot 10^{-3}$ |
| | 10000 | $2.7 \cdot 10^{-3}$ | 5.1 | 1.3 | 0.80 | 0.98 | $1.4 \cdot 10^{-5}$ | $1.7 \cdot 10^{-5}$ |
| | 50000 | $4.7 \cdot 10^{-4}$ | 6.3 | 1.7 | 2.10 | 1.0 | $1.7 \cdot 10^{-7}$ | $2.0 \cdot 10^{-7}$ |
| 0.3 | 1000 | $5.3 \cdot 10^{-2}$ | 2.7 | 1.6 | 1.70 | 1.0 | $8.0 \cdot 10^{-4}$ | $1.9 \cdot 10^{-3}$ |
| | 10000 | $1.8 \cdot 10^{-3}$ | 3.1 | 1.3 | 0.79 | 0.99 | $8.8 \cdot 10^{-6}$ | $1.2 \cdot 10^{-5}$ |
| | 50000 | $5.5 \cdot 10^{-5}$ | 4.8 | 1.4 | 1.10 | 0.99 | $1.0 \cdot 10^{-7}$ | $1.3 \cdot 10^{-7}$ |
| 0.4 | 1000 | $5.3 \cdot 10^{-2}$ | 2.4 | 1.9 | 2.90 | 0.98 | $6.6 \cdot 10^{-4}$ | $1.4 \cdot 10^{-3}$ |
| | 10000 | $3.7 \cdot 10^{-4}$ | 4.7 | 1.6 | 1.70 | 0.50 | $6.2 \cdot 10^{-6}$ | $8.7 \cdot 10^{-6}$ |
| | 50000 | $1.8 \cdot 10^{-5}$ | 6.2 | 1.6 | 1.70 | 0.70 | $7.2 \cdot 10^{-8}$ | $1.0 \cdot 10^{-7}$ |
| 0.5 | 1000 | $4.0 \cdot 10^{-2}$ | 3.7 | 2.6 | 6.90 | 0.70 | $5.4 \cdot 10^{-4}$ | $1.1 \cdot 10^{-3}$ |
| | 10000 | $7.8 \cdot 10^{-3}$ | 2.3 | 1.7 | 2.09 | 0.99 | $4.6 \cdot 10^{-6}$ | $7.0 \cdot 10^{-6}$ |
| | 50000 | $6.2 \cdot 10^{-6}$ | 1.7 | 1.2 | 0.55 | 0.80 | $5.4 \cdot 10^{-8}$ | $8.0 \cdot 10^{-8}$ |

It is seen that the lower bound fits the asymptotic value of ruin probability to a good accuracy. The accuracy of the upper bound is low which can be explained partly by the fact that the choice of optimal values of $a$, $t$, $\kappa$, and $\epsilon$ was organized as an item-by-item examination of *a few* values of these parameters and the function $G$ was taken only in the mentioned form. One should not forget however that unknown ruin probability *certainly* lies between these bounds.

The second table corresponds to the case $\sigma = 0.989$ ($b_2 = 2.72$).

| $\rho$ | $x$ | $\bar{\Psi}(x)$ | $a$ | $t$ | $\kappa$ | $\epsilon$ | $\underline{\Psi}(x)$ | $\Psi_{EV}(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 10 | $6.1 \cdot 10^{-1}$ | 2.9 | 3.2 | 4.0 | 1.0 | $1.8 \cdot 10^{-1}$ | $9.5 \cdot 10^{-3}$ |
| | 50 | $8.1 \cdot 10^{-2}$ | 2.1 | 2.5 | 2.35 | 1.0 | $1.4 \cdot 10^{-2}$ | $5.4 \cdot 10^{-3}$ |
| | 100 | $1.1 \cdot 10^{-2}$ | 3.6 | 2.4 | 2.10 | 1.0 | $4.0 \cdot 10^{-4}$ | $2.8 \cdot 10^{-5}$ |
| 0.2 | 10 | $4.0 \cdot 10^{-1}$ | 1.4 | 2.4 | 2.20 | 1.0 | $4.0 \cdot 10^{-2}$ | $4.8 \cdot 10^{-2}$ |
| | 50 | $3.5 \cdot 10^{-2}$ | 3.7 | 1.8 | 1.22 | 1.0 | $3.4 \cdot 10^{-4}$ | $2.7 \cdot 10^{-4}$ |
| | 100 | $5.5 \cdot 10^{-3}$ | 4.9 | 2.7 | 2.8 | 1.0 | $7.7 \cdot 10^{-6}$ | $1.4 \cdot 10^{-5}$ |
| 0.3 | 10 | $3.7 \cdot 10^{-1}$ | 1.2 | 2.5 | 2.4 | 1.0 | $1.0 \cdot 10^{-2}$ | $3.1 \cdot 10^{-2}$ |
| | 50 | $2.9 \cdot 10^{-2}$ | 3.7 | 1.8 | 1.2 | 1.0 | $7.2 \cdot 10^{-5}$ | $1.8 \cdot 10^{-4}$ |
| | 100 | $5.1 \cdot 10^{-3}$ | 5.6 | 2.1 | 1.6 | 1.0 | $5.4 \cdot 10^{-6}$ | $9.3 \cdot 10^{-6}$ |
| 0.4 | 10 | $3.0 \cdot 10^{-1}$ | 1.6 | 1.9 | 1.4 | 0.98 | $3.2 \cdot 10^{-3}$ | $2.4 \cdot 10^{-2}$ |
| | 50 | $2.7 \cdot 10^{-2}$ | 4.2 | 1.7 | 1.10 | 1.0 | $5.5 \cdot 10^{-5}$ | $1.3 \cdot 10^{-4}$ |
| | 100 | $5.1 \cdot 10^{-3}$ | 3.8 | 3.5 | 4.8 | 0.98 | $4.0 \cdot 10^{-6}$ | $7.0 \cdot 10^{-6}$ |
| 0.5 | 10 | $2.9 \cdot 10^{-1}$ | 1.3 | 2.1 | 1.6 | 0.98 | $1.5 \cdot 10^{-3}$ | $1.9 \cdot 10^{-2}$ |
| | 50 | $2.3 \cdot 10^{-2}$ | 2.7 | 2.4 | 3.25 | 0.46 | $4.6 \cdot 10^{-5}$ | $1.1 \cdot 10^{-4}$ |
| | 100 | $2.2 \cdot 10^{-3}$ | 2.7 | 1.7 | 1.1 | 1.0 | $3.1 \cdot 10^{-6}$ | $5.6 \cdot 10^{-6}$ |

It can be seen that here the values calculated with the help of the asymptotic formula in some cases do not lie between upper and lower bounds. This can be explained by the fact that the behaviour of ruin probability for moderate values of $x$ should be calculated with the accounting of the first (Cramér's) summand, which presents in the lower and upper bounds derived above. Namely this summand begin dominating when $\sigma$ decays. This fact is particularly keen for $\sigma = 0.53$ ($b_2 = 1.32$); see the following table.

| $\rho$ | $x$ | $\bar{\Psi}(x)$ | $a$ | $t$ | $\kappa$ | $\epsilon$ | $\underline{\Psi}(x)$ | $\Psi_{EV}(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 10 | $1.2 \cdot 10^{-1}$ | 2.9 | 7.9 | 9.51 | 1.0 | $7.9 \cdot 10^{-2}$ | $1.2 \cdot 10^{-5}$ |
| | 20 | $1.8 \cdot 10^{-2}$ | 1.1 | 6.9 | 7.55 | 1.0 | $1.3 \cdot 10^{-2}$ | $1.5 \cdot 10^{-8}$ |
| | 30 | $2.4 \cdot 10^{-3}$ | 1.3 | 6.9 | 7.55 | 1.0 | $2.2 \cdot 10^{-3}$ | $1.9 \cdot 10^{-10}$ |
| 0.3 | 10 | $8.3 \cdot 10^{-2}$ | 1.1 | 6.0 | 6.02 | 1.0 | $2.6 \cdot 10^{-2}$ | $7.9 \cdot 10^{-6}$ |
| | 20 | $6.1 \cdot 10^{-3}$ | 1.4 | 6.0 | 6.02 | 1.0 | $2.0 \cdot 10^{-3}$ | $1.0 \cdot 10^{-8}$ |
| | 30 | $5.5 \cdot 10^{-4}$ | 1.8 | 5.9 | 5.9 | 1.0 | $1.5 \cdot 10^{-4}$ | $1.3 \cdot 10^{-10}$ |
| 0.4 | 10 | $5.6 \cdot 10^{-2}$ | 1.3 | 5.4 | 5.14 | 1.0 | $9.2 \cdot 10^{-3}$ | $6.0 \cdot 10^{-6}$ |
| | 20 | $3.0 \cdot 10^{-3}$ | 1.8 | 5.4 | 5.14 | 1.0 | $3.4 \cdot 10^{-4}$ | $7.6 \cdot 10^{-9}$ |
| | 30 | $1.6 \cdot 10^{-4}$ | 0.1 | 6.5 | 7.84 | 0.3 | $1.2 \cdot 10^{-5}$ | $9.5 \cdot 10^{-11}$ |
| 0.5 | 10 | $4.2 \cdot 10^{-2}$ | 1.5 | 5.0 | 4.60 | 1.0 | $3.5 \cdot 10^{-3}$ | $4.8 \cdot 10^{-6}$ |
| | 20 | $1.8 \cdot 10^{-3}$ | 0.1 | 6.7 | 8.19 | 0.35 | $6.7 \cdot 10^{-5}$ | $6.1 \cdot 10^{-9}$ |
| | 30 | $3.3 \cdot 10^{-5}$ | 0.1 | 7.6 | 10.89 | 0.32 | $1.2 \cdot 10^{-6}$ | $7.6 \cdot 10^{-11}$ |

For $\rho = 0.1$ the upper bound was trivial ($= 1$) and for this reason we do not give the values of corresponding parameters in the table. In this case, upper and lower bounds are pretty close and the asymptotic approximation does not work. This can be explained by the fact that the first summand (the Cramér one) totally dominates here.

**Example 4.** *Pareto distribution of claim sizes*

Let $B(u)$ be a Pareto distribution with mean 1:

$$B(u) = \begin{cases} 1 - (1-1/t)^t/u^t, & \text{for } u \geq 1 - 1/t, \\ 0, & \text{in opposite case} \end{cases},$$

$t > 1$ is the paremeter of the distribution. In particular, $b_2 = (t-1)^2/(t(t-2))$ decreases when $t$ increases. We consider that $t > 3$ (more precisely, $t = 3.01$ corresponding to $b_2 = 1.33$) in order for claim sizes to have a finite third moment. Therefore, d.f. $F$ has a finite second moment $\varphi_2 < \infty$. Let $G(u) = 1 + a_1 u + a_2 u^2$, where $a_1 \geq 0$ and $a_2 \geq 0$ are the parameters of $G$ to be chosen. Evidently, $G \in \mathcal{G}^s$.

The following table contains the results of calculations. In it, $a, a_1, \epsilon$ are the values of the chosen parameters where $\bar{\Psi}(x)$ is minimal. An optimal value of $a_2$ is equal to 0 identically. When calculating lower bounds, we used exact values of truncated moments $\varphi_1(b)$ and $\varphi_2(b)$ since they can be calculated easily.

| $\rho$ | $x$ | $\bar{\Psi}(x)$ | $a$ | $a_1$ | $\epsilon$ | $\underline{\Psi}(x)$ | $\Psi_{EV}(x)$ |
|---|---|---|---|---|---|---|---|
| 0.1 | 5 | $6.1 \cdot 10^{-1}$ | 0.7 | 0 | 1.0 | $3.9 \cdot 10^{-1}$ | $5.8 \cdot 10^{-2}$ |
| | 10 | $6.1 \cdot 10^{-1}$ | 1.5 | 0 | 1.0 | $1.8 \cdot 10^{-1}$ | $1.4 \cdot 10^{-2}$ |
| | 50 | $1.0 \cdot 10^{-1}$ | 1.1 | 0 | 1.0 | $6.0 \cdot 10^{-4}$ | $5.7 \cdot 10^{-4}$ |
| | 100 | $1.9 \cdot 10^{-2}$ | 1.8 | 0 | 1.0 | $5.2 \cdot 10^{-5}$ | $1.4 \cdot 10^{-4}$ |
| | 500 | $3.8 \cdot 10^{-5}$ | 2.3 | 1.1 | 0.4 | $4.6 \cdot 10^{-6}$ | $5.5 \cdot 10^{-6}$ |
| 0.2 | 5 | $6.7 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $1.6 \cdot 10^{-1}$ | $2.9 \cdot 10^{-2}$ |
| | 10 | $5.2 \cdot 10^{-1}$ | 1.1 | 0 | 1.0 | $3.9 \cdot 10^{-2}$ | $7.2 \cdot 10^{-3}$ |
| | 50 | $5.4 \cdot 10^{-2}$ | 1.7 | 0 | 1.0 | $6.4 \cdot 10^{-5}$ | $2.8 \cdot 10^{-4}$ |
| | 100 | $9.2 \cdot 10^{-3}$ | 1.0 | 0.1 | 0.2 | $3.8 \cdot 10^{-5}$ | $7.0 \cdot 10^{-5}$ |
| | 500 | $1.6 \cdot 10^{-5}$ | 3.3 | 2.4 | 0.5 | $2.2 \cdot 10^{-6}$ | $2.8 \cdot 10^{-6}$ |
| 0.3 | 5 | $6.8 \cdot 10^{-1}$ | 1.9 | 0 | 1.0 | $7.3 \cdot 10^{-2}$ | $1.9 \cdot 10^{-2}$ |
| | 10 | $4.6 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $9.4 \cdot 10^{-3}$ | $4.8 \cdot 10^{-3}$ |
| | 50 | $3.9 \cdot 10^{-2}$ | 1.9 | 0 | 1.0 | $5.8 \cdot 10^{-5}$ | $1.9 \cdot 10^{-4}$ |
| | 100 | $2.6 \cdot 10^{-3}$ | 1.1 | 0.2 | 0.3 | $2.7 \cdot 10^{-5}$ | $4.7 \cdot 10^{-5}$ |
| | 500 | $1.0 \cdot 10^{-5}$ | 3.7 | 3.1 | 0.5 | $1.4 \cdot 10^{-6}$ | $1.8 \cdot 10^{-6}$ |
| 0.4 | 5 | $6.6 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $3.5 \cdot 10^{-2}$ | $1.5 \cdot 10^{-2}$ |
| | 10 | $4.2 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $2.5 \cdot 10^{-3}$ | $3.6 \cdot 10^{-3}$ |
| | 50 | $3.2 \cdot 10^{-2}$ | 2.1 | 0 | 1.0 | $5.0 \cdot 10^{-5}$ | $1.4 \cdot 10^{-4}$ |
| | 100 | $1.1 \cdot 10^{-3}$ | 1.1 | 0.3 | 0.3 | $2.0 \cdot 10^{-5}$ | $3.5 \cdot 10^{-5}$ |
| | 500 | $7.3 \cdot 10^{-6}$ | 3.9 | 3.6 | 0.5 | $9.8 \cdot 10^{-7}$ | $1.4 \cdot 10^{-6}$ |
| 0.5 | 5 | $6.4 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $1.8 \cdot 10^{-2}$ | $1.2 \cdot 10^{-2}$ |
| | 10 | $3.9 \cdot 10^{-1}$ | 1.3 | 0 | 1.0 | $7.6 \cdot 10^{-4}$ | $2.9 \cdot 10^{-3}$ |
| | 50 | $2.8 \cdot 10^{-2}$ | 2.3 | 0 | 1.0 | $4.3 \cdot 10^{-5}$ | $1.1 \cdot 10^{-4}$ |
| | 100 | $6.5 \cdot 10^{-4}$ | 1.3 | 0.4 | 0.3 | $1.6 \cdot 10^{-5}$ | $2.8 \cdot 10^{-5}$ |
| | 500 | $5.7 \cdot 10^{-6}$ | 4.1 | 4.1 | 0.5 | $7.3 \cdot 10^{-7}$ | $1.1 \cdot 10^{-6}$ |

It is interesting to compare this table with the table presented for a lognormal case with $\sigma = 0.53$ when the two first moments of claim sizes are the same in both cases. It can be seen that in the Pareto case the asymptotic approximation works better than in the lognormal case although its values do not lie between the lower and upper bounds. In this case, ruin probability decreases slower which is natural since the tail of the Pareto distribution is heavier than the tail of the lognormal distribution. This explains the difference. One can see that the accuracy of two-sided bounds is not bad. The fact that $a_1 = 0$ in some lines is explained by the fact that the optimal upper bound is uniform as defined in Lemma 7.

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