Monetary risk measures on Orlicz spaces produced by set-valued risk measures and random measures

Dimitrios G. Konstantinides and Christos E. Kountzakis

Abstract In this article we study the construction of coherent or convex risk measures defined either on an Orlicz heart, either on an Orlicz space, with respect to a Young loss function. The Orlicz heart is taken as a subset of $L^0(\Omega, \mathscr{F}, \mu)$ endowed with the pointwise partial ordering. We define set-valued risk measures related to this partial ordering. We also derive monetary risk measures both by the class of coherent set-valued risk measures defined on them. We also use random measures related to heavy-tailed distributions in order to define monetary risk measures on Orlicz spaces, whose properties are also compared to the previous ones.

Key words: Set-valued risk measure; random measure; Young loss function; Orlicz Heart; Orlicz Space

1 Orlicz Hearts and Orlicz Spaces in Finance

We consider two periods of time (0 and 1) and a non-empty set of states of the world Ω which is supposed to be an infinite set. The true state $\omega \in \Omega$ that the investors face is contained in some $A \in \mathscr{F}$, where $\mathscr{F}$ is some σ-algebra of subsets of Ω which gives the information about the states that may occur at time-period 1. A financial position is a $\mathscr{F}$-measurable random variable $x : \Omega \to \mathbf{R}$. This random variable is the profile of this position at time-period 1. We suppose that the probability of any state of the world to occur is given by a probability measure $\mu : \mathscr{F} \to [0,1]$. We remind of the use of Orlicz Spaces in Finance, through the notion of *loss function*.

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Definition 1. A function $l : \mathbb{R} \to \mathbb{R}$ is a *loss function* if (a) l is increasing and convex, (b) $l(0) = 0$ and $l(x) \geq x$.

Such a function is used to measure the *expected loss* $\mathbb{E}_\mu(l(-x))$ of a financial position x. Related to the neutral evaluation of losses $\mathbb{E}_\mu(-x)$ under the state-of-the world measure, the expected loss $\mathbb{E}_\mu(l(-x))$ puts greater weights on high losses than gains. Since l is convex, its convex conjugate $l^*(y) = \sup_{x\in\mathbb{R}}\{xy - l(x)\}$ is well defined (see also [6, p.4]. Let E be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbf{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(E, \geq)$ where E is a linear space and $\geq$ is a binary relation on E satisfying the following properties:

(i) $x \geq x$ for any $x \in E$ (reflexive)
(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive)
(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbf{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of E),

is called *partially ordered linear space*. The partially ordered vector space E is a *vector lattice* if for any $x, y \in E$, the supremum $x \vee y$ and the infimum $x \wedge y$ of $\{x, y\}$ with respect to the partial ordering in E. If so, $|x| = \sup\{x, -x\}$ is the *absolute value* of x and if E is also a normed space such that $\| \, |x| \, \| = \|x\|$ for any $x \in E$, then E is called *normed lattice*. If a normed lattice is a Banach space, then it is called *Banach lattice*. If we consider the *pointwise partial ordering* on $L^0(\Omega, \mathscr{F}, \mu)$, then the *Orlicz Heart* $X_l := \{x \in L^0 | \mathbb{E}_\mu[l(c|x|)] < \infty, \, \forall \, c > 0\}$ endowed with the *l-Luxemburg* norm

$$\|x\|_l := \inf\left\{a > 0 \,\middle|\, \mathbb{E}_\mu\left[l\left(\frac{|x|}{a}\right)\right] \leq 1\right\},$$

is a Banach lattice. The norm-dual of X_l is the $X_l^* := \{y \in L^0 | \mathbb{E}_\mu[l^*(c|y|)] < \infty, \, \exists \, c > 0\}$ Orlicz Space, endowed with the Orlicz norm $\|y\|_{l^*} := \sup\{\mathbb{E}_\mu(xy) : \|x\|_l \leq 1\}$, being equivalent to the l^*-Luxemburg norm (see also in [4]). Since $l(x) \geq x, x \in \mathbb{R}_+$, then $X_l \subseteq L^1$. We also denote by $M_{l,l^*}(\mu)$ the probalility measures on $(\Omega, \mathscr{F})$, which are ansolutely continuous with respect to μ and their densities lie in X_{l^*} (see [6, p.5]). First, we remind the definitions of monetary risk measures on X_l, being alike to the ones met in [2, 5, 7].

Definition 2. A real-valued function $\rho : X_l \to \mathbb{R}$ is a *coherent monetary risk measure* if it is satisfying

(i) $\rho(x + a\mathbf{1}) = \rho(x) - a$ (Translation Invariance)
(ii) $\rho(\lambda x + (1-\lambda)x) \leq \lambda\rho(x) + (1-\lambda)\rho(y)$ for any $\lambda \in [0,1]$ (Convexity)
(iii) $\rho(\lambda x) = \lambda\rho(x)$ for any $x \in X_l$ and any $\lambda \in \mathbb{R}_+$ (Positive Homogeneity)
(iv) $y \geq x$ implies $\rho(y) \leq \rho(x)$ (Monotonicity)

where $x, y \in X_l$, is called *convex*.

Definition 3. A correspondence $\rho : X_l \to 2^{X_l}$ is a **coherent risk correspondence** if it satisfies the properties

(i) $\rho(x+a\mathbf{1})=\rho(x)-\{a\mathbf{1}\}, a\in\mathbb{R}$ (Translation Invariance)
(ii) $\rho(\lambda x+(1-\lambda)x)\subseteq\lambda\rho(x)+(1-\lambda)\rho(y)$ for any $\lambda\in[0,1]$ (Convexity)
(iii) $\rho(\lambda x)=\lambda\rho(x)$ for any $x\in X_l$ and any $\lambda\in\mathbb{R}_+$ (Positive Homogeneity)
(iv) $y\geq x$ implies $\rho(y)\subseteq\rho(x)$ (Monotonicity)

where $x,y\in X_l$.

Definition 4. A financial position x is **safe with respect to** $rho : X_l\rightarrow 2^{X_l}$, if for this $x\in X_l$, $\rho(x)\subseteq -X_{l,+}$.

Definition 5. The wedge of the safe financial positions $\mathscr{A}_\rho=\{x\in X_l|\rho(x)\subseteq -X_{l,+}\}$ is called **acceptance set** of ρ.

Suppose that C is a wedge of a linear space E. A linear functional f of E is called *positive functional* of C if $f(x)\geq 0$ for any $x\in C$. f is a *strictly positive functional* of C if $f(x)>0$ for any $x\in C\setminus\{0\}$. A linear functional f of E, where E is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a>0$ such that $f(x)\geq a\|x\|$ for any $x\in C$.

The set of functions $C^0=\{f\in E^*|f(x)\geq 0\,,\ \forall x\in C\}$ is the *dual wedge of C in* E^*. If for two wedges K,C of E $K\subseteq C$ holds, then $C^0\subseteq K^0$. If C is a cone, then a set $B\subseteq C$ is called *base* of C if for any $x\in C\setminus\{0\}$ there exists a unique $\lambda_x>0$ such that $\lambda_x x\in B$.

The set $B_f=\{x\in C|f(x)=1\}$, where f is a strictly positive functional of C is the *base of C defined by f*. B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0\notin\overline{B}$ then C is called *well-based*. If C is well-based, then a bounded base of C defined by a $g\in E^*$ exists. Also, $f\in E^*$ is a uniformly monotonic functional of C if and only if $f\in intC^0$, where $intC^0$ denotes the norm-interior of C^0.

Theorem 1. *The **derivative monetary coherent risk measure** $\hat{\rho}: X_l\rightarrow\mathbb{R}\cup\{+\infty\}$, which arises from $\rho : X_l\rightarrow 2^{X_l}$ is defined as follows:*

$$\hat{\rho}(x)=\sup_{\pi\in\mathscr{A}_\rho^0\cap M_{l,l^*(\mu)}}\pi(-x), x\in X_l.$$

The consistence range of ρ is used for the definition of a pricing functional $\psi : E\rightarrow\mathbb{R}$, which satisfies the properties of a *generalized price* indicated in [1, Lem.3.3]. These properties are listed in the following

Definition 6. A functional $\psi : E\rightarrow\mathbb{R}$ is a **generalized price** if and only if satisfies the following properties:

(i) If $x\geq y$, then $\psi(x)\geq\psi(y)$ (E_+-Monotonicity)
(ii) $\psi(x+y)\geq\psi(x)+\psi(y)$ (Super-Additivity)
(iii) $\psi(a\cdot x)=a\psi(x), x\in E, a\in\mathbb{R}_+$ (Positive Homogeneity)

We obtain the following duality between monetary coherent risk measures and generalized prices.

Theorem 2. *If $\psi : X_l \to \mathbb{R}$ is a* ***generalized price*** *which is normalized at the numeraire financial position $\mathbf{1} \in X_{l,+}$ ($\psi(\mathbf{1}) = 1$) and satsifies the Translation Invariance $\psi(x + a\mathbf{1}) = \psi(x) + a$, $a \in \mathbb{R}$, then $-\psi = \rho_\psi$ is a monetary coherent risk measure. On the other hand if $\rho : X_l \to \mathbb{R}$ is a monetary coherent risk measure, then $-\rho = \psi_\rho : E \to \mathbb{R}$ is a generalized price functional.*

2 Monetary Risk Measures arising from Random Measures

We consider the probability space $(\Omega, \mathscr{F}, \mu)$ and a measurable space $(E, \mathscr{E})$. A *random measure* (see [8]) ξ on $(E, \mathscr{E})$ over the probability space $(\Omega, \mathscr{F}, \mu)$ is a map $\xi : \mathscr{E} \times \Omega \to \mathbb{R}_+$, such that

1. the map $\omega \mapsto \xi(A, \omega)$ is a random variable for any $A \in \mathscr{E}$
2. the map $A \mapsto \xi(A, \omega)$ is a measure on $\mathscr{E}$, μ- almost surely in Ω.

The measurable $(E, \mathscr{E})$ is actually $(S, \mathscr{B}_S)$, where $\mathscr{B}$ is the Borel σ-algebra on S. If we would like to create a $\mathscr{V}$-related coherent $\rho : X_l \to \mathbb{R}$, where $\mathscr{V}$ is a family of heavy-tailed distributions (see [3]), such that the densities of $\xi(y), y \in S$ belong to $M_{l,l^*}(\mu)$, where $S \subseteq Y$ is compact and Y is a finite-dimensional topological manifold Y (see [9]) of parameters of $\mathscr{V}$.

Theorem 3. *The following risk measure is coherent on X_l*

$$\mathbb{ES}_{a,\mathscr{V},S,l}(x) = \sup_{0 \leq \frac{d\mu_{\xi(y)}}{d\mu} \leq \frac{1}{a},\, y \in S} \mathbb{E}_{\mu_{\xi(y)}}(-x), \qquad x \in X_l\,.$$

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