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MONOTONE TRAJECTORY PRINCIPLE FOR A COMPLEX RENEWAL SYSTEM WITH ONE REPAIR UNIT

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A complex renewal system with one repair unit has been shown to obey the monotone trajectory principle which states that under "quick" renewal of defective elements, the failure intensity of a system is equivalent to the intencity of filures by the monotone trajectory law when none of the defective elements is renewed in a time interval beginning from the commencement of the busy period up to the instant when the system fails. It is found that for the monotone trajectory principle to hold, it is sufficient that for fixed failure intensities the probabilities of repair time of elements tend to zero.

1. INTRODUCTION

In [1] a complex renewal system was studied and the following limit theorem was proved: if the m repair time of elements tends to zero, then $\lim \mathrm{P}\{\lambda(0)q\tau > x\} = e^{-x}$, where τ is the time to the first failur the system, $\lambda(0)$ is the total intensity of failure of elements in a system in full order, and q is the probabi for the system to fail in a busy period. By a busy period we understand the time interval on which at le one element of the system is defective.

Imposing more stringent limitations on the higher moments of repair time, paper [1] gives the estim

$$q \sim q_0,$$

where q_0 is the probability for the system to fail by the monotone trajectory law in a busy period w none of the defective elements is renewed in the interval beginning from the start of the busy period till system fails. About quarter of a century ago Solov'ev and Kovalenko reported that relation (1) holds fc renewable system under certain natural conditions. Today relation (1) is known as the monotone traject principle. In [2], for a general reservation model with renewal, if there is one repair unit, the monot trajectory principle has been shown to hold, provided the probability of repair time tends to zero. Our a in this paper is to extend this result to a complex renewable system with one repair unit.

2. DESCRIPTION OF A SYSTEM. NOTATION

Let us consider a system consisting of N elements and one repair unit which renews the elements order they arrive for repair. If there are s defective elements, the state of the system is given by the vec $\alpha = (i_1, \dots, i_s)$, where $i_1, \dots, i_s$ are the ordinal numbers of the defective elements in the order of occurra of failure. If all the elements of a system are in good order, we formally write $\alpha = 0$. Let α' and $\bar{\alpha}$ den the vectors $\alpha' = (i_2, \dots, i_s)$ and $\bar{\alpha} = (i_1, \dots, i_{s-1})$. If β denotes the vector $(i_{s+1}, \dots, i_{s+r})$, then $\alpha\beta$ denotes vector $(i_1, \dots, i_s, i_{s+1}, \dots, i_{s+r})$. Let $\alpha\beta^0$ stand for the set of all vectors of the type $(i_1, \dots, i_s, i'_{s+1}, \dots, i'_{s+}$ where the vector $(i'_{s+1}, \dots, i'_{s+r}) \in \beta^0$ is a permutation of the vector β.

We denote the set of all states of the system by E. Let us subdivide the set E into classes by number of defective elements: $E = E^0 \cup E^1 \cup \dots \cup E^{(N)}$. Let $E_- \subset E$ denote the set of those states in wh the system is defective and E_+ the complement of E_-. Let $E_+^{(s)} = E^{(s)} \cap E_+$, $E_-^{(s)} = E^{(s)} \cap E_-$, $n = min$ $E_-^{(s)} \neq \varnothing\}$, $(n+m) = \max\{s : E_+^{(s)} \neq \varnothing\}$. Furthermore, let $\Gamma = \{\alpha : \alpha \in E_-, \bar{\alpha} \in E_+\}$ and $\beta^{0\alpha} = \alpha\beta^0 \cap$

The failure intensity of the ith element, if at this instant the system exists in the state α, is $\lambda_i(\alpha) > 0$, on the assumption that $\sum_{i=1}^N \lambda_i(0) = 1$. The repair time η_i of the ith element is a random variable with the distribution function $C_i(x)$.

Operation of this system is a special type of regeneration process (see [3]). Let q_α denote the probability that the system fails in a busy period, under the condition that at the initial moment of the busy period the system was in the state α and the element i_1, the first to arrive for repair, was taken up for renewal. Then we have $q = \sum_{i=1}^N \lambda_i(0) q_i$. We assume that $\sum_{i=1}^N \lambda_i(0) = 1$. For the state $\alpha\beta$, $\alpha \cap \beta = \varnothing$, we introduce the following notation: $p_\alpha(\beta^0, x)$ is the probability that in the time interval x all elements of the set β^0 fail and none of the remaining elements fail before the last of the elements of β^0 fails. The following inequality holds:

$$p_\alpha(\beta^0, x) \leqslant \prod_{i_k \in \beta} (1 - e^{-\bar{\lambda}_{i_k} x}), \quad \text{where} \quad \bar{\lambda}_{i_k} = \max_{\substack{\alpha \in E_+ \\ i_k \in \alpha}} \lambda_{i_k}(\alpha). \tag{2}$$

and if, additionally, $\alpha\beta \in E^{(N)}$, then

$$p_\alpha(\beta^0, x) \geqslant \prod_{i_k \in \beta} (1 - e^{-\underline{\lambda}_{i_k} x}), \quad \text{where} \quad \underline{\lambda}_{i_k} = \min_{\substack{\alpha \in E_+ \\ i_k \in \alpha}} \lambda_{i_k}(\alpha). \tag{3}$$

If the order of arrival of the elements for renewal is defined by the vector β, we obtain a Markovian pure death process. Let us introduce the following notation: $\pi_\alpha(\beta, x)$ is the probability that in the initial state α the pure death process suffered at least r jumps in the time interval x, where the first r jumps correspond to the order of occurrance of failures of the elements $i_{s+1}, \dots, i_{s+r}$. Thus $p_\alpha(\beta^0, x) = \sum_{\beta \in \beta^0} \pi_\alpha(\beta, x)$. The probability $\pi_\alpha(\beta, x)$ can be represented as the product:

$$\pi_\alpha(\beta, x) = \pi_{\alpha,\beta}^N \cdot \pi_\beta^\alpha(x), \tag{4}$$

where $\pi_{\alpha,\beta}^N$ is the probability that from the set $\{i_{s+1}, \dots, i_N\}$ of serviceable elements, the elements $i_{s+1}, \dots, i_{s+r}$ fail in succession before the commencement of repair of the ith element (it is assumed that the repair of the ith element is not completed until the i_{s+1}th element fails). Let $\pi_\beta^\alpha(x)$ denote the probability that pure death process with intensities $\lambda_{i_{s+1}}(\alpha), \lambda_{i_{s+2}}(\alpha_{i_{s+1}}), \dots, \lambda_{i_{s+r}}(\alpha\beta - i_{s+r})$ suffers at least r jumps up to the time instant x.

Now let a_α be the probability that in the time interval, when the i_1th element is being repaired, none of the elements fails, $a_{\alpha,\beta}$ be the probability that up to the time of completion of repair of the i_1th element the elements $i_{s+1}, \dots, i_{s+r}$ fail in succession, and b_α be the probability that the system fails before the completion of renewal of the ith element.

Since the repair commencement instants form a Markovian chain, the following system of equations holds:

$$q_\alpha = b_\alpha + \sum_{r=1}^{n+m-1} \sum_{\beta \in E_+^{(r)}} a_{\alpha,\beta} q_\beta, \quad s = 1;$$

$$q_\alpha = b_\alpha + a_\alpha q_{\alpha'} + \sum_{r=1}^{n+m-s} \sum_{\beta \in E^{(r)}} a_{\alpha,\beta} q_{\alpha'\beta}, \quad s > 1.$$

For $Q_s = \sum_{\alpha \in E_+^{(s)}} q_\alpha$ the above system yields

$$Q_1 \leqslant \sum_{\alpha \in E_+^{(1)}} b_\alpha + \sum_{r=1}^{n+m-1} \sum_{\alpha \in E_+^{(1)}} \max_{\omega_r^1(\alpha)} a_{\alpha,\beta} Q_r, \quad s = 1;$$

$$Q_s \leqslant \sum_{\alpha \in E_+^{(s)}} b_\alpha + \sum_{i_1=1}^{N} \max_{s(\alpha)} a_\alpha Q_{s-1} + \sum_{r=1}^{n+m-s} \sum_{i_1=1}^{N} \max_{s(\alpha)} \max_{\omega_r^s(\alpha)} a_{\alpha,\beta} Q_{s+r-1}, \quad 2 \leqslant s \leqslant n+m; \tag{5}$$

$$z(\alpha) = \begin{bmatrix} i_1 \notin \alpha' \\ i_1\alpha' \in E_+^{(s)} \end{bmatrix}, \quad \omega_r^s(\alpha) = \begin{bmatrix} \beta \in E_+^{(r)} \\ \alpha \cap \beta = \varnothing \\ \alpha\beta \in E_+^{(r+s)} \end{bmatrix}.$$

3. AUXILIARY PROPOSITIONS

Lemma 1.

(a)
$$a_{\alpha,\beta} \leqslant \int_0^\infty \prod_{i_k \in \beta} (1 - e^{-\bar{\lambda}_{i_k} x}) dG_{i_1}(x),$$

(b)
$$\sum_{r=\max(1,n-s)}^{n+m-s+1} \sum_{\beta^{0\alpha} \subset E^{(r)}} d_1^{\beta^0} \int_0^\infty \prod_{i_k \in \beta} (1 - e^{-\underline{\lambda}_{i_k} x}) dG_{i_1}(x) \leqslant b_\alpha \leqslant$$

$$\sum_{r=\max(1,n-s)}^{n+m-s+1} \sum_{\beta^{0\alpha} \subset E^{(r)}} d_2^{\beta^0} \int_0^\infty \prod_{i_k \in \beta} (1 - e^{-\bar{\lambda}_{i_k} x}) dG_{i_1}(x),$$

where
$$d_1^{\beta^0} = d_\alpha^{\beta^0} \frac{\min_{\beta \in \beta^0} \pi_{\alpha,\beta}^N}{\max_{\beta \in \beta^0} \pi_{\alpha,\beta}^{s+r}}, \quad d_\alpha^{\beta^0} = \min_{x>0} \frac{\sum_{\beta \in \beta^{0\alpha}} \pi_\alpha(\beta, x)}{\sum_{\beta \in \beta^0} \pi_\alpha(\beta, x)}, \quad d_2^{\beta^0} = \frac{\max_{\beta \in \beta^0} \pi_{\alpha,\beta}^N}{\min_{\beta \in \beta^0} \pi_{\alpha,\beta}^{s+r}},$$

Proof. (a) Obviously, we have

$$a_{\alpha,\beta} \leqslant \int_0^\infty \pi_\alpha(\beta, x) dG_{i_1}(x), \quad \underset{(3)}{\overset{(2)}{\Rightarrow}} a_{\alpha,\beta} \leqslant \int_0^\infty p_\alpha(\beta^0, x) dG_{i_1}(x).$$

(b) Let $b_\alpha^{(s+r)}$ be the probability that the system fails before the completion of repair of the i_1th element, where the first failure state belongs to $E_-^{(s+r)}$, and $b_\alpha(\beta^{0\alpha})$ be the probability that the system fails before the completion of repair of the i_1th element, where the first failure state belongs to $\alpha\beta^0$, and $b_\alpha(\beta)$ be the probability that the system fails before the completion of repair of the i_1th element, where the trajectory is given by the vector β. Then

$$b_\alpha = \sum_{r=\max(1,n-s)}^{n+m-s+1} b_\alpha^{(s+r)}, \quad b_\alpha^{(s+r)} = \sum_{\beta^{0\alpha} \subset E^{(r)}} b_\alpha(\beta^{0\alpha}),$$

$$b_\alpha(\beta^{0\alpha}) = \sum_{\beta \in \beta^{0\alpha}} b_\alpha(\beta), \quad b_\alpha(\beta) = \int_0^\infty \pi_\alpha(\beta, x) dG_{i_1}(x) \quad \Rightarrow$$

$$b_\alpha(\beta^{0\alpha}) = \int_0^\infty \sum_{\beta \in \beta^{0\alpha}} \pi_\alpha(\beta, x) dG_{i_1}(x).$$

To evaluate the lower bound, let us use the fact that for $\beta^{0\alpha} \neq \varnothing \Rightarrow 0 < d_\alpha^{\beta^0} < 1$. Thus

$$d_\alpha^{\beta^0} \sum_{\beta \in \beta^0} \pi_\alpha(\beta, x) \leqslant \sum_{\beta \in \beta^{0\alpha}} \pi_\alpha(\beta, x) \leqslant \sum_{\beta \in \beta^0} \pi_\alpha(\beta, x), \overset{(4)}{\Rightarrow}$$

$$\min_{\beta \in \beta^0} \pi_{\alpha,\beta}^N \sum_{\beta \in \beta^0} \pi_\beta^\alpha(x) \leqslant \sum_{\beta \in \beta^0} \pi_\alpha(\beta, x) \leqslant \max_{\beta \in \beta^0} \pi_{\alpha,\beta}^N \sum_{\beta \in \beta^0} \pi_\beta^\alpha(x). \tag{6}$$

In order to evaluate $\sum_{\beta\in\beta^o}\pi^\alpha_\beta(x)$, let us consider a new system consisting if only $(s+r)$ elements of the vector $\alpha\beta$. For this system let $\pi^*_\alpha(\beta,x)$ denote the probability similar to the probability $\pi_\alpha(\beta,x)$ for the initial system. Thus, we have

$$\pi^*_\alpha(\beta,x)=\pi^{s+r}_{\alpha,\beta}\pi^\alpha_\beta(x),$$

here $\pi^\alpha_\beta(x)$ is the same as in (4), because the corresponding pure death process for these r jumps is the same in both systems. On the other hand, we have the estimate

$$\prod_{i_k\in\beta}(1-e^{-\underline{\lambda}_{i_k}x})\leqslant\sum_{\beta\in\beta^o}\pi^*_\alpha(\beta,x)\leqslant\prod_{i_k\in\beta}(1-e^{-\overline{\lambda}_{i_k}x}),$$

$$\overset{(7)}{\underset{(6)}{\Rightarrow}} d_1^{\beta^o}\prod_{i_k\in\beta}(1-e^{-\underline{\lambda}_{i_k}x})\leqslant\sum_{\beta\in\beta^{o*}}\pi_\alpha(\beta,x)\leqslant d_2^{\beta^o}\prod_{i_k\in\beta}(1-e^{-\overline{\lambda}_{i_k}x}).$$

This completes the proof of our assertion.

Let us introduce the notation

$$\gamma=\int_0^\infty(1-e^{-x})\sum_{i=1}^N dG_i(x),\quad \mu_r=\int_0^\infty\frac{(1-e^{-x})^{r+1}}{\gamma}\sum_{i=1}^N dG_i(x).$$

Lemma 2.

$$\text{(a)}\sum_{i_1=1}^N\max_{s(\alpha)}\max_{w^s_r(\alpha)}a_{\alpha,\beta}\leqslant\gamma\mu_{r-1},\quad 1\leqslant s\leqslant n+m;$$

$$\text{(b)}c_1^s(\max(1,n-s))\gamma\mu_{\max(0,n-s-1)}\leqslant\sum_{\alpha\in E_+^{(s)}}b_\alpha\leqslant\left(\sum_{r=\max(1,n-s)}^{n+m-s+1}c_2^s(r)\right)\gamma\mu_{\max(0,n-s-1)},$$

$$\textit{where}\quad c_1^s(r)=\min_{1\leqslant i_1\leqslant N}\sum_{s(\alpha)}\sum_{\beta^{o*}\subset E^{(r)}}d_1^{\beta^o}\prod_{i_k\in\beta}\underline{\lambda}_{i_k},\quad c_2^s(r)=\max_{1\leqslant i_1\leqslant N}\sum_{s(\alpha)}\sum_{\beta^{o*}\subset E^{(r)}}d_2^{\beta^o}.$$

Proof. (a) For $0<\lambda<1$, we have

$$\lambda(1-e^{-x})\leqslant1-e^{-\lambda x}\leqslant1-e^{-x}\Rightarrow\sum_{i_1=1}^N\max_{s(\alpha)}\max_{w^s_r(\alpha)}a_{\alpha,\beta}$$

$$\leqslant\sum_{i_1=1}^N\max_{s(\alpha)}\max_{w^s_r(\alpha)}\int_0^\infty(1-e^{-x})^r dG_{i_1}(x)\leqslant\int_0^\infty(1-e^{-x})^r\sum_{i_1=1}^N dG_{i_1}(x)=\gamma\mu_{r-1}.$$

(b)

$$\sum_{r=\max(1,n-s)}^{n+m-s+1}\sum_{\alpha\in E_+^{(s)}}\sum_{\beta^{o*}\subset E^{(r)}}d_1^{\beta o}\prod_{i_k\in\beta}\underline{\lambda}_{i_k}\int_0^\infty(1-e^{-x})^r dG_{i_1}(x)\leqslant\sum_{\alpha\in E_+^{(s)}}b_\alpha$$

$$\leqslant\sum_{r=\max(1,n-s)}^{n+m-s+1}\sum_{\alpha\in E_+^{(s)}}\sum_{\beta^{o*}\subset E^{(r)}}d_2^{\beta o}\int_0^\infty(1-e^{-x})^r dG_{i_r}(x)$$

$$\Rightarrow\sum_{r=\max(1,n-s)}^{n+m-s+1}c_1^s(r)\gamma\mu_{r-1}\leqslant\sum_{\alpha\in E_+^{(s)}}b_\alpha\leqslant\sum_{r=\max(1,n-s)}^{n+m-s+1}c_2^s(r)\gamma\mu_{r-1}.$$

This completes the proof of our assertion.

Lemma 3. $\mu_k\cdot\mu_\ell\leqslant\mu_{k+\ell},\ \forall k,\ell\geqslant0.$

Proof. Let us recall the Chebyshev inequality ([3], p.36). If $G(x)$ is a distribution function and the functions $\psi(x) \geqslant 0$, $\varphi(x) \geqslant 0$ do not monotonously increase, then

$$\int_{-\infty}^{\infty} \varphi(x)\psi(x)dG(x) \geqslant \int_{-\infty}^{\infty} \varphi(x)dG(x) \int_{-\infty}^{\infty} \psi(x)dG(x).$$

For $G(x)$ let us substitute the following destribution function

$$\int_0^x \frac{(1-e^{-t})}{\gamma} \sum_{i=1}^N dG_i(t)$$

and substitute $(1-e^{-x})^k$ and $(1-e^{-x})^\ell$ for $\varphi(x)$ and $\psi(x)$, respectively.

4. MAIN RESULT

Inequalities (5) can be "roughened" as follows:

$$Q_1 \leqslant \left(\sum_{r=n-1}^{n+m} c_2^1(r)\right) \gamma\mu_{n-2} + \sum_{r=1}^{n+m-1} \gamma\mu_{r-1}Q_r,$$

$$Q_s \leqslant \left(\sum_{r=\max(1,n-s)}^{n+m-s+1} c_2^s(r)\right) \gamma\mu_{\max(0,n-s-1)} + NQ_{s-1} + \sum_{r=1}^{n+m-1} \gamma\mu_{r-1}Q_{s+r-1}, \quad 2 \leqslant s \leqslant n+m. \qquad (8)$$

Theorem. *The following inequality holds*

$$\sum_{i=1}^N \lambda_i(0)b_{(i)} \leqslant q \leqslant \sum_{i=1}^N \lambda_i(0)b_{(i)}(1+c\gamma),$$

where

$$c = \frac{\bar\lambda}{\underline{\lambda} c_1^1(n-1)} \sum_{r=1}^{n+m-1} \bar{x}_r, \quad \bar{x}_r = \sum_{k=0}^{r-1} N^k \sum_{\ell=\max(1,n-r+k)}^{n+m-r+k+1} c_2^{r-k}(\ell),$$

$$\bar\lambda = \max_{1\leqslant i\leqslant N} \lambda_i(0), \quad \underline{\lambda} = \min_{1\leqslant i\leqslant N} \lambda_i(0), \quad 1 \leqslant r \leqslant n+m.$$

Hence, it follows that $g \sim \sum_{i=1}^N \lambda_i(0)b_{(i)}$ *as* $\eta_i \xrightarrow{\mathbf{P}} 0$, $1 \leqslant i \leqslant N$.

Proof. Let us use the notation

$$x_s = Q_s[\gamma\mu_{\max(0,n-s-1)}]^{-1}.$$

Then, when $\gamma \to 0$, the system of inequalities (8) yields

$$x_1 = \sum_{r=1}^{n+m} c_2^1(r) + O(\gamma),$$

. . . .

$$x_\ell = \sum_{r=1}^{n+m-\ell+1} c_2^\ell(r) + \dots + N^{\ell-n+1}\sum_{r=1}^{m+2} c_2^{n-1}(r) + N^{\ell-n+2}\sum_{r=2}^{m+3} c_2^{n-2}(r) + \dots + N^{\ell-1}\sum_{r=n-1}^{n+m} c_2^1(r) + O(\gamma),$$

. . . .

$$x_{m+n-1} = \sum_{r=1}^{2} c_2^{n+m-1}(r) + \dots + N^m\sum_{r=1}^{m+2} c_2^{n-1}(r) + N^{m+1}\sum_{r=2}^{m+3} c_2^{n-2}(r) + \dots + N^{n+m-2}\sum_{r=n-1}^{n+m} c_2^1(r) + O(\gamma)$$

and, using the notation $\bar{x}_r$, we obtain

$$Q_s = \bar{x}_s \gamma \mu_{\max(0,n-s-1)} + O(\gamma^2 \mu_{\max(0,n-s-1)}),$$
$$\Rightarrow q = \sum_{i=1}^{N} \lambda_i(0) q_{(i)} = \sum_{i=1}^{N} \lambda_i(0) b_{(i)} + \sum_{i=1}^{N} \lambda_i(0) \sum_{r=1}^{n+m-1} \sum_{\omega_r^1(i)} a_{i,\beta} q_\beta .$$

To evaluate the upper bound, note that

$$\sum_{i=1}^{N} \lambda_i(0) \sum_{r=1}^{n+m=1} \sum_{\omega_r^1(i)} a_{i,\beta} q_\beta \leqslant \sum_{r=1}^{n+m-1} \bar{\lambda} \sum_{i=1}^{N} \max_{\omega^1(i)} a_{i,\beta} Q_r . \tag{9}$$

By virtue of lemmas, formula (9), and the expression for c, we obtain

$$q = \sum_{i=1}^{N} \lambda_i(0) b_{(i)} + \sum_{i=1}^{N} \lambda_i(0) b_{(i)} (c\gamma + o(\gamma)).$$

Since the condition $\gamma \to 0$ is equivalent to the condition $\eta_i \xrightarrow{P} 0$, $1 \leqslant i \leqslant N$, we obtain the assertion of the theorem.

Remark. Even if one of the quantities η_i does not tend to zero in probability, then the probability q does not tend to zero and the mean busy period also does not tend to zero. In such a case the time up to the first failure of the system will not exhibit the exponential asymptotic behavior, so the estimate of the probability q loses meaning. Therefore, the conditions for the monotone trajectory principle in the theorem are indeed essential.

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