Journal of Mathematical Sciences, Vol. 93, No. 4, 1999

COMPARISON OF RUIN PROBABILITY ESTIMATES IN THE PRESENCE OF HEAVY TAILS

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Four methods of estimation of the ruin probability in the presence of heavy tails are compared in accordance with their effectiveness on the Pareto-distributed claim sizes. The first method, proposed by Embrechts and Veraverbeke, provides an asymptotic expression when the initial capital tends to infinity. The second method, proposed by Goovaerts and De Vylder, provides an algorithm for two-sided estimation based on the solution of the renewal equation through discretization. Its advantage is the perfect applicability in cases with small initial capital. The third method, proposed initially by Willmot and improved by Kalashnikov, provides an upper bound of the ruin probability with the help of a test function. The last method, proposed by Kalashnikov, provides two-sided bounds for the ruin probability using truncation of random variables to avoid the Cramér condition. The last two methods enable one to handle cases with intermediate initial capital.

1. Introduction

Consider as initial objects a sequence $(X_n)_{n\geq 1}$ of nonnegative independent identically distributed random variables (i.i.d.r.v.'s) and a random integer variable ν. Denote by X a generic r.v. (distributed as X_1) with distribution function $F(y)$ and consider the normalizing assumption that $\mathbf{E}X = 1$. Let ν be independent of $(X_n)_{n\geq 1}$ and have a geometric distribution with parameter q:

$$\mathbf{P}(\nu = k) = q(1-q)^{k-1}, \quad k \geq 1.$$

Then construct the geometric sum $S_\nu = X_1 + \cdots + X_\nu$. Consider now the following expression of the ruin probability $\Psi_q(y)$ through the compound geometric distribution:

$$\Psi_q(y) = (1-q)\mathbf{P}(S_\nu > y) = \mathbf{P}(S_{\nu-1} > y), \quad y \geq 0.$$

If the number of renewals within $[0, y]$ (including the zero point $S_0 = 0$) is

$$N(y) = \min\{n\colon\ S_n > y, n \geq 1\}, \quad y \geq 0,$$

the ruin probability can be written as (see, for example, [4])

$$\Psi_q(y) = \mathbf{E}(1-q)^{N(y)}.$$

Definition 1. Say that the geometric sum S_ν governed by a generic summand X with $\mathbf{E}X = 1$ satisfies the Cramér condition at point $0 < q < 1$, if there exists a constant $R > 0$ such that

$$(1-q)\mathbf{E}\exp(RX) = 1. \tag{1}$$

In order to satisfy the Cramér condition, X should have a finite exponential moment

$$\mathbf{E}\exp(\lambda X) < \infty, \quad \lambda > 0. \tag{2}$$

Cramér's condition plays an important role in various probabilistic problems, and we use it to derive two-sided bounds of Ψ_q having the same tail behavior. If lower $\underline{\Psi}_q(y)$ and upper $\overline{\Psi}_q(y)$ bounds of the ruin probability $\Psi_q(y)$ are found, let us define the relative accuracy of these estimations as

$$\delta(y) \equiv \frac{\overline{\Psi}_q(y) - \underline{\Psi}_q(y)}{\overline{\Psi}_q(y) + \underline{\Psi}_q(y)}. \tag{3}$$

We need the following class of r.v.

Definition 2. A random variable X with distribution function F belongs to the class $\mathcal{J}$, called subexponential, if

$$\lim_{y\to\infty} \frac{1 - F^{(2)}(y)}{1 - F(y)} = 2,$$

Proceedings of the Seminar on Stability Problems for Stochastic Models, Hajdúszoboszló, Hungary, 1997, Part III.

 1072-3374/99/9304-0552$22.00

where $F^{(2)}$ denotes the square convolution of F. Note that many heavy-tailed random variables, including Pareto and lognormal, belong to this class of subexponential r.v.'s (see [2]).

Definition 3. A function G belongs to the class Θ_c if it is differentiable, convex for $u \geq 0$, its derivative $g(u)$ is concave for $u \geq 0$, and $g(u) \to \infty$ for $u \to \infty$.

Definition 4. A function H belongs to the class $\mathcal{G}_s$ if it is expressed as $H(u) = \exp(\Lambda(u))$, where $\Lambda(0) = 0$, $\lim_{u\to\infty} \Lambda(u) = \infty$, $\Lambda(u)/u$ does not increase, and $\lim_{u\to\infty} \Lambda(u)/u = 0$, such that

$$\lim_{u\to\infty} \frac{H(u)}{(1+u)^s} = \infty.$$

Let us now introduce for any $b > 0$, called the truncation parameter, the sequence of truncated i.i.d.r.v.'s $\tilde{X}_n(b) \equiv X_n \wedge b$, $n \geq 1$, in correspondence with the sequence $(X_n)_{n\geq 1}$ with generic distribution function

$$\tilde{F}(u) = \begin{cases} F(u), & \text{if } u \leq b, \\ 1, & \text{if } u > b. \end{cases}$$

The truncated moments are denoted as

$$\tilde{m}_i(b) \equiv \mathbf{E}\tilde{X}^i(b),$$

$$\tilde{m}_G(b) \equiv \mathbf{E}G(\tilde{X}(b)),$$

for any properly defined function G. Further, the complement sequence of nonnegative i.i.d.r.v.'s $Y_n(b) \equiv X_n I_{\{X>b\}}$, $n \geq 1$, is introduced having generic distribution

$$T(u) = \begin{cases} F(b), & \text{if } u \leq b, \\ F(u), & \text{if } u > b. \end{cases}$$

2. Ruin Probability Estimates for Heavy-Tailed Claim Sizes

Now let us get rid of the assumption that the generic summand X of a geometric sum has finite exponential moment. From [3] take the following asymptotic expression.

Theorem 1. *Let X be a subexponential r.v. with distribution function $F(x)$. Then*

$$\Psi_q(y) \sim \frac{1-q}{q}[1 - F(y)], \quad y \to \infty.$$

This asymptotic estimation works well enough for large values of y, but it does not provide any information about its accuracy. To overcome this defect, look for two-sided bounds for the ruin probability to give a better description of the accuracy level.

In [4] (see also [1]), an algorithm that provides two-sided bounds $l(y)$ and $u(y)$ of the ruin probability

$$\mathrm{l}(y) \leq \Psi_q(y) \leq \mathrm{u}(y)$$

is found, applicable mostly for small values of y. For this take $d > 0$ an arbitrary small positive number serving as the span of discretization of F and consider

$$\underline{f}_k = F((k+1)d) - F(kd), \quad k = 0, 1, \ldots, n,$$

$$\overline{f}_k = F(kd) - F((k-1)d), \quad k = 0, 1, \ldots, n.$$

Now take recursively for $i \geq 0$

$$\underline{g}_0 = \frac{q}{1-(1-q)\underline{f}_0}, \qquad \underline{g}_i = \frac{1-q}{1-(1-q)\underline{f}_0} \sum_{k=1}^{i} \underline{f}_k \underline{g}_{i-k}, \qquad i = 1, 2, \ldots, n-1,$$

$$\overline{g}_0 = q, \qquad \overline{g}_i = (1-q) \sum_{k=1}^{i} \overline{f}_k \overline{g}_{i-k}, \qquad i = 1, 2, \ldots, n.$$

Thus, finally find

$$\mathrm{l}(nd) = 1 - \sum_{i=0}^{n-1} \underline{g}_i,$$

$$\mathrm{u}(nd) = 1 - \sum_{i=0}^{n} \overline{g}_i.$$

It is clear that

$$\lim_{d\to 0}(\mathrm{u}(y) - \mathrm{l}(y)) = 0$$

for any fixed y. But the speed of this convergence falls when q becomes small or x is large.

The third method that permits one to obtain the upper bound of $\Psi_q(y)$ follows a general method by the use of the so-called *test functions.* Consider a class $\mathcal{T}$ of monotone functions T: $[0,\infty) \to [0,\infty)$ such that

$$T(x+y) \le T(x)T(y) \quad \text{for any} \quad x, y \ge 0, \quad T(0) = 1.$$

As an example, for such a function from a class $\mathcal{T}$ take any function T of the form

$$T(y) = \exp\left(p\int_0^y \lambda(u)\,du\right),$$

where $p > 0$ and $\lambda(u) \ge 0$ is a decreasing function.

The following theorem from [7] represents a generalization of the corresponding result in [12].

THEOREM 2. *If $T \in \mathcal{T}$ and*

$$(1-q)\mathbf{E}T(X) \le 1,$$

then

$$\Psi_q(y) \le \frac{\tilde{m}_T(y)}{T(y)}.$$

The fourth method found in different versions in [7], [9], and [10] provides two-sided estimates using test functions too. Next, the following two cases are examined separately. The first case requires the existence of the second moment $m_2 \equiv \mathbf{E}X^2 < \infty$, but in the second case only the condition $m_G \equiv \mathbf{E}G(X) < \infty$ is required for appropriate functions $G \in \Theta_c$ and $G \in \mathcal{G}_1$. The second case is wider but more complicated. These estimates lead to the Rényi theorem, but they disclose additionally the tail behavior of the d.f. W_q. Let us denote

$$q' = -\log(1-q).$$

Now the corresponding lower estimation is stated in the form of the following theorem.

THEOREM 3. *If $m_2 < \infty$, then*

$$\Psi_q(y) \ge \underline{\Psi}_q(y) \equiv e_1(y) + r(y_1),$$

where

$$e_1(y) = \exp\left(-\frac{q'y}{\tilde{m}_1(y/2)} - \frac{q'\tilde{m}_2(y/2)}{\tilde{m}_1^2(y/2)}\right),$$

$$r(x) \equiv [1 - F(y)]\frac{(1-q)q}{(q')^2}\left(1 + \exp(-q'x) - \frac{2}{q'x}(1 - \exp(-q'x))\right),$$

$$y_1 = \frac{y}{2} + \frac{m_2 - 1}{2}\left(1 - \sqrt{1 + \frac{2y}{m_2 - 1}}\right).$$

Now the upper estimation is stated in the following theorem. For $0 < \theta \le 1$, let us denote

$$\lambda \equiv \frac{\Lambda(\theta y)}{\theta y)}, \qquad E(y) \equiv \exp(\lambda\tilde{X}(a/q)), \qquad d_{GE} \equiv (1-q)\mathbf{E}G(Y(a/q))\mathbf{E}E(y).$$

Theorem 4. *If $m_2 < \infty$ and $m_G < \infty$ for some function $G \in \mathcal{G}_s$, $s \geq 0$, $\lambda \leq q$ for $0 < \theta \leq 1$, and the parameters a and θ properly chosen in order to satisfy*

$$d_{GE} < 1,$$

where

$$\mathbf{E}E(y) \leq E_1(y) \equiv 1 + \lambda + \frac{1}{2} m_2 M(a) \lambda^2, \qquad M(a) = \frac{2}{a^2}(e^a - 1 - a),$$

then

$$\Psi_q(y) \leq \overline{\Psi}_q(y) \equiv e^1(y) + \mathbf{1}(y - a/q)\left(r(y, E_1, G) \vee \frac{1-q}{q}[1 - F(a/q)] \right),$$

where

$$e^1(y) = \exp(-R_1(a) y), \qquad R_1(a) = \frac{1}{\tilde{m}_2(a/q) M(a)} \left(-\tilde{m}_1(a/q) + \sqrt{\tilde{m}_1^2(a/q) + 2q\tilde{m}_2(a/q) M(a)} \right),$$

$$\mathbf{1}(u) \equiv \begin{cases} 0, & \text{if } u < 0, \\ 1, & \text{if } u \geq 0, \end{cases}$$

$$r(y, E, G) = \frac{1-q}{q}(1 - F(\theta y)) + \frac{q d_{GE}}{1 - d_{GE}} G^{-1/\theta}(\theta y).$$

Remark 1. As $\Psi_q(y)$ is a monotone nonincreasing function of y and q, it holds that

$$\Psi_q(y) \leq \min_{u \leq y} \overline{\Psi}_q(u), \qquad \Psi_q(y) \leq \min_{q' \leq q} \overline{\Psi}_{q'}(y). \tag{4}$$

3. Example with Pareto Distribution

Let us now take a concrete example inspired from collective risk theory with a generic claim size Z following the Pareto distribution. Our goal is to provide numerical experiments and especially to choose appropriate test functions G and T.

Now consider the case where the common d.f. F of the normed summands X_i in the geometric sum has the form

$$F(y) = \begin{cases} \dfrac{(t-1)y}{ty^*}, & \text{if } y < y^*, \\ 1 - \dfrac{1}{t}\left(\dfrac{y^*}{y}\right)^{t-1}, & \text{if } y \geq y^*, \end{cases}$$

where $t > 2$ and $y^* \equiv 2(t-2)/(t-1)$. This case appears in the Sparre Andersen risk model with exponential interclaim times and normed claim sizes Z_i which follow the Pareto distribution

$$B(z) = \begin{cases} 0, & z < (t-1)/t, \\ 1 - \dfrac{(t-1)^t}{t^t z^t}, & z \geq (t-1)/t, \end{cases}$$

with moments

$$\mathbf{E}Z^k = \frac{(t-1)^k}{(t-k)t^{k-1}}.$$

Here the second parameter of the Pareto distribution was determined from the normalizing condition $\mathbf{E}Z = 1$. This normalization enables the initial capital x of the insurance company to be expressed in comparison with the mean claim size. Because of the other normalization of the summands in the geometric sum, considered at the beginning of the introduction, the argument in Ψ_q has to be divided by the mean value of the conditional ladder height (see [7]),

$$\mathbf{E}Y = \frac{\mathbf{E}Z^2}{2\mathbf{E}Z} = \frac{t-1}{ty^*},$$

of this risk process, since in this case they play the role of the geometric summands. In collective risk theory, the ruin probability relates to the previous geometric parameter q through the following equation:

$$\Psi_q(0) = q,$$

and the relative safety loading is equal to

$$\rho \equiv \frac{q}{1-q}.$$

This parameter, together with the initial capital x, plays the role of the input data for the risk analysis. Typically, we are interested in small values of

$$\Psi_q\left(\frac{ty^*}{(t-1)}x\right).$$

Hence will be found one- and two-sided bounds following from the earlier results. Let us list these estimates, which will be used in numerical examples, and equip them with new notations just to indicate them in the tables.

Application of Theorem 1. Consider the asymptotic expression

$$\Psi_q\left(\frac{ty^*}{(t-1)}x\right) \sim A(x) \equiv \frac{1-F(y)}{\rho} = \begin{cases} \frac{1}{\rho}\left(1-\frac{(t-1)y}{ty^*}\right), & \text{if } y < y^*, \\ \frac{1}{\rho t}\left(\frac{y^*}{y}\right)^{t-1}, & \text{if } y \geq y^*, \end{cases}$$

where

$$y \equiv \frac{ty^*}{(t-1)}x.$$

The function $A(x)$ thus defined shows the asymptotic behavior of $\Psi_q(y)$ for $y \to \infty$.

Application of the recursive algorithm. Let us calculate the two-sided bounds

$$l(x) = \mathrm{l}\left(\frac{ty^*}{(t-1)}x\right), \qquad u(x) \equiv \mathrm{u}\left(\frac{ty^*}{(t-1)}x\right)$$

such that

$$l(x) \leq \Psi_q\left(\frac{ty^*}{(t-1)}x\right) \leq u(x),$$

using the straightforward recursive algorithm with span $d = 1$. The relative accuracy of these bounds is obtained as

$$\delta(x) = \frac{u(x)-l(x)}{u(x)+l(x)}.$$

Evidently, F is a heavy-tailed d.f. Moreover, it is subexponential. The density f of F is equal to

$$f(y) = \frac{t-1}{ty^*}\left(\frac{y^*}{y}\right)^{t1(y-y^*)} = \begin{cases} \frac{(t-1)^2}{2t(t-2)}, & \text{if } y < y^*, \\ \frac{2^{t-1}(t-2)^{t-1}}{t(t-1)^{t-2}y^t}, & \text{if } y \geq y^*, \end{cases}$$

and the failure rate function is

$$\lambda_F(y) = \frac{f(y)}{1-F(y)} = \begin{cases} \frac{(t-1)^2}{2t(t-2)-y(t-1)^2}, & \text{if } y < y^*, \\ \frac{t-1}{y}, & \text{if } y \geq y^*. \end{cases}$$

Application of Theorem 2. Since for $y \geq y^*$ the d.f. F satisfies the DFR (Decreasing Failure Rate) property, let us take the test function $T(y)$ in the form

$$T(y) = \exp\left(p\int_0^y \lambda_F\left(u \vee \frac{t-1}{\varepsilon}\right)du\right),$$

where $\varepsilon < (t-1)/y^*$ and $p < 1$ are parameters that satisfy the relation

$$(1-q)\mathbf{E}T(X) \leq 1.$$

Thus,

$$T(y) = \begin{cases} \exp(p\varepsilon y), & y < (t-1)/\varepsilon, \\ \left(\dfrac{y\varepsilon e}{t-1}\right)^{(t-1)p}, & y \geq (t-1)/\varepsilon, \end{cases}$$

and for $t > 3$

$$\begin{aligned} \mathbf{E}T(X) \leq & \frac{(t-1)^2(e^{\delta\varepsilon y^*}-1)}{2t(t-2)\delta\varepsilon} + \frac{(t-1)y^{*t-1}}{t}\left(\frac{e^{\delta\varepsilon y^*}}{(t-1)y^{*t-1}} - \frac{\varepsilon^{t-1}e^{\delta(t-1)}}{(t-1)^t}\right. \\ & + \frac{\delta\varepsilon e^{\delta\varepsilon y^*}}{(t-1)(t-2)y^{*t-2}} - \frac{\delta\varepsilon^{t-1}e^{\delta(t-1)}}{(t-2)(t-1)^{t-1}} + \frac{\delta^2\varepsilon^2 e^{\delta\varepsilon y^*}}{(t-1)(t-2)(t-3)y^{*t-3}} \\ & \left. - \frac{\delta^2\varepsilon^{t-1}e^{\delta(t-1)}}{(t-2)(t-3)(t-1)^{t-2}} + \frac{\delta^2\varepsilon^2(e^{\delta(t-1)} - e^{\delta\varepsilon y^*})}{(t-1)(t-2)(t-3)y^{*t-3}}\right) + \frac{y^{*t-1}\varepsilon^{t-1}e^{\delta(t-1)}}{t(1-\delta)(t-1)^{t-1}}. \end{aligned}$$

Then, by Theorem 2,

$$\Psi_q\left(\frac{ty^*}{(t-1)}x\right) \leq W(x) \equiv \frac{\mathbf{E}T(X\wedge y)}{1-q}.$$

The estimate $W(x)$ depends on two parameters ε and p for any given q. These parameters ε and p are properly chosen to minimize $W(x)$.

Note that the d.f. $F(y)$ has a finite mean $m_1 = 1$ and truncated first moment

$$\tilde{m}_1(b) = \begin{cases} b - \dfrac{(t-1)^2 b^2}{4t(t-2)}, & \text{if } b < y^*, \\ 1 - \dfrac{2^{t-1}(t-2)^{t-2}}{t(t-1)^{t-1}b^{t-2}}, & \text{if } b \geq y^*. \end{cases}$$

For $t > 3$, there also exist the second moments

$$m_2 = \mathbf{E}X^2 = \frac{4(t-2)^2}{3(t-3)(t-1)} < \infty,$$

$$\tilde{m}_2(b) = \begin{cases} b^2 - \dfrac{b^3(t-1)^2}{3t(t-2)}, & \text{if } b < y^*, \\ \dfrac{4(t-2)^2}{3(t-1)(t-3)} - \dfrac{2^t(t-2)^{t-1}}{t(t-3)(t-1)^{t-1}b^{t-3}}, & \text{if } b \geq y^*. \end{cases}$$

Application of Theorem 3 for $m_2 < \infty$. For $t > 3$, the condition $m_2 < \infty$ is satisfied and hence $\tilde{m}_2(b) < \infty$, so the lower estimation $L(x)$ has the form

$$\Psi_q\left(\frac{ty^*}{(t-1)}x\right) \geq L(x) \equiv \exp\left(-\frac{q'y}{\tilde{m}_1(y/2)} - \frac{q'\tilde{m}_2(y/2)}{\tilde{m}_1^2(y/2)}\right) + r\left(\frac{y}{2} + \frac{m_2-1}{2}\left\{1 - \sqrt{1 + \frac{2y}{m_2-1}}\right\}\right),$$

where the previous expressions for the moments are to be substituted.

Application of Theorem 4 for $m_2 < \infty$. For the upper bound when $t > 3$ the test function G can be taken in the form

$$G(y) = \begin{cases} (1 + \alpha y^{0\delta})^{y/y^0}, & y < y^0, \\ 1 + \alpha y^\delta, & y \geq y^0, \end{cases}$$

with two free parameters $1 < \delta < t-1$, $\alpha > 0$, and y^0 is the unique positive solution of the equation

$$\frac{\delta\alpha y}{1 + \alpha y^\delta} - \log(1 + \alpha y^\delta) = 0.$$

Note that for this choice of y^0 the $\Lambda(y)/y = \log(1 + \alpha y^\delta)/y$ does not increase for any $y > 0$, which follows from the relation

$$\left(\frac{\Lambda(y)}{y}\right)' = \begin{cases} 0, & y < y^0, \\ \dfrac{1}{y^2}\left(\dfrac{\delta\alpha y^\delta}{1+\alpha y^\delta} - \log(1+\alpha y^\delta)\right), & y \geq y^0. \end{cases}$$

Hence, $G \in \mathcal{G}_{\delta-} \subset \mathcal{G}_1$. Further calculations give

$$m_G \leq \frac{(t-1)(B^{y^*}-1)}{ty^* \log B} + \frac{(t-1)y^{*(t-1)}(B^{y^0}-B^{y^*})}{ty^{0t}\log B} + \frac{y^{*(t-1)}}{ty^{0(t-1)}} + \frac{\alpha(t-1)y^{*(t-1)}}{t(t-1-\delta)y^{0(t-1-\delta)}} < \infty,$$

where

$$B \equiv (1+\alpha y^{0\delta})^{1/y^0}.$$

Then

$$\Psi_q\left(\frac{ty^*}{(t-1)}x\right) \leq U(x) \equiv \exp(-R_1(a)y) + \frac{1-F(\theta y)}{\rho} + \frac{qd_{GE}}{1-d_{GE}} G^{-1/\theta}(\theta y).$$

4. Numerical Results

Let us now present some numerical results for Pareto claim sizes with $t = 3.01 > 3$. The results are placed into ten tables. Each table is referred to a constant value of the initial capital x and concretely to the values $x = 5000, 1000, 500, 100, 50$. In the first column is the relative safety loading for the corresponding values $\rho = 0.01, 0.02, \ldots, 0.1, 0.2, \ldots, 0.5$. Thus it is possible to see how the estimations in different regions of parameters work.

During the numerical procedure, parameters a, θ, α, and δ are chosen to minimize the upper estimation, under given q and x. This optimization is carried out by a sorting approach.

Tables 1, 3, 5, 7, 9, and 11 contain figures of $l(x)$, $u(x)$, $\delta(x)$, and $A(x)$ against ρ. One can see that, in this case, it is possible to obtain tight lower and upper bounds with the help of recursive algorithm. It is noteworthy that asymptotic approximation $A(x)$ lies out of these bounds in the chosen range of argument x.

Tables 2, 4, 6, 8, 10, and 12 contain figures of $L(x)$, $U(x)$ (together with optimal values of the parameters a, θ, α, and δ), and $W(x)$ (with optimal values of p and ε) against ρ.

TABLE 1. $t = 3.01$, $x = 5000$.

ρ	$l(x)$	$u(x)$	$\delta(x)$	$A(x)$
0.01	$5.586 \cdot 10^{-7}$	$5.917 \cdot 10^{-7}$	0.029	$5.421 \cdot 10^{-7}$
0.02	$2.751 \cdot 10^{-7}$	$2.828 \cdot 10^{-7}$	0.014	$2.711 \cdot 10^{-7}$
0.03	$1.825 \cdot 10^{-7}$	$1.859 \cdot 10^{-7}$	0.009	$1.807 \cdot 10^{-7}$
0.04	$1.365 \cdot 10^{-7}$	$1.384 \cdot 10^{-7}$	0.007	$1.355 \cdot 10^{-7}$
0.05	$1.091 \cdot 10^{-7}$	$1.103 \cdot 10^{-7}$	0.005	$1.084 \cdot 10^{-7}$
0.06	$9.080 \cdot 10^{-8}$	$9.162 \cdot 10^{-8}$	0.004	$9.036 \cdot 10^{-8}$
0.07	$7.777 \cdot 10^{-8}$	$7.838 \cdot 10^{-8}$	0.004	$7.745 \cdot 10^{-8}$
0.08	$6.802 \cdot 10^{-8}$	$6.848 \cdot 10^{-8}$	0.003	$6.777 \cdot 10^{-8}$
0.09	$6.043 \cdot 10^{-8}$	$6.080 \cdot 10^{-8}$	0.003	$6.024 \cdot 10^{-8}$
0.1	$5.437 \cdot 10^{-8}$	$5.467 \cdot 10^{-8}$	0.003	$5.421 \cdot 10^{-8}$
0.2	$2.715 \cdot 10^{-8}$	$2.722 \cdot 10^{-8}$	0.001	$2.711 \cdot 10^{-8}$
0.3	$1.809 \cdot 10^{-8}$	$1.812 \cdot 10^{-8}$	0.001	$1.807 \cdot 10^{-8}$
0.4	$1.356 \cdot 10^{-8}$	$1.358 \cdot 10^{-8}$	0.001	$1.355 \cdot 10^{-8}$
0.5	$1.085 \cdot 10^{-8}$	$1.086 \cdot 10^{-8}$	0.001	$1.084 \cdot 10^{-8}$

TABLE 2. $t = 3.01$, $x = 5000$.

ρ	$L(x)$	$U(x)$	a	θ	α	δ	$W(x)$	p	ε
0.01	$5.039 \cdot 10^{-7}$	$1.455 \cdot 10^{-6}$	09.9	0.65	0.40	1.38	$2.786 \cdot 10^{-4}$	0.88	0.01
0.02	$2.576 \cdot 10^{-7}$	$7.265 \cdot 10^{-7}$	10.9	0.65	0.13	1.56	$9.933 \cdot 10^{-5}$	0.99	0.01
0.03	$1.718 \cdot 10^{-7}$	$4.827 \cdot 10^{-7}$	11.4	0.65	0.11	1.61	$3.474 \cdot 10^{-5}$	0.96	0.02
0.04	$1.283 \cdot 10^{-7}$	$3.607 \cdot 10^{-7}$	11.8	0.65	0.10	1.65	$2.527 \cdot 10^{-5}$	0.92	0.03
0.05	$1.020 \cdot 10^{-7}$	$2.876 \cdot 10^{-7}$	12.0	0.65	0.09	1.68	$1.585 \cdot 10^{-5}$	0.96	0.03
0.06	$8.436 \cdot 10^{-8}$	$2.389 \cdot 10^{-7}$	12.3	0.65	0.10	1.68	$1.313 \cdot 10^{-5}$	0.93	0.04
0.07	$7.174 \cdot 10^{-8}$	$2.041 \cdot 10^{-7}$	12.5	0.66	0.09	1.71	$1.024 \cdot 10^{-5}$	0.95	0.04
0.08	$6.227 \cdot 10^{-8}$	$1.780 \cdot 10^{-7}$	12.6	0.66	0.09	1.72	$8.636 \cdot 10^{-6}$	0.93	0.05
0.09	$5.490 \cdot 10^{-8}$	$1.578 \cdot 10^{-7}$	12.7	0.66	0.09	1.73	$7.563 \cdot 10^{-6}$	0.94	0.05
0.1	$4.900 \cdot 10^{-8}$	$1.417 \cdot 10^{-7}$	12.9	0.66	0.10	1.73	$5.855 \cdot 10^{-6}$	0.96	0.05
0.2	$2.257 \cdot 10^{-8}$	$6.942 \cdot 10^{-8}$	13.6	0.66	0.12	1.77	$2.454 \cdot 10^{-6}$	0.94	0.09
0.3	$1.395 \cdot 10^{-8}$	$4.565 \cdot 10^{-8}$	14.0	0.67	0.15	1.78	$1.419 \cdot 10^{-6}$	0.95	0.11
0.4	$9.753 \cdot 10^{-9}$	$3.384 \cdot 10^{-8}$	14.3	0.67	0.16	1.80	$1.000 \cdot 10^{-6}$	0.96	0.12
0.5	$7.315 \cdot 10^{-9}$	$2.683 \cdot 10^{-8}$	14.4	0.67	0.19	1.80	$7.319 \cdot 10^{-7}$	0.96	0.14

One can note that the bound $W(x)$ is mostly worse than the bound $U(x)$. Next we take these bounds for $x = 1000$.

TABLE 3. $t = 3.01$, $x = 1000$.

ρ	$l(x)$	$u(x)$	$\delta(x)$	$A(x)$
0.01	$1.625 \cdot 10^{-5}$	$9.283 \cdot 10^{-5}$	0.701	$1.379 \cdot 10^{-5}$
0.02	$7.451 \cdot 10^{-6}$	$8.813 \cdot 10^{-6}$	0.084	$6.894 \cdot 10^{-6}$
0.03	$4.836 \cdot 10^{-6}$	$5.356 \cdot 10^{-6}$	0.051	$4.596 \cdot 10^{-6}$
0.04	$3.580 \cdot 10^{-6}$	$3.854 \cdot 10^{-6}$	0.037	$3.447 \cdot 10^{-6}$
0.05	$2.842 \cdot 10^{-6}$	$3.011 \cdot 10^{-6}$	0.029	$2.758 \cdot 10^{-6}$
0.06	$2.356 \cdot 10^{-6}$	$2.471 \cdot 10^{-6}$	0.024	$2.298 \cdot 10^{-6}$
0.07	$2.012 \cdot 10^{-6}$	$2.095 \cdot 10^{-6}$	0.020	$1.970 \cdot 10^{-6}$
0.08	$1.756 \cdot 10^{-6}$	$1.819 \cdot 10^{-6}$	0.018	$1.723 \cdot 10^{-6}$
0.09	$1.558 \cdot 10^{-6}$	$1.607 \cdot 10^{-6}$	0.015	$1.532 \cdot 10^{-6}$
0.1	$1.400 \cdot 10^{-6}$	$1.439 \cdot 10^{-6}$	0.014	$1.379 \cdot 10^{-6}$
0.2	$6.945 \cdot 10^{-7}$	$7.041 \cdot 10^{-7}$	0.007	$6.894 \cdot 10^{-7}$
0.3	$4.619 \cdot 10^{-7}$	$4.661 \cdot 10^{-7}$	0.005	$4.596 \cdot 10^{-7}$
0.4	$3.460 \cdot 10^{-7}$	$3.483 \cdot 10^{-7}$	0.003	$3.447 \cdot 10^{-7}$
0.5	$2.766 \cdot 10^{-7}$	$2.781 \cdot 10^{-7}$	0.003	$2.758 \cdot 10^{-7}$

TABLE 4. $t = 3.01$, $x = 1000$.

ρ	$L(x)$	$U(x)$	a	θ	α	δ	$W(x)$	p	ε
0.01	$9.103 \cdot 10^{-6}$	$5.662 \cdot 10^{-5}$	2.6	0.56	0.47	1.17	$4.800 \cdot 10^{-3}$	0.88	0.01
0.02	$5.532 \cdot 10^{-6}$	$2.289 \cdot 10^{-5}$	6.0	0.60	1.93	1.11	$1.906 \cdot 10^{-3}$	0.84	0.02
0.03	$3.918 \cdot 10^{-6}$	$1.442 \cdot 10^{-5}$	7.6	0.62	1.42	1.22	$7.750 \cdot 10^{-4}$	0.96	0.02
0.04	$3.009 \cdot 10^{-6}$	$1.065 \cdot 10^{-5}$	8.1	0.62	0.77	1.34	$4.952 \cdot 10^{-4}$	0.92	0.03
0.05	$2.431 \cdot 10^{-6}$	$8.468 \cdot 10^{-6}$	8.5	0.62	0.59	1.40	$3.533 \cdot 10^{-4}$	0.96	0.03
0.06	$2.033 \cdot 10^{-6}$	$7.026 \cdot 10^{-6}$	8.7	0.62	0.44	1.46	$2.655 \cdot 10^{-4}$	0.93	0.04
0.07	$1.742 \cdot 10^{-6}$	$6.002 \cdot 10^{-6}$	8.9	0.62	0.40	1.49	$2.209 \cdot 10^{-4}$	0.95	0.04
0.08	$1.520 \cdot 10^{-6}$	$5.235 \cdot 10^{-6}$	9.1	0.62	0.38	1.51	$1.746 \cdot 10^{-4}$	0.93	0.05
0.09	$1.346 \cdot 10^{-6}$	$4.640 \cdot 10^{-6}$	9.3	0.62	0.36	1.53	$1.579 \cdot 10^{-4}$	0.94	0.05
0.1	$1.206 \cdot 10^{-6}$	$4.165 \cdot 10^{-6}$	9.4	0.62	0.31	1.56	$1.303 \cdot 10^{-4}$	0.96	0.05
0.2	$5.641 \cdot 10^{-7}$	$2.039 \cdot 10^{-6}$	10.2	0.63	0.29	1.64	$5.104 \cdot 10^{-5}$	0.94	0.09
0.3	$3.503 \cdot 10^{-7}$	$1.336 \cdot 10^{-6}$	10.6	0.63	0.30	1.68	$3.038 \cdot 10^{-5}$	0.95	0.11
0.4	$2.456 \cdot 10^{-7}$	$9.887 \cdot 10^{-7}$	10.9	0.63	0.33	1.70	$2.125 \cdot 10^{-5}$	0.94	0.14
0.5	$1.845 \cdot 10^{-7}$	$7.821 \cdot 10^{-7}$	11.1	0.64	0.36	1.71	$1.604 \cdot 10^{-5}$	0.95	0.15

TABLE 5. $t = 3.01$, $x = 500$.

ρ	$l(x)$	$u(x)$	$\delta(x)$	$A(x)$
0.01	$8.435 \cdot 10^{-5}$	$8.227 \cdot 10^{-3}$	0.98	$5.554 \cdot 10^{-5}$
0.02	$3.277 \cdot 10^{-5}$	$1.248 \cdot 10^{-4}$	0.58	$2.777 \cdot 10^{-5}$
0.03	$2.058 \cdot 10^{-5}$	$2.719 \cdot 10^{-5}$	0.14	$1.851 \cdot 10^{-5}$
0.04	$1.501 \cdot 10^{-5}$	$1.777 \cdot 10^{-5}$	0.08	$1.388 \cdot 10^{-5}$
0.05	$1.182 \cdot 10^{-5}$	$1.342 \cdot 10^{-5}$	0.06	$1.111 \cdot 10^{-5}$
0.06	$9.742 \cdot 10^{-6}$	$1.079 \cdot 10^{-5}$	0.05	$9.256 \cdot 10^{-6}$
0.07	$8.288 \cdot 10^{-6}$	$9.031 \cdot 10^{-6}$	0.04	$7.934 \cdot 10^{-6}$
0.08	$7.211 \cdot 10^{-6}$	$7.765 \cdot 10^{-6}$	0.04	$6.942 \cdot 10^{-6}$
0.09	$6.382 \cdot 10^{-6}$	$6.811 \cdot 10^{-6}$	0.03	$6.171 \cdot 10^{-6}$
0.1	$5.724 \cdot 10^{-6}$	$6.066 \cdot 10^{-6}$	0.029	$5.554 \cdot 10^{-6}$
0.2	$2.819 \cdot 10^{-6}$	$2.898 \cdot 10^{-6}$	0.014	$2.777 \cdot 10^{-6}$
0.3	$1.870 \cdot 10^{-6}$	$1.904 \cdot 10^{-6}$	0.009	$1.851 \cdot 10^{-6}$
0.4	$1.399 \cdot 10^{-6}$	$1.418 \cdot 10^{-6}$	0.007	$1.388 \cdot 10^{-6}$
0.5	$1.117 \cdot 10^{-6}$	$1.130 \cdot 10^{-6}$	0.005	$1.111 \cdot 10^{-6}$

TABLE 6. $t = 3.01$, $x = 500$.

ρ	$L(x)$	$U(x)$	a	θ	α	δ	$W(x)$	p	ε
0.01	$5.514 \cdot 10^{-4}$	$1.061 \cdot 10^{-3}$	0.6	0.43	0.03	1.41	$1.636 \cdot 10^{-2}$	0.88	0.01
0.02	$1.669 \cdot 10^{-5}$	$1.268 \cdot 10^{-4}$	2.4	0.53	0.53	1.16	$6.143 \cdot 10^{-3}$	0.84	0.02
0.03	$1.306 \cdot 10^{-5}$	$7.250 \cdot 10^{-5}$	4.1	0.57	1.19	1.13	$2.952 \cdot 10^{-3}$	0.96	0.02
0.04	$1.059 \cdot 10^{-5}$	$5.036 \cdot 10^{-5}$	5.6	0.58	1.97	1.11	$1.783 \cdot 10^{-3}$	0.92	0.03
0.05	$8.818 \cdot 10^{-6}$	$3.858 \cdot 10^{-5}$	6.5	0.59	1.90	1.16	$1.345 \cdot 10^{-3}$	0.96	0.03
0.06	$7.515 \cdot 10^{-6}$	$3.146 \cdot 10^{-5}$	6.9	0.60	1.38	1.24	$9.695 \cdot 10^{-4}$	0.93	0.04
0.07	$6.524 \cdot 10^{-6}$	$2.661 \cdot 10^{-5}$	7.2	0.60	1.08	1.30	$8.291 \cdot 10^{-4}$	0.95	0.04
0.08	$5.749 \cdot 10^{-6}$	$2.309 \cdot 10^{-5}$	7.5	0.60	0.94	1.34	$6.373 \cdot 10^{-4}$	0.93	0.05
0.09	$5.127 \cdot 10^{-6}$	$2.040 \cdot 10^{-5}$	7.6	0.60	0.80	1.38	$5.823 \cdot 10^{-4}$	0.90	0.06
0.1	$4.619 \cdot 10^{-6}$	$1.827 \cdot 10^{-5}$	7.8	0.60	0.71	1.41	$4.907 \cdot 10^{-4}$	0.92	0.06
0.2	$2.215 \cdot 10^{-6}$	$8.888 \cdot 10^{-6}$	8.7	0.61	0.48	1.55	$1.884 \cdot 10^{-4}$	0.92	0.10
0.3	$1.386 \cdot 10^{-6}$	$5.814 \cdot 10^{-6}$	9.2	0.61	0.48	1.60	$1.136 \cdot 10^{-4}$	0.95	0.11
0.4	$9.758 \cdot 10^{-7}$	$4.297 \cdot 10^{-6}$	9.4	0.61	0.49	1.63	$7.716 \cdot 10^{-5}$	0.93	0.15
0.5	$7.348 \cdot 10^{-7}$	$3.393 \cdot 10^{-6}$	9.7	0.62	0.51	1.65	$5.947 \cdot 10^{-5}$	0.94	0.16

TABLE 7. $t = 3.01$, $x = 100$.

ρ	$l(x)$	$u(x)$	$\delta(x)$	$A(x)$
0.01	$7.060 \cdot 10^{-2}$	$3.775 \cdot 10^{-1}$	0.68	$1.418 \cdot 10^{-3}$
0.02	$7.370 \cdot 10^{-3}$	$1.455 \cdot 10^{-1}$	0.90	$7.092 \cdot 10^{-4}$
0.03	$1.538 \cdot 10^{-3}$	$5.736 \cdot 10^{-2}$	0.95	$4.728 \cdot 10^{-4}$
0.04	$6.732 \cdot 10^{-4}$	$2.325 \cdot 10^{-2}$	0.94	$3.546 \cdot 10^{-4}$
0.05	$4.349 \cdot 10^{-4}$	$9.764 \cdot 10^{-3}$	0.91	$2.837 \cdot 10^{-4}$
0.06	$3.273 \cdot 10^{-4}$	$4.300 \cdot 10^{-3}$	0.86	$2.364 \cdot 10^{-4}$
0.07	$2.642 \cdot 10^{-4}$	$2.024 \cdot 10^{-3}$	0.77	$2.026 \cdot 10^{-4}$
0.08	$2.220 \cdot 10^{-4}$	$1.042 \cdot 10^{-3}$	0.65	$1.773 \cdot 10^{-4}$
0.09	$1.917 \cdot 10^{-4}$	$5.998 \cdot 10^{-4}$	0.52	$1.576 \cdot 10^{-4}$
0.1	$1.687 \cdot 10^{-4}$	$3.885 \cdot 10^{-4}$	0.39	$1.418 \cdot 10^{-4}$
0.2	$7.693 \cdot 10^{-5}$	$9.164 \cdot 10^{-5}$	0.09	$7.092 \cdot 10^{-5}$
0.3	$4.986 \cdot 10^{-5}$	$5.541 \cdot 10^{-5}$	0.05	$4.728 \cdot 10^{-5}$
0.4	$3.689 \cdot 10^{-5}$	$3.980 \cdot 10^{-5}$	0.04	$3.547 \cdot 10^{-5}$
0.5	$2.928 \cdot 10^{-5}$	$3.107 \cdot 10^{-5}$	0.03	$2.837 \cdot 10^{-5}$

TABLE 8. $t = 3.01$, $x = 100$.

ρ	$L(x)$	$U(x)$	a	θ	α	δ	$W(x)$	p	ε
0.01	$2.148 \cdot 10^{-1}$	$5.574 \cdot 10^{-2}$	3.1				$2.640 \cdot 10^{-1}$	0.88	0.01
0.02	$4.689 \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$	5.7				$9.299 \cdot 10^{-2}$	0.84	0.02
0.03	$1.041 \cdot 10^{-2}$	$2.028 \cdot 10^{-2}$	1.8				$5.235 \cdot 10^{-2}$	0.81	0.03
0.04	$2.376 \cdot 10^{-3}$	$8.313 \cdot 10^{-3}$	2.3				$3.494 \cdot 10^{-2}$	0.92	0.03
0.05	$5.802 \cdot 10^{-4}$	$4.092 \cdot 10^{-3}$	0.8	0.39	0.17	1.22	$2.434 \cdot 10^{-2}$	0.88	0.04
0.06	$1.741 \cdot 10^{-4}$	$2.452 \cdot 10^{-3}$	1.0	0.41	0.24	1.21	$1.960 \cdot 10^{-2}$	0.93	0.04
0.07	$8.120 \cdot 10^{-5}$	$1.710 \cdot 10^{-3}$	1.3	0.44	0.37	1.17	$1.559 \cdot 10^{-2}$	0.89	0.05
0.08	$5.949 \cdot 10^{-5}$	$1.311 \cdot 10^{-3}$	1.5	0.45	0.43	1.18	$1.287 \cdot 10^{-2}$	0.93	0.05
0.09	$5.393 \cdot 10^{-5}$	$1.063 \cdot 10^{-3}$	1.8	0.46	0.56	1.16	$1.067 \cdot 10^{-2}$	0.90	0.06
0.1	$5.190 \cdot 10^{-5}$	$8.942 \cdot 10^{-4}$	2.0	0.47	0.68	1.15	$9.488 \cdot 10^{-3}$	0.87	0.07
0.2	$3.788 \cdot 10^{-5}$	$3.331 \cdot 10^{-4}$	4.2	0.53	1.65	1.16	$3.663 \cdot 10^{-3}$	0.92	0.10
0.3	$2.721 \cdot 10^{-5}$	$2.011 \cdot 10^{-4}$	5.3	0.55	1.89	1.23	$2.204 \cdot 10^{-3}$	0.92	0.13
0.4	$2.042 \cdot 10^{-5}$	$1.438 \cdot 10^{-4}$	5.7	0.56	1.71	1.31	$1.509 \cdot 10^{-3}$	0.89	0.18
0.5	$1.595 \cdot 10^{-5}$	$1.116 \cdot 10^{-4}$	6.0	0.56	1.64	1.36	$1.135 \cdot 10^{-3}$	0.90	0.20

In the case $\rho = 0.01, 0.02, 0.03, 0.04$, the optimal upper bound $U(x)$ is taken through the uniform estimation, so the test function and the parameter θ are absent.

TABLE 9. $t = 3.01$, $x = 50$.

ρ	$l(x)$	$u(x)$	$\delta(x)$	$A(x)$
0.01	$2.570 \cdot 10^{-1}$	$6.100 \cdot 10^{-1}$	0.41	$5.713 \cdot 10^{-3}$
0.02	$7.444 \cdot 10^{-2}$	$3.754 \cdot 10^{-1}$	0.67	$2.857 \cdot 10^{-3}$
0.03	$2.485 \cdot 10^{-2}$	$2.332 \cdot 10^{-1}$	0.81	$1.904 \cdot 10^{-3}$
0.04	$9.843 \cdot 10^{-3}$	$1.463 \cdot 10^{-1}$	0.87	$1.428 \cdot 10^{-3}$
0.05	$4.715 \cdot 10^{-3}$	$9.266 \cdot 10^{-2}$	0.90	$1.143 \cdot 10^{-3}$
0.06	$2.711 \cdot 10^{-3}$	$5.933 \cdot 10^{-2}$	0.91	$9.522 \cdot 10^{-4}$
0.07	$1.807 \cdot 10^{-3}$	$3.843 \cdot 10^{-2}$	0.91	$8.162 \cdot 10^{-4}$
0.08	$1.339 \cdot 10^{-3}$	$2.520 \cdot 10^{-2}$	0.90	$7.142 \cdot 10^{-4}$
0.09	$1.063 \cdot 10^{-3}$	$1.676 \cdot 10^{-2}$	0.88	$6.348 \cdot 10^{-4}$
0.1	$8.848 \cdot 10^{-4}$	$1.131 \cdot 10^{-2}$	0.85	$5.713 \cdot 10^{-4}$
0.2	$3.417 \cdot 10^{-4}$	$7.243 \cdot 10^{-3}$	0.36	$2.857 \cdot 10^{-4}$
0.3	$2.134 \cdot 10^{-4}$	$2.854 \cdot 10^{-3}$	0.14	$1.904 \cdot 10^{-4}$
0.4	$1.553 \cdot 10^{-4}$	$1.864 \cdot 10^{-3}$	0.09	$1.428 \cdot 10^{-4}$
0.5	$1.221 \cdot 10^{-4}$	$1.399 \cdot 10^{-3}$	0.07	$1.143 \cdot 10^{-4}$

TABLE 10. $t = 3.01$, $x = 50$.

ρ	$L(x)$	$U(x)$	a	θ	α	δ	$W(x)$	p	ε
0.01	$4.561 \cdot 10^{-1}$	$2.478 \cdot 10^{-2}$	4.2				$5.135 \cdot 10^{-1}$	0.88	0.01
0.02	$2.097 \cdot 10^{-1}$	$2.235 \cdot 10^{-2}$	5.7				$2.811 \cdot 10^{-1}$	0.84	0.02
0.03	$9.713 \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$					$1.618 \cdot 10^{-1}$	0.81	0.03
0.04	$4.534 \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$					$1.072 \cdot 10^{-1}$	0.79	0.04
0.05	$2.134 \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$					$8.028 \cdot 10^{-2}$	0.77	0.05
0.06	$1.013 \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$					$6.365 \cdot 10^{-2}$	0.84	0.05
0.07	$4.854 \cdot 10^{-3}$	$1.947 \cdot 10^{-2}$	2.0				$5.036 \cdot 10^{-2}$	0.82	0.06
0.08	$2.358 \cdot 10^{-3}$	$1.411 \cdot 10^{-2}$	2.2				$4.228 \cdot 10^{-2}$	0.80	0.07
0.09	$1.169 \cdot 10^{-3}$	$1.080 \cdot 10^{-2}$	2.4				$3.599 \cdot 10^{-2}$	0.84	0.07
0.1	$5.984 \cdot 10^{-4}$	$8.675 \cdot 10^{-3}$	2.7				$3.128 \cdot 10^{-2}$	0.82	0.08
0.2	$6.354 \cdot 10^{-5}$	$2.141 \cdot 10^{-3}$	1.9	0.45	0.94	1.10	$1.284 \cdot 10^{-2}$	0.89	0.11
0.3	$5.772 \cdot 10^{-5}$	$1.153 \cdot 10^{-3}$	2.8	0.48	1.45	1.12	$7.671 \cdot 10^{-3}$	0.88	0.15
0.4	$5.018 \cdot 10^{-5}$	$7.734 \cdot 10^{-4}$	3.5	0.50	1.82	1.15	$5.167 \cdot 10^{-3}$	0.89	0.18
0.5	$4.307 \cdot 10^{-5}$	$5.766 \cdot 10^{-4}$	3.9	0.51	2.00	1.19	$3.933 \cdot 10^{-3}$	0.90	0.20

Here the absence of the value for the parameter a means that the upper bound is taken from the previous cases using (4).

Acknowledgments

I am indebted to V. V. Kalashnikov for his encouragement and valuable comments during the preparation of this work. I am also thankful to H. Schmidli and to the Laboratory of Applied Mathematics at the Technical University of Crete for their support.

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