A CLASS OF HEAVY TAILED DISTRIBUTIONS

D. G. KONSTANTINIDES

ABSTRACT. In this paper we present a class of heavy tailed distributions which provide simple asymptotes for the ruin probability in the classical risk model under a constant interest force. We examine the properties of this class in comparison with the standard subexponential distributions.

Key words and phrases: Constant interest force; Ruin probability; Subexponential distribution.

1. MOTIVATION.

We cosider a sequence of i.i.d. non-negative ramdom variables $(Z_k)_{k\geq 1}$(claim sizes) with a common distrubution function $B(x) \equiv 1-\overline{B}(x) = P(Z_1 \leq x)$, $x \geq 0$, and a finite expectation b. Let us introduce the integrated tail distribution, denoted by

$$F(x) = \frac{1}{b}\int_0^x \overline{B}(z)\,dz, \quad x \geq 0\,. \tag{1.1}$$

Definition 1.1. *A distribution F on $[0,\infty)$ is said to belong to the class $\mathcal{E}$ if for some $v > 1$*

$$\overline{F}^*(v) \equiv \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1. \tag{1.2}$$

According to this definition, we can find the following property of the class $\mathcal{E}$:

Lemma 1.2. *Let $F \in \mathcal{E}$, then there exists some $\alpha > 0$ and $c > 0$ such that*

$$\overline{F}(x) \leq cx^{-\alpha} \tag{1.3}$$

holds for all large $x > 0$.

Proof. For the fixed $v > 1$ taken from (1.2), we write

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = y.$$

Hence, there exists a small enough $\varepsilon > 0$ such that $y + \varepsilon < 1$ and there exists a $x_0 = x_0(v, y, \varepsilon) > 0$ such that for any $x \geq x_0$, it holds

$$\frac{\overline{F}(vx)}{\overline{F}(x)} \leq y + \varepsilon < 1. \tag{1.4}$$

Now we start to prove the relation (1.3). For $x \geq x_0$, we write

$$n = \left[\frac{\log(x/x_0)}{\log v}\right],$$

where $[a]$ denotes the integer part of the number a. After several applications of the inequality (1.4), we find:

$$\begin{aligned} \overline{F}(x) &\leq (y+\varepsilon)\overline{F}(x/v) \leq \cdots \leq (y+\varepsilon)^n \overline{F}(x/v^n) \leq \\ &\leq \exp\left\{\frac{\log(y+\varepsilon)}{\log v} \log x/x_0\right\} \overline{F}(x_0). \end{aligned}$$

Hence (1.3) follows with

$$c = \exp\left\{-\frac{\log(y+\varepsilon)}{\log v}\log x_0\right\}\overline{F}(x_0) > 0, \qquad \alpha = -\frac{\log(y+\varepsilon)}{\log v} > 0.$$

This ends the proof. □

We can see that this class is similar with the class $\mathcal{D}$ of distributions with doninatedly varying tails. A distribution F on $[0,\infty)$ belongs to $\mathcal{D}$ if for some $v \in (0,1)$

$$\overline{F}^*(v) < \infty. \tag{1.5}$$

or equivalently if for some $v > 1$

$$\overline{F}_*(v) \equiv \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} > 0. \tag{1.6}$$

Lemma 1.3. *Let $F \in \mathcal{D}$, then there exists some $\beta > 0$ and $d > 0$ such that*

$$\overline{F}(x) \geq dx^{-\beta} \tag{1.7}$$

holds for all large $x > 0$.

Proof. For the fixed $v > 1$ in (1.6), we write

$$\liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = z.$$

Hence, for a small $\varepsilon > 0$ such that $z - \varepsilon > 0$, it holds for all large x, say $x \geq x_0 = x_0(v,z,\varepsilon) > 0$, that

$$1 > \frac{\overline{F}(vx)}{\overline{F}(x)} \geq z - \varepsilon > 0. \tag{1.8}$$

Now we aim to prove the result in (1.7). For a number $x \geq x_0$, we write

$$n = \left[\frac{\log x/x_0}{\log v} + 1\right],$$

One easily checks the following steps by (1.8):

$$\begin{aligned} \overline{F}(x) &\geq (z-\varepsilon)\,\overline{F}(x/v) \geq \cdots \geq (z-\varepsilon)^n \overline{F}(x/v^n) \geq \\ &\geq \exp\left\{\frac{\log(z-\varepsilon)}{\log v} \log x/x_0\right\} \overline{F}(x_0). \end{aligned}$$

Hence (1.7) follows with

$$d = \exp\left\{-\frac{\log(z-\varepsilon)}{\log v}\log x_0\right\}\overline{F}(x_0) > 0, \qquad \beta = -\frac{\log(z-\varepsilon)}{\log v} > 0.$$

The proof is over. □

Let us remark that neither $\mathcal{E}$ nor $\mathcal{D}$ are restricted in the frame of the heavy tailed distributions. For example we remind that the distribution F on $[0,\infty)$ belongs to the class $\mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2, \tag{1.9}$$

where F^{2*} denotes the convolution of F with itself. It well known that both $\mathcal{E}$ and $\mathcal{D}$ are not covered by the class $\mathcal{S}$. However on the subexponential part of these two classes is concetrated the interest of the risk theory. Let us introduce the notation

$$\mathcal{A} = \mathcal{E} \cap \mathcal{S}.$$

Now we return to the fact that the distribution F is the integrated tail distribution corresponding to a claim distribution B. A distribution B on $[0,\infty)$ is said to belong to the class $\mathcal{S}^*$ if it has a finite expectation b and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{B}(x-z)}{\overline{B}(x)} \overline{B}(z) dz = 2b.$$

In this case we know that $B \in \mathcal{S}$ and $F \in \mathcal{S}$.

Definition 1.4. *A distribution B on $[0,\infty)$ is said to belong to the class $\mathcal{A}^*$ if it has a finite expectation b and its e.d.f.*

$$F(x) = \frac{1}{b} \int_0^x \overline{B}(z)\, dz, \quad x \geq 0,$$

belongs to $\mathcal{A}$.

In this paper we concetrate our attention on the the classes $\mathcal{A}$ and $\mathcal{A}^*$. The initial motivation of this investigation comes from an issue arised in risk theory. Namely a risk model was considered, where the claim arrival times constitute a homogeneous Poisson process $(N(t))_{t\geq 0}$, which is independent of $(Z_k)_{k\geq 1}$ and has an intensity $\lambda > 0$. Therefore, the compound Poisson process $X(t) = \sum_{k=1}^{N(t)} Z_k$ is representing in actuarial context the total claim amount accumulated up to time $t \geq 0$, with $X(t) = 0$ when $N(t) = 0$. Let us denote by c the constant gross premium rate and assume that there exists a constant interest force $r > 0$ which affects the risk process. With $u > 0$ as initial surplus of the insurance company, the total surplus $U_r(t)$ up to time t, is given as follows

$$U_r(t) = ue^{rt} + c\int_0^t e^{rz} dz - \int_0^t e^{r(t-z)}\, dX(z), \quad t \geq 0.$$

Let us define the ruin probability

$$\psi_r(u) = \mathbf{P}\left(\inf_{t\geq 0} U_r(t) < 0 \mid U_r(0) = u\right), \quad u \geq 0.$$

So the class $\mathcal{A}$ appears in Konstantinides et al. (2002) as the space of distributions, where the following asymtpotics for $\psi_r(u)$ holds

$$\psi_r(u) \sim \frac{\lambda}{r} \int_u^\infty \overline{B}(z)\frac{dz}{z}, \quad u \to \infty. \tag{1.10}$$

Simultaneously this issue was solved with the help of a local limit theory presented in the papers Asmussen-Kalashnikov et al. (2002) and Asmussen-Korshunov et al. (2002) in the frame of the class $\mathcal{S}^*$. Thus arises the question of comparison of the two classes and their applications.

The paper is organized as follows. In the next section we collect some properties of the classes $\mathcal{E}$ and $\mathcal{D}$. In the third section we study the classes $\mathcal{A}$ and $\mathcal{D}\cap\mathcal{S}$. In the forth section we examine inclusion criteria for the classes $\mathcal{A}^*$ and $\mathcal{S}^*$.

2. Comparison of the classes $\mathcal{E}$ and $\mathcal{D}$.

We write

$$M_1 = \liminf_{x\to\infty} xq(x) \quad \text{and} \quad M_2 = \limsup_{x\to\infty} xq(x)$$

From Konstantinides et al. (2002) we have the following result:

Lemma 2.1. *Let F be a d.f. supported on $[0,\infty)$ with a density function $f(x)$ which is eventually non-increasing. Then the following statements are equivalent:*

I_1. (1.2) holds for some $v > 1$;

I_2. (1.2) holds for any $v > 1$;

I_3. the hazard rate function of F, $q(x) = f(x)/\overline{F}(x)$, satisfies

$$M_1 = \liminf_{x\to\infty} xq(x) > 0. \tag{2.1}$$

Corollary 2.2. *1. $F \in \mathcal{A}$, if*

$$0 < M_1 \le M_2 < \infty.$$

2. $F \in \mathcal{S} \setminus \mathcal{E}$, if

$$0 = M_1 \le M_2 < \infty.$$

3. $F \in \mathcal{E} \setminus \mathcal{S}$, if

$$0 < M_1 \le M_2 = \infty.$$

4. $F \notin \mathcal{E} \cup \mathcal{S}$, if

$$0 = M_1 < M_2 = \infty.$$

Remark 2.3. *Clearly, if $\overline{F}^*(v) = 1$ for some $v > 1$ then it holds for all $v > 1$. Furthermore, we can obtain that*

$$\overline{F}_*(v) < 1 \;\; \exists v > 1 \iff \overline{F}_*(v) < 1 \;\; \forall v > 1 \iff \limsup_{x\to\infty} xq(x) > 0.$$

So we have that if $\overline{F}_*(v) = 1$ *for some* $v > 1$ *then it holds for all* $v > 1$. *From these discussions we can classify all possibilities of the values of* $\overline{F}_*(v)$ *and* $\overline{F}^*(v)$ *into three cases:*

1. $\overline{F}_*(v) = \overline{F}^*(v) = 1 \quad \forall v > 1$;
2. $\overline{F}_*(v) < 1$ *but* $\overline{F}^*(v) = 1 \quad \forall v > 1$;
3. $\overline{F}^*(v) < 1 \quad \forall v > 1$.

The first case indicates that $\overline{F}(x)$ *is slowly varying as* $x \to \infty$. *The third case is just the fundamental assumption of the present paper.*

In order to describe the relation between the two classes $\mathcal{D}$ and $\mathcal{E}$, let see the following obvious result:

Corollary 2.4. *1.* $F \in \mathcal{D} \cap \mathcal{E}$, *if and only if*

$$0 < \overline{F}_*(v) \le \overline{F}^*(v) < 1, \qquad \forall v > 1.$$

2. $F \in \mathcal{D} \setminus \mathcal{E}$, *if and only if*

$$0 < \overline{F}_*(v) \le \overline{F}^*(v) = 1, \qquad \forall v > 1.$$

3. $F \in \mathcal{E} \setminus \mathcal{D}$, *if and only if*

$$0 = \overline{F}_*(v) \le \overline{F}^*(v) < 1, \qquad \forall v > 1.$$

4. $F \notin \mathcal{E} \cup \mathcal{D}$, *if and only if*

$$0 = \overline{F}_*(v) < \overline{F}^*(v) = 1, \qquad \forall v > 1.$$

Let us now remind the concept of the lower and upper Matuszewska indices. The upper and lower Matuszewska index J_F^+ and J_F^- are defined as follows

$$J_F^+ = J^+(X) = \inf\left\{-\frac{\log \overline{F}_*(v)}{\log v} : v > 1\right\} = -\lim_{v\to\infty} \frac{\log \overline{F}_*(v)}{\log v}, \tag{2.2}$$

$$J_F^- = J^-(X) = \sup\left\{-\frac{\log \overline{F}^*(v)}{\log v} : v > 1\right\} = -\lim_{\lambda\to\infty} \frac{\log \overline{F}^*(v)}{\log v}. \tag{2.3}$$

In the terminology of Bingham et al. (1987), here the quantities J_F^+ and J_F^- are the upper and lower Matuszewska indices of the non-negative and non-decreasing function $f(x) = \left(\overline{F}(x)\right)^{-1}$, $x \ge 0$. The latter equalities in (2.3) and (2.2) are due to Theorem 2.1.5 in Bingham et al. (1987). Without any confusion we simply call the $J_F^{\pm}$ as the upper/lower Matuszewska index of the d.f. F. For more details of the Matuszewska indices, see Chapter 2.1 of Bingham *et al.* (1987), Cline & Samorodnitsky (1994).

By the definitions of the Matuszewska indices and the classes $\mathcal{D}$ and $\mathcal{E}$, we immediately obtain the following result, which clearly illustrate the symmetrical positions of the classes $\mathcal{D}$ and $\mathcal{E}$:

Lemma 2.5. *$F \in \mathcal{E}$ if and only if $J_F^- > 0$; $F \in \mathcal{D}$ if and only if $J_F^+ < \infty$;*

Proof. From the assumption,

$$-\lim_{v\to\infty} \frac{\ln \overline{F}^*(v)}{\ln v} > 0,$$

and for v large enough, it follows that

$$\ln \left[\overline{F}^*(v)\right] < 0,$$

which means that there exists a finite $v > 1$ for $F \in \mathcal{E}$. □

Remark 2.6. *It is worth to recall that if the upper Matuszewska index J_F^+ is finite then $F \in \mathcal{D}$. This indicates the symmetry with respect to the Matuszewska indices between the classes $\mathcal{E}$ and $\mathcal{D}$.*

3. The class $\mathcal{A}$.

Let us point out that the class $\mathcal{A}$ covers most of the well-known subexponential d.f.'s. Indeed, by the definition one easily checks:

Remark 3.1. *The distributions Pareto, Lognormal, Weibull, Loggamma, Burr, Benktander I and II are members of the class $\mathcal{A}$.*

Furthermore, according to Pitman (1980) Th.II , we easily obtain the following criteria:

Corollary 3.2. *Suppose that the hazard rate function $q(x) = Q'(x)$ exists and eventually decrease to 0. We have*

1. $F \in \mathcal{A}$ if and only if

$$\lim_{x\to\infty} \int_0^x \exp\{yq(y)\} f(y)dy = 1, \tag{3.1}$$

and

$$\liminf_{x\to\infty} xq(x) > 0. \tag{3.2}$$

2. If

$$\int_0^\infty \exp\{yq(y)\} f(y)dy < \infty, \tag{3.3}$$

and (3.2) hold, then $F \in \mathcal{A}$.

Theorem 3.3. *Let F be a d.f. supported on $[0,\infty)$ with an eventually decreasing density function $f(x)$. If for some $v > 1$*

$$0 < \overline{F}_*(v) \leq \overline{F}^*(v) < 1, \tag{3.4}$$

then $F \in \mathcal{A} \cap \mathcal{D}$.

Proof. It suffices to prove $F \in \mathcal{S}$. Noting that the left-hand side of inequality (3.4) is just the definition of $F \in \mathcal{D}$, we obtain that $F \in \mathcal{D} \cap \mathcal{L} \subset \mathcal{S}$ (see Klüppelberg (1988) Cor. 3.4). □

Let us look for some easily verifiable conditions for the inclusion in $\mathcal{A}$. We formulate them in terms of the hazard function $Q = -\ln \overline{F}$ and its derivative q.

Definition 3.4. *A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{I}$ of intermediate varying tails, if*

$$\lim_{v \downarrow 1} \overline{F}_* (v) = 1.$$

In the following example appears that (3.3) is not a sufficient condition for $F \in \mathcal{A}$.

Example 3.5. *Let F be a d.f. with a density function that*

$$f(x) = \begin{cases} x^{-1} \ln^{-2} x, & x > e, \\ 0, & x \le e. \end{cases}$$

Clearly, for $y > e$, $\overline{F}(y) = \int_y^\infty x^{-1} \ln^{-2} x dx = \ln^{-1} y$. On one hand, we have

$$\begin{aligned} \int_0^\infty \exp\{yq(y)\} f(y) dy &= \int_e^\infty \exp\left\{ \frac{\ln^{-2} y}{\ln^{-1} y} \right\} y^{-1} \ln^{-2} y dy \\ &\le \int_e^\infty \exp\left\{ \ln^{-1} e \right\} \cdot y^{-1} \ln^{-2} y dy = e. \end{aligned}$$

So condition (3.3) is fulfilled. On the other hand, it is clear that $F \notin \mathcal{A}$ since, for any $v > 0$,

$$\lim_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 1.$$

Proposition 3.6. *Let F and G be two d.f.'s supported on $[0, \infty)$ and $F \in \mathcal{A}$. We have $G \in \mathcal{A}$ if one of the following conditions holds:*

1. $\overline{G}(x) \sim C\overline{F}(x)$ for some $C > 0$;

2. $\overline{G}(x) \sim \sum_{k=0}^\infty p_k \overline{F^{*k}}(x)$, where, $\{p_k, k \ge 0\}$ is a sequence of non-negative numbers satisfying that $0 < \sum_{k=0}^\infty p_k (1 + \varepsilon)^k < \infty$ for some $\varepsilon > 0$.

Proof. *1.*By a similar approach as used in the proof of Theorem 3 in Teugels (1975) we obtain that if $F \in \mathcal{S}$ then $G \in \mathcal{S}$. The remaining proof of $G \in \mathcal{A}$ is trivial.

2. It is well-known that, if $F \in \mathcal{S}$ then $G \in \mathcal{S}$ and $\overline{G}(x) \sim \overline{F}(x) \sum_{k=0}^\infty k p_k$; see for example Lemma 1 in Chover et al. (1973) and Theorem 2.13 in Cline (1987). Then from the first part we obtain the assertion $G \in \mathcal{A}$. □

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{L}$ (long-tailed d.f.'s) if

$$\lim_{x \to \infty} \frac{\overline{F}(x + z)}{\overline{F}(x)} = 1$$

for some $z > 0$.

Theorem 3.7. *Let F be an absolutely continuous d.f. supported on $[0,\infty)$. We have $F \in \mathcal{A}$ if one of the following conditions holds:*

1. $0 < \liminf_{x\to\infty} xq(x) \le \limsup_{x\to\infty} xq(x) < \infty$*;*

2. $q \in \mathcal{R}_0$ *and eventually decreases to* 0 *as* $x \to \infty$*, and* $Q(x) - xq(x) \in \mathcal{R}_{-1}$*;*

3. $q \in \mathcal{R}_{-\gamma}$ *for* $\gamma \in (0,1)$*;*

4. q *is eventually decreasing, there exist some* $\delta \in (0,1)$ *and* $v \in (1,\infty)$ *such that* $Q(xz) \le z^{\delta}Q(x)$ *for all* $x \ge v$*,* $z \ge 1$*, and* $\liminf_{x\to\infty} xq(x) \ge (2-2^{\delta})^{-1}$*.*

Proof. From Proposition 3.8 and Corollary 3.9 in Goldie and Klüppelberg (1998) we take that $F \in \mathcal{S}^* \subset \mathcal{S}$. Furthermore, anyone of the four conditions above implies (2.1). Therefore by Lemma 3.7 we obtain that $F \in \mathcal{A}$. □

The following example comes from Su and Tang (2003). It shows that there exist suitable subexponential d.f.'s which satisfy the request in the case 2 on page 453 of Konstantinides et al. (2002).

Example 3.8. *Let X be a r.v. distributed by*

$$p_n = P(X = 2^{n^\alpha}) = c_0 n^{-\beta} 2^{-n^\alpha}, \qquad n \ge 0,$$

where $\alpha > 1$*,* $\beta > 1$ *and* $c_0 > 0$ *is such that* $\sum_{n=1}^{\infty} p_n = 1$*. Then the e.d.f. F of the r.v. X satisfies*

$$F \in \mathcal{S}, \qquad F \notin \mathcal{A}.$$

Proof. Clearly, for any large enough $x > 0$ and some $n(x) \ge 0$ such that $2^{(n-1)^\alpha} \le x < 2^{n^\alpha}$, $\overline{B}(x) = P(X \ge 2^{n^\alpha}) \sim P(X = 2^{n^\alpha})$. Su and Tang (2003). proved that $B \in \mathcal{M}^*$, that means

$$\limsup_{x\to\infty} \frac{x\overline{B}(x)}{b\overline{F}(x)} < \infty,$$

hence $F \in \mathcal{S}$ follows based on the discussions there. The proof of the following assertion is straightforward: for any $v > 1$,

$$\overline{F}_*(v) = 0, \qquad \overline{F}^*(v) = 1. \tag{3.5}$$

It yields that $F \notin \mathcal{A} \cup \mathcal{D}$. □

4. Inclusion Criteria for the Classes $\mathcal{A}^*$ and $\mathcal{S}^*$.

Example 4.1. *Let τ be a geometric r.v. $P(\tau = n) = (1-q)q^n$, $0 < q < 1$ and $n \ge 0$. Then, for arbitrarily fixed v, $1 < v < 1/q$, the d.f. $B = B_v$ of r.v. $Z = v^\tau$ satisfies $B \in \mathcal{A}^*$ but $B \notin \mathcal{S}^*$.*

Proof. Clearly, the d.f. B has a finite expectation. Further, we have that d.f. $B = B_v$ of r.v. $Z = v^\tau$ satisfies $B \in \mathcal{A}^*$ but $B \notin \mathcal{S}^*$.

$$\lim_{x\to\infty} \frac{\overline{B}(vx)}{\overline{B}(x)} = \lim_{x\to\infty} \frac{P(v^\tau > vx)}{P(v^\tau > x)} = \lim_{x\to\infty} \frac{P(\tau > \log x/\log v + 1)}{P(\tau > \log x/\log v)} = \lim_{x\to\infty} \frac{P(\tau > x+1)}{P(\tau > x)} = q.$$

From this we get to know: 1. $B \in \mathcal{D}$ and therefore its e.d.f. $F = F_v \in \mathcal{S}$ (see Embrechts and Omey (1984)); 2. $B \notin \mathcal{L}$ and therefore $B \notin \mathcal{S}^*$ (recall Theorem 3.2(b) in Klüppelberg (1988)); 3. for any fixed $\varepsilon > 0$ and all large $x > 0$,

$$(1-\varepsilon)q\overline{B}(x) \leq \overline{B}(vx) \leq (1+\varepsilon)q\overline{B}(x). \tag{4.1}$$

Integration on (4.1) from x to ∞ yields $(1-\varepsilon)qv\overline{F}(x) \leq \overline{F}(vx) \leq (1+\varepsilon)qv\overline{F}(x)$, which implies that

$$\lim_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = qv < 1,$$

i.e. (1.2) holds. Hence $B \in \mathcal{A}^*$. □

We provide now some verifiable conditions for $B \in \mathcal{A}^*$.

Theorem 4.2. *Let B be a d.f. supported on $[0,\infty)$. If*

$$0 < \liminf_{x\to\infty} \frac{x\overline{B}(x)}{\int_x^\infty \overline{B}(z)dz} \leq \limsup_{x\to\infty} \frac{x\overline{B}(x)}{\int_x^\infty \overline{B}(z)dz} < \infty, \tag{4.2}$$

then $B \in \mathcal{A}^*$.

Proof. By Lemma 3.7, the left-hand side of (4.2) guarantees the condition (1.2) for some $v > 1$. Meanwhile, by Cor. 3.4 in Klüppelberg (1988) the right-hand side of (4.2) implies that $F \in \mathcal{S}$. This proves that $F \in \mathcal{A}$. □

Corollary 4.3. *For any $1 < \alpha \leq \beta < \infty$, $ERV(-\alpha,-\beta) \subset \mathcal{A}^*$.*

Proof. Let $B \in ERV(-\alpha,\ -\beta)$. From the definition of $ERV(-\alpha,\ -\beta)$, it follows that for any $\varepsilon > 0$ and all sufficiently large $x > 0$ that $\overline{B}(2x) \leq (1+\varepsilon)2^{-\alpha}\overline{B}(x)$. We choose $0 < \varepsilon < 2^{\alpha-1} - 1$ in the following inequalities:

$$\begin{aligned}\int_x^\infty \overline{B}(z)dz &= \sum_{k=0}^\infty \int_{2^k x}^{2^{k+1}x} \overline{B}(z)dz \leq \sum_{k=0}^\infty 2^k x \overline{B}(2^k x) \\ &\leq \sum_{k=0}^\infty 2^k x \left((1+\varepsilon)2^{-\alpha}\right)^k \overline{B}(x) = Cx\overline{B}(x),\end{aligned}$$

where the constant C satisfies

$$C = \sum_{k=0}^\infty 2^k \left((1+\varepsilon)2^{-\alpha}\right)^k = \sum_{k=0}^\infty \left((1+\varepsilon)2^{1-\alpha}\right)^k < \infty.$$

It follows that, for all sufficiently large $x > 0$,

$$0 < \frac{1}{C} \leq \frac{x\overline{B}(x)}{\int_x^\infty \overline{B}(z)dz} \leq \frac{x\overline{B}(x)}{\int_x^{2x} \overline{B}(z)dz} \leq \frac{\overline{B}(x)}{\overline{B}(2x)} \leq (1+\varepsilon)2^\beta < \infty,$$

i.e. the condition (4.2) holds. Thus, by Theorem 4.2 we obtain $B \in \mathcal{A}^*$. □

Remark 4.4. *From Corollary 4.3 we see that the asymptotics (1.10) holds if the d.f. B of the claim size belongs to the class $ERV(-\alpha,\ -\beta)$ for some $1 < \alpha \leq \beta < \infty$. This improves the main results in Klüppelberg & Stadtmüller (1998) since the corresponding assumption in that paper is $B \in \mathcal{R}_{-\gamma}$ for some $\gamma > 1$.*

Let us consider now the criteria based on the hazard rate function. According to Theorem 3.6 in Klüppelberg (1988) and Lemma 3.3 in Konstantinides et al. (2002), we obtain immediately

Theorem 4.5. *Suppose that the hazard rate function $q_B = Q'_B$ exists and eventually decrease to* 0*. We have*

1. $B \in \mathcal{A}^*$ if and only if

$$\lim_{x\to\infty} \int_0^x \exp\{yq_B(x)\}\overline{B}(y)dy = \mu < \infty, \tag{4.3}$$

and

$$\liminf_{x\to\infty} xq_F(x) > 0. \tag{4.4}$$

2. If (4.4) and

$$\int_0^\infty \exp\{yq_B(y)\}\overline{B}(y)dy < \infty \tag{4.5}$$

hold, then $B \in \mathcal{A}^*$.

Clearly, condition (4.4) can be implied by

$$\liminf_{x\to\infty} xq_B(x) > 0. \tag{4.6}$$

References

[1] Asmussen, S., Kalashnikov, V., Klüppelberg, C., Konstantinides, D., Tsitsiashvili, G.Sh. (2002) A local limit theorem for random walk maxima with heavy-tails. *Statistics and Probability Letters.* **56**, 399–404.

[2] Asmussen, S., Korshunov, D., Foss (2002) Asymptotics for sums of random variables with local subexponential behaviour. *Journal of Theoretical Probability.*

[3] Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987) *Regular Variation.* Cambridge University Press, Cambridge.

[4] Chover, J., Ney, P., Wainger, S. (1973) Functions of Probability Measures. *J. Analyse Math.* **26**, 255–302.

[5] Cline, D.B.H. (1987) Convolutions of Distributions with Exponential and Subexponential Tails. *J. Austral. Math. Soc. Ser. A* **43**, 347–365.

[6] Cline, D.B.H., Samorodnitsky, G. (1994) Subexponentiality of the product of independent random variables. *Stochastic Process. Appl.* **49**, no. **1**, 75–98.

[7] Embrechts, P., Omey, E. (1984) A Property of Longtailed Distributions. *J. Appl. Probab.* **21**, 80–87.

[8] Goldie, C.M., Klüppelberg, C. (1998) Subexponential Distributions. In: Adler, R., Feldman, R. and Taqqu, M.S. (Eds.) *A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy-Tailed Distributions.* 435–462, Birkhäuser, Boston.

[9] Konstantinides, D.G., Tang, Q.H., Tsitsiashvili, G.Sh. (2002) Estimates for the Ruin Probability in the Classical Risk Model with Constant Interest Force in the Presence of Heavy Tails. *Insurance: Mathematics and Economics.* **31,** no. 3, 447–460.

[10] Klüppelberg, C. (1988) Subexponential distributions and integrated tails. *J. Appl. Probab.* **25**, no. 1, 132–141.

[11] Klüppelberg, C., Stadtmüller, U. (1998) Ruin probabilities in the presence of heavy-tails and interest rates. *Scandinavian Actuarial Journal*, 49–58.

[12] Su, C., Tang, Q.H. (2003) Characterizations on heavy-tailed distributions by means of hazard rate. *Acta. Math. Appl. Sinica*, to appear.

[13] Teugels, J.L. (1975) The Class of Subexponential Distributions. *Ann. Probab.* **3**, 1001–1011.

Department of Statistics and Actuarial Science, University of the Aegean, Karlovassi, GR-83 200 Samos, Greece
E-mail address: konstant@aegean.gr