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Extremal Subexponentiality in Ruin Probabilities

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In this article, we consider risk models with a heavy-tailed parametric claim distribution from the subexponential class $\mathscr{S}$ with at least two parameters. We choose the proper convergence of a parameter, such that the tail of the claims distribution becomes heavier, and then we study the limit behavior of the ruin probability. Further, we employ an appropriate functional normalization in order to keep intact the distributional properties.

Keywords Ergodicity condition; Explosive behavior; Heavy-tailed distributions; Heavytailedness parameters.

Mathematics Subject Classification Primary 91B30; Secondary 91B70.

1. Introduction

The risk models provide tools for handling problems in the insurance business and can be evaluated with respect to this aspect as more or less adequate. In order to evaluate the accuracy of the risk models, we check their limit behavior.

Here and henceforth, the symbol $\sim$ means that the quotient of both sides tends to 1. Clearly, for two positive functions $f(\cdot)$ and $g(\cdot)$, the relation $f(x) \sim g(x)$ amounts to the conjunction of the relations $\limsup f(x)/g(x) \leq 1$ and $\liminf f(x)/g(x) \geq 1$, which are denoted as $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$, respectively. For two distributions F_1 and F_2 on $[0, \infty)$, denote by $F_1 * F_2$ their convolution. Furthermore, we write $F^{1*} = F$ and $F^{n*} = F^{(n-1)*} * F$ for every $n = 2, 3, \ldots$.

A distribution $F(x) = 1 - \overline{F}(x)$ on $[0, \infty)$ is said to belong to the class $\mathscr{S}$ if the relation

$$\lim_{x \to \infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2,$$

holds.

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In this article, some parametric models in the subexponential class of distributions $\mathscr{S}$ are considered. The relation between the parameters is chosen appropriately, in such a way that the mean claim remains fixed but the other parameters change so that the distribution of the claims reaches extremal subexponential behavior. Considering the possibility to change the parameters and especially through an appropriate convergence of the parameters, we find that the tail of the claim size distribution becomes heaviest and then we explore the effect of extreme heavytailedness. As far we stay in the frame of the subexponential class of claim distributions, we speak about extreme subexponentiality. We concentrate on the ruin probability in the context of the classical risk model.

We consider random claim occurrences modeled as homogeneous Poisson process $\{N(t), t \geq 0\}$ with intensity $\lambda > 0$, independent of the claim sizes Z_k, $k = 1, 2, \ldots$ from distribution $B(x) = \mathbf{P}[Z \leq x]$ and tail $\overline{B}(x) = 1 - B(x)$. Let us assume that, there exist its density function $b(x)$ and its mean $b_1 = \mathbf{E}[Z] < \infty$. We denote the expected claim per time unit by $\rho = \lambda b_1$ and the total claim amount up to time t by

$$X(t) = \sum_{k=1}^{N(t)} Z_k.$$

We also need the notation

$$\overline{F}(x) = \frac{1}{b_1} \int_0^x \overline{B}(z) dz \tag{1.1}$$

for the integrated tail of the claim size distribution. The income for the insurance business comes from the premium received with constant rate equal to $c > 0$. Then the ruin probability can be written in the form

$$\psi(x, \infty) = \mathbf{P}\Big(\max_{s \geq 0} [X(s) - cs] > x \Big).$$

In the classical risk model, the starting point for the calculation of the ruin probability is the Pollachek-Khinchin formula (see, e.g., Asmussen, 2000, Eq. III(2.1))

$$\psi(x, \infty) = \left(1 - \frac{\rho}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\rho}{c}\right)^n \overline{F^{n*}}(x). \tag{1.2}$$

If $F \in \mathscr{S}$, the following alternative asymptotic expression was suggested in Embrechts and Veraverbeke (1982):

$$\psi(x, \infty) \sim \frac{\rho}{c - \rho} \overline{B}_I(x) = \frac{\rho}{(c - \rho) b_1} \int_x^{\infty} \overline{F}(y) dy,$$

as $x \to \infty$.

In insurance and finance we often deal with large claims that can be described by heavy-tailed distributions. The asymptotic behavior of the ruin probability represents only an approximation of its real value. Numerical calculations can show that the accuracy of these asymptotic formulas can be quite diverse, especially in the range that is relevant to the practical purposes (see, for example, Kalashnikov, 1997; Konstantinidis, 1999).

It is worth mentioning that heavy-tailed distributions play an important and increasing role during the last years because of occasional appearance of huge claims. The key-idea consists in proposing other approximations that work in the area of practically significant values of the corresponding parameters. To this end, the classification of the distributions describing large claims could be further developed.

Let us start with the following five parametric families of distributions, widely used in insurance mathematics.

Example 1.1. The Pareto distribution:

$$\overline{B}(x) = \left(\frac{\kappa}{\kappa + x}\right)^{\alpha} \mathbf{1}_{[x>0]},$$

with $\alpha > 1$, $\kappa > 0$. The mean takes the form $b_1 = \frac{\kappa}{\alpha-1}$ for any $\alpha > 1$.

Example 1.2. The Lognormal distribution:

$$b(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\},$$

with parameters $-\infty < \mu < \infty$ and $\sigma^2 > 0$. The mean takes the form $b_1 = e^{\mu+\sigma^2/2}$.

Example 1.3. The Weibull distribution:

$$\overline{B}(x) = e^{-cx^{\tau}},$$

with parameters $c > 0$, $0 < \tau < 1$ and the mean is $b_1 = \frac{\Gamma(1+1/\tau)}{c^{1/\tau}}$, where Γ denotes the Gamma function.

Example 1.4. The Loggamma distribution with density function:

$$b(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)}[\ln(1 + x)]^{\beta-1}(1 + x)^{-\alpha-1},$$

with parameters $\alpha > 0$, $\beta > 0$ and mean $b_1 = \left(\frac{\alpha}{\alpha-1}\right)^{\beta} - 1$.

Example 1.5. The Burr distribution:

$$\overline{B}(x) = \left(\frac{\kappa}{\kappa + x^{\tau}}\right)^{\alpha},$$

with parameters $\kappa > 0$, $\tau > 0$, $\alpha > 1/\tau$ and the mean equals to $b_1 = \frac{\kappa^{1/\tau}}{\Gamma(\alpha)}\Gamma(1 + 1/\tau)\Gamma(\alpha - 1/\tau)$ for $\alpha > 1/\tau$.

2. The Heuristics

Firstly, we treat the Pareto distribution, in which the parameter α can be considered as the parameter indicating the heavytailedness. However, if the other parameter k

remains fixed and α tends to its minimal value 1, then the mean claim size grows to infinity. Therefore, the integrated tail claim distribution from (1.1) is meaningless and the Pollaczeck–Khinchine formula does not work. Hence, in order to keep a balance between the two parameters during the convergence of α, we require the safety loading to be fixed. This means that the mean claim $b_1 = \kappa/(\alpha - 1)$ remains constant. Therefore, we introduce a normalization

$$b_1 = 1, \tag{2.1}$$

which will remain the same for the rest of the examples listed above. Thus, we obtain $\rho = \lambda$. Relation (2.1) implies the following value for the second parameter κ

$$\kappa = \alpha - 1. \tag{2.2}$$

The most heavy tail of the claim distribution appears when the heavytailedness parameter α decreases to 1. It should be kept in mind that this kind of ergodic control is not unique (see Kalashnikov, 1997).

We repeat these steps in the case of Lognormal distribution. It follows that the normalization (2.1) implies

$$\mu = -\frac{\sigma^2}{2},$$

and that the value of the heavytailedness parameter σ tends to infinity $\sigma \longrightarrow \infty$, caused by the need to identify the superheavy tailed distribution.

In the next example of the Weibull distribution, this pattern of normalization (2.1) renders

$$c = \left[\Gamma\left(1 + \frac{1}{\tau}\right)\right]^{\tau}.$$

The most heavy-tailed distribution comes when the heavytailedness parameter τ tends to 0.

Further, when we consider the Loggamma distribution according to the normalization procedure (2.1), we take

$$\alpha = \frac{2^{1/\beta}}{2^{1/\beta} - 1}.$$

Now, the heavytailedness parameter β tends to 0.

Finally, in the case of the Burr distribution the normalization (2.1) leads to

$$\kappa = \left(\frac{\tau\Gamma(\alpha)}{\Gamma(1/\tau)\Gamma(\alpha - 1/\tau)}\right)^{\tau}.$$

When τ is fixed, the parameter of heavytailedness α tends to $1/\tau$.

Remark 2.1. If $B(x)$ belongs to the Pareto, Lognormal, Weibull, Loggamma, or Burr distribution family, then for any $x > 0$ its tail tends to zero

$$\overline{B}(x) \to 0, \tag{2.3}$$

as the heavytailedness parameter tends to its limit ($\alpha \to 1$, $\sigma \to \infty$, $\tau \to 0$, $\beta \to 0$, $\alpha \to 1/\tau$, respectively). This effect of the convergence of the parameter of heavytailedness to its extremal value can be interpreted in the following way. As the tail becomes heavier, its contour becomes parallel to the axis. This fact together with the finite mean gives the constant zero value for the tail, which contradicts with the typical decreasing form for the distribution tail. This limit deformation can be understood as the "explosion" of the tail.

Indeed, for the Pareto distribution family, after substitution from (2.2) we obtain

$$\overline{B}_\alpha(x) \sim \frac{(\alpha-1)^\alpha}{x^\alpha} \to 0,$$

as $x \to \infty$ for the asymptotic equivalence ($\sim$) and $\alpha \to 1$ for the limit ($\to$).

Similarly, in case of the Lognormal distribution family, for any $x > 0$ holds

$$\overline{B}_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln x}{\sigma}+\frac{\sigma}{2}}^{\infty} \exp[-y^2/2]dy \to 0,$$

as $\sigma \to \infty$.

For the Weibull distribution family by using Stirling's formula it can be found that

$$cx^\tau = \left[x\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau \sim (\sqrt{2\pi})^\tau \left(\frac{1}{\tau}\right)^{\tau/2} \frac{x^\tau}{\tau e} \to \infty,$$

as $\tau \to 0$, whence limit (2.3) follows immediately.

For the Loggamma distribution family for any $\varepsilon \in (0, x)$, the following sequence of relations is true:

$$\overline{B}_\beta(x) = \frac{1}{\Gamma(\beta)} \int_{A_\beta(x)}^{\infty} w^{\beta-1}e^{-w}dw \le \frac{1}{\Gamma(\beta)} \int_{\ln(1+x)}^{\infty} w^{\beta-1}e^{-w}dw$$
$$\le 1 - \frac{1}{\Gamma(\beta)(1+\varepsilon)} \int_0^{\ln(1+x)} w^{\beta-1}dw = 1 - \frac{\ln^\beta(1+x)}{\beta\Gamma(\beta)(1+\varepsilon)} \to \frac{\varepsilon}{1+\varepsilon},$$

as $\beta \to 0$, where

$$A_\beta(x) = \frac{2^{1/\beta}}{2^{1/\beta}-1}\ln(x+1).$$

At the last step we used a well-known fact

$$\Gamma(\alpha) \sim \frac{1}{\alpha}, \tag{2.4}$$

as $\alpha \to 0$.

For the Burr distribution family for fixed τ relation (2.4) gives

$$\kappa \sim \tau^\tau \left(\alpha - \frac{1}{\tau}\right)^\tau \to 0,$$

as $\alpha \to 1/\tau$, when limit (2.3) follows immediately.

However, around these critical values of the parameters ($\alpha \to 1$, $\sigma \to \infty$, $\tau \to 0$, $\beta \to 0$, $\alpha \to 1/\tau$, respectively), where the ergodicity condition fails, the ruin probability jumps to 1. The practical difficulty from this unpleasant effect is demonstrated by the fact, that the ruin probability on these neighborhoods does not depend on the initial capital. This is definitely unacceptable in risk analysis. It is worth mentioning that a similar situation appears in asymptotic formulas for the ruin probability with claim distribution with heavy tails, where the asymptotics do not depend on another important parameter of the risk business, the premium rate c.

3. Superheavy Subexponential Tails

The next Lemma can be found in Tsitsiashvili and Konstantinides (2001). For the sake of convenience, we repeat the argument.

Lemma 3.1 (Tsitsiashvili and Konstantinides, 2001). *In the classical risk model, if $B(x)$ belongs to the Pareto, Lognormal, Weibull, Loggamma, or Burr distribution family, then for any $x > 0$, the ruin probability tends to a constant*

$$\psi(x, \infty) \to \frac{\rho}{c},$$

as the heavytailedness parameter tends to its limit ($\alpha \to 1$, $\sigma \to \infty$, $\tau \to 0$, $\beta \to 0$, $\alpha \to 1/\tau$, *respectively*).

Proof. Let us consider the Pareto distribution family. According to the convergence (2.3), mentioned in the Remark 2.1, it follows that for any $\varepsilon \in (0, x)$ we can find a constant $\alpha_0 > 1$ such that

$$\overline{B}(\varepsilon) \leq \frac{\varepsilon}{x - \varepsilon},$$

for any $\alpha \in (1, \alpha_0)$. So from the Pollaczeck–Khinchine formula (1.2), the following chain of inequalities can be obtained

$$\begin{aligned}\frac{\rho}{c} = \psi(0, \infty) \geq \psi(x, \infty) &\geq \frac{\rho}{c}\overline{F}(x) = \frac{\rho}{c}\left(1 - \int_0^{\varepsilon} \overline{B}(y)dy - \int_{\varepsilon}^{x} \overline{B}(y)dy\right) \\ &\geq \frac{\rho}{c}[1 - \varepsilon - (x - \varepsilon)\overline{B}(\varepsilon)] \geq \frac{\rho}{c}[1 - 2\varepsilon],\end{aligned}$$

for any $\alpha \in (1, \alpha_0)$, which, due to the arbitrariness of ε, gives the desired convergence. Easily we can verify that the same argument holds for the rest of the distribution families.

We see that the super heavy limit of the claim distribution in Remark 2.1

$$\lim_{x \to \infty} B(x) = 1,$$

does not meet the requirements of a distribution function (as a non decreasing function from zero to one) and the super heavy limit of the ruin probability in Lemma 3.1

$$\lim_{x\to\infty} \psi(x, \infty) = \frac{\rho}{c},$$

does not meet the features of a ruin probability, as a non increasing function with respect to the initial capital x, as used to interpret the fact that more initial surplus reduces the chance of a ruin. These deformations of the standard form of the distribution and the ruin probability can be explained through an explosive behavior during the convergence to the limit.

In order to keep these standard forms in the course of the limit passage, and to preserve our intuitive perception for them, we fall back on application of functional normalization. Namely, we consider the heavytailedness parameter as a function of the limit variable, let say $\alpha(x) > 1$, for any $x \geq 0$ in the Pareto case, such that $\alpha(x) \downarrow 1$ when $x \to \infty$. The interpretation of this kind of functional normalization can be the following. The change of the initial surplus x affects the parameter $\alpha(x)$ of the Pareto distribution in such a way that infinite surplus means infinitely heavy tail. However, we shall need in the next result more restrictive convergence, namely the extra condition

$$[\alpha(x) - 1]x \to \infty,$$

as $x \to \infty$, to make the convergence appropriate. Then the corresponding tail and ruin probability regain their standard form, and we call them normalized tail and normalized ruin probability. Hopefully, the correct choice of the heavytailedness parameter convergence disclose their limit behavior.

The same approach applies to the remaining distributions. For example, in the Lognormal case the heavytailedness parameter $\sigma(x)$ tend to infinity as $x \to \infty$ but we need some more restriction $\sigma(x) < 2x$ for any $x > 0$ to have the convergence in an appropriate form. Only for the case of Loggamma, such an extra restriction is not needed.

The next statement is the main result of the article, attempting to explore the delicate interaction between the insurance company and its clients. For example a strong company with large reserves eventually attracts more demanding clients with extremely heavy-tailed claim distributions. This situation inspires the special limit conditions of the theorem.

Theorem 3.1. *If B belongs to the Pareto, Lognormal, Weibull, Loggamma, or Burr distribution families and its heavytailedness parameter tends to its limit as* $x \to \infty$ *in the following way*:

$$\alpha(x) \downarrow 1, \quad [\alpha(x) - 1]x \to \infty,$$

$$\sigma(x) \to \infty, \quad \sigma(x) < 2x, \quad \forall x > 0,$$

$$\tau(x) \downarrow 0, \quad x\tau(x) > 1, \quad \forall x > 0,$$

$$\beta(x) \downarrow 0,$$

$$\alpha(x) \downarrow 1/\tau, \quad \frac{1}{\tau\alpha(x) - 1} = o(x^{1/\tau}),$$

respectively, then their normalized tails have the following asymptotic behavior

$$\overline{B}([\alpha(x)-1]x) \sim \left(\frac{1}{x\,\alpha(x)}\right)^{\alpha(x)},$$

$$\overline{B}\left(\exp\left\{\sigma(x)x-\frac{1}{2}\sigma^2(x)\right\}\right) \sim \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left[-\frac{y^2}{2}\right]dy,$$

$$\overline{B}\left(\sqrt{\frac{\tau(x)}{2\pi}}[ex\tau(x)]^{1/\tau(x)}\right) \sim e^{-x},$$

$$\overline{B}\left(\exp\left\{\frac{x}{2^{1/\beta(x)}}\left(2^{1/\beta(x)}-1\right)\right\}-1\right) \sim \frac{1}{\Gamma[\beta(x)]}\int_x^\infty y^{\beta(x)-1}e^{-y}dy,$$

$$\overline{B}([\tau\alpha(x)-1](x-1)^{1/\tau}) \sim \left(\frac{1}{x}\right)^{\alpha(x)},$$

as $x \to \infty$. Furthermore, in the classical risk model, the corresponding normalized ruin probabilities have the following asymptotic behavior:

$$\psi([\alpha(x)-1]x) \sim \frac{\rho}{c-\rho}\int_x^\infty \frac{\alpha(z)-1+z\alpha'(z)}{(z[\alpha(z)-1]\alpha(z[\alpha(z)-1]))^{\alpha(z[\alpha(z)-1])}}dz,$$

$$\psi\left(\exp\left\{\sigma(x)x-\frac{1}{2}\sigma^2(x)\right\}\right) \sim \frac{\rho}{\sqrt{2\pi}(c-\rho)}\int_x^\infty \frac{\sigma(z)+\sigma'(z)[z-\sigma(z)]}{\exp\left\{-z\sigma(z)+\frac{1}{2}\sigma^2(z)\right\}} \times\left[\int_{\exp(z\sigma(z)-\frac{1}{2}\sigma^2(z))}^\infty \exp\left(-\frac{w^2}{2}\right)dw\right]dz,$$

$$\psi\left(\sqrt{\frac{\tau(x)}{2\pi}}[e\tau(x)x]^{1/\tau(x)}\right) \sim \frac{\rho}{\sqrt{2\pi}(c-\rho)}\int_x^\infty e^{-\frac{\sqrt{\tau(z)}}{\sqrt{2\pi}}}[ez\tau(z)]^{\frac{2}{\tau(z)}}\tau'(z) \times\left\{\frac{1}{2\sqrt{\tau(z)}}-e\frac{z\tau'(z)+\tau(z)}{\tau^2(z)}\log[ez\tau(z)]\right\}dz,$$

$$\psi\left(\exp\left\{\frac{x}{2^{1/\beta(x)}}(2^{1/\beta(x)}-1)\right\}-1\right) \sim \frac{\rho}{c-\rho}\int_x^\infty \frac{(1-2^{-1/\beta(z)})\exp\{(1-2^{-1/\beta(z)})z\}}{\Gamma[\beta\ (\exp\ [(1-2^{-1/\beta(z)})z]-1)]} \cdot\int_{\exp[(1-2^{-1/\beta(z)})z]-1}^\infty w^{\beta(\exp[(1-2^{-1/\beta(z)})z]-1)}e^{-w}dw\,dz,$$

$$\psi([\tau\alpha(x)-1](x-1)^{1/\tau}) \sim \frac{\rho}{c-\rho}\int_x^\infty \left(\frac{1}{[\tau\alpha(z)-1](z-1)^{1/\tau}}\right)^{\alpha([\tau\alpha(z)-1](z-1)^{1/\tau})} \times[\tau\alpha'(z)(z-1)^{1/\tau}]dz,$$

as $x \to \infty$.

Proof. We look for a bivariate auxiliary-factor function $f[x,\alpha(x)]$, such that the function $B(xf[x,\alpha(x)])$ keeps its distributional form. This implies the requirements:

(1) $xf[x,\alpha(x)] \downarrow 0$, as $x \downarrow 0$,
(2) $xf[x,\alpha(x)] \uparrow \infty$, as $x \to \infty$.

The auxiliary-factor function will be used to change the scale of the variable x, in such a way that the properties of $B([xf(x, \alpha(x)])$ and $\psi([xf(x, \alpha(x)])$ could be disclosed.

Let us start with the Pareto case to find the auxiliary-factor function

$$f[x, \alpha(x)] = \alpha(x) - 1,$$

under the restriction $[\alpha(x) - 1]x \to \infty$, meeting these requirements and serving as a candidate for our purposes.

Next, we fix the value of $\alpha(x) = \alpha > 1$ and we obtain the following asymptotic expression for the ruin probability when $x \to \infty$ uniformly for any $\alpha > 1$,

$$\lim_{x\to\infty} \sup_{\alpha>1} \left| \frac{\psi(xf[x, \alpha])}{\frac{\rho}{c-\rho}\overline{F}(xf[x, \alpha])} - 1 \right| = \lim_{x\to\infty} \sup_{\alpha>1} \left| \frac{\psi(x[\alpha - 1])}{\frac{\rho}{c-\rho}\overline{F}(x[\alpha - 1])} - 1 \right| = 0. \tag{3.1}$$

Indeed, as far $F(x[\alpha - 1])$ represents a subexponential distribution function and $\frac{c-\rho}{2c} > 0$, there exists some constant $K = K\left(\frac{c-\rho}{2c}\right)$ (for example, see Embrechts et al., 1997, Lemma 1.3.5) such that for any integer $N \geq 1$ we obtain

$$\begin{aligned}
\psi(x[\alpha - 1]) &= \frac{c-\rho}{c} \sum_{n=0}^{N-1} \left(\frac{\rho}{c}\right)^n \overline{F}^{n*}(x[\alpha - 1]) + \frac{c-\rho}{c} \sum_{n=N}^{\infty} \left(\frac{\rho}{c}\right)^n \overline{F}^{n*}(x[\alpha - 1]) \\
&\sim \frac{c-\rho}{c} \sum_{n=0}^{N-1} \left(\frac{\rho}{c}\right)^n n\overline{F}(x[\alpha - 1]) + \frac{c-\rho}{c} \sum_{n=N}^{\infty} \left(\frac{\rho}{c}\right)^n \overline{F}^{n*}(x[\alpha - 1]) \\
&\leq \left[\frac{\rho}{c-\rho} - \left(\frac{c}{c-\rho} + N - 1\right)\left(\frac{\rho}{c}\right)^N \right. \\
&\qquad \left. + \frac{c-\rho}{c} \sum_{n=N}^{\infty} K\left(\frac{\rho}{c}\left[1 + \frac{c-\rho}{2c}\right]\right)^n \right] \overline{F}(x[\alpha - 1])
\end{aligned}$$

as $x \to \infty$. Therefore, for any real $M > 1$ we find

$$\begin{aligned}
&\lim_{x\to\infty} \sup_{\alpha\in[M,\infty)} \left| \frac{\psi(x[\alpha - 1])}{\frac{\rho}{c-\rho}\overline{F}(x[\alpha - 1])} - 1 \right| \\
&\quad \leq \left| \left(1 + N\frac{c-\rho}{\rho}\right)\left(\frac{\rho}{c}\right)^N \right| + \left| \frac{2K(c-\rho)}{\rho}\left(\frac{c+\rho}{2c}\right)^N \right|.
\end{aligned}$$

Now we take the limit for $N \to \infty$ and the limit for $M \to 1$ and we reach the asymptotic relation (3.1).

Now we see that the expression $1 - \overline{B}(xf[x, \alpha(x)]) = 1 - \overline{B}(x[\alpha(x) - 1])$ keeps the distributional form because $x[\alpha(x) - 1] \to \infty$ as $x \to \infty$ and consequently

$$\psi(xf[x, \alpha(x)]) \sim \frac{\rho}{c-\rho} \int_x^{\infty} \overline{B}(zf[z, \alpha(z)])\left(z\frac{d(f[z, \alpha(z)])}{dz} + f[z, \alpha(z)]\right)dz.$$

Further, we continue with the rest cases under the following auxiliary-factor functions, respectively

$$f[x, \sigma(x)] = \frac{1}{x}\exp\left\{\sigma(x)x - \frac{1}{2}\sigma^2(x)\right\},$$

$$f[x, \tau(x)] = \frac{1}{x}\sqrt{\frac{\tau(x)}{2\pi}}[e\tau(x)x]^{1/\tau(x)},$$

$$f[x, \beta(x)] = \frac{1}{x}\left[\exp\left\{\frac{x}{2^{1/\beta(x)}}(2^{1/\beta(x)} - 1)\right\} - 1\right],$$

$$f[x, \alpha(x)] = \frac{1}{x}\left[\tau\alpha(x) - 1\right](x-1)^{1/\tau}.$$

It remains to repeat the uniform convergence. After the substitution, we obtain the other ruin probability formulas.

All of the five examples of heavy-tailed distributions outlined in the previous statements belong to the class of subexponential distributions $\mathscr{S}$. Two members of $\mathscr{S}$ which serve as exceptions to the Lemma 3.1 are given below.

Example 3.1. The Benktander I distribution with tail

$$\overline{B}_p(x) = \left[1 + \frac{2p}{\alpha}\ln(1+x)\right]\exp\{-p\ln^2(1+x) - (\alpha+1)\ln(1+x)\}$$

with $\alpha > 0$, $p > 0$, $x > 0$.

Example 3.2. The Benktander II distribution with tail

$$\overline{B}_p(x) = \frac{1}{(1+x)^{1-p}}\exp\left\{\frac{\alpha}{p}[1-(1+x)^p]\right\},$$

with $\alpha > 0$, $p \in [0, 1]$, $x > 0$.

We consider again the classical risk model. In both cases, the normalization condition (2.1) gives

$$\alpha = 1$$

and the heavytailedness parameter tends to zero

$$p \to 0.$$

Let us denote by $\psi^*(x)$ the ruin probability in the classical risk model in which the tail of claim is from the Pareto distribution $\overline{P}(x) = 1/(1+x)^2$, for $x > 0$ (that claim distribution coincides with the Burr(2, 1)).

The next result can be found in Tsitsiashvili and Konstantinides (2001) and we remind the argument for the sake of convenience.

Theorem 3.2 (Tsitsiashvili and Konstantinides, 2001). *In the classical risk model, if $B(x)$ belongs to either Benktander I or Benktander II distribution family, then its tail converges in $\mathscr{L}^1$ to $\overline{P}(x)$, as $p \to 0$, and the corresponding ruin probability $\psi_p(x, \infty)$ converges weakly to the function $\psi^*(x)$, representing the stationary distribution tail of the waiting time in the $M/G/1/\infty$ queuing system with service time distribution $P(x)$*

$$\psi_p(x, \infty) \Rightarrow \psi^*(x),$$

as $p \to 0$.

Proof. Firstly, let us take the example of the Benktander I distribution. Here, for any $T > 0$

$$\begin{aligned}
&\int_0^\infty |\overline{B}_p(x) - \overline{P}(x)|dx \\
&\leq \int_0^T |\overline{B}_p(x) - \overline{P}(x)|dx + \int_T^\infty \overline{P}(x)dx + \int_T^\infty \overline{B}_p(x)dx \\
&\leq T \sup_{[0,T]} \overline{P}(x)|[1 + 2p\ln(1+x)]\exp[-p\ln^2(1+x)] - 1| + \frac{1}{1+T} \\
&\quad + \int_T^\infty \left\{ \exp[-2\ln(1+x)] + 2p\ln(1+x)\exp[-p\ln^2(1+x) - 2\ln(1+x)] \right\} dx \\
&\leq T \sup_{[0,T]} \{|\exp[-p\ln^2(1+x)] - 1| + 2p\ln(1+x)\} + 2\left(\frac{1}{1+T} + (1+T)e^{-T}\right).
\end{aligned}$$

Further, for any $\varepsilon > 0$ there exists a T_ε and a $p_0 > 0$ such that for any $p \in (0, p_0)$ the last expression becomes less than ε, so

$$\int_0^\infty |\overline{B}_p(x) - \overline{P}(x)|dx \leq T \sup_{[0,T]} \{|\exp[-p\ln^2(1+x)] - 1| + 2p\ln(1+x)\} + \varepsilon \longrightarrow \varepsilon,$$

as $p \longrightarrow 0$ and thus the convergence in $\mathscr{L}^1$ for the Benktander I case is obtained. Similarly for the Benktander II case, considering the uniform convergence over any finite interval $[0, T]$

$$\exp\left(\frac{1 - e^{p\ln(1+x)}}{p}\right) \to \frac{1}{1+x},$$

as $p \to 0$, the convergence of the claim tail in $\mathscr{L}^1$ is confirmed.

In both cases the convergence of the ruin probability can be verified from a well-known result of the stability theory (see, for example, Kalashnikov, 1978; Kalashnikov and Tsitsiashvili, 1973). Indeed, in classical risk model the convergence of $\overline{B}(x)$ to $\overline{P}(x)$ in $\mathscr{L}^1$ implies the convergence of the ruin probability $\psi(x, \infty)$ to the function $\psi^*(x)$, which represents the ruin probability with claim distribution $P(x)$ (or, in other words, it represents the stationary distribution tail of the waiting time in the $M/G/1/\infty$ queuing system with service time distribution $P(x)$).

Remark 3.1. For the distributions from the five examples of Theorem 3.1, the convergence in $\mathscr{L}^1$ does not hold and therefore this argument from the stability theory is not applicable.

Remark 3.2. It is possible to prove that the function $\psi^*(x)$ is continuous and so weak convergence in Theorem 3.2 can be replaced by uniform convergence on half-line ($x \geq 0$); see Embrechts et al. (1997).

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