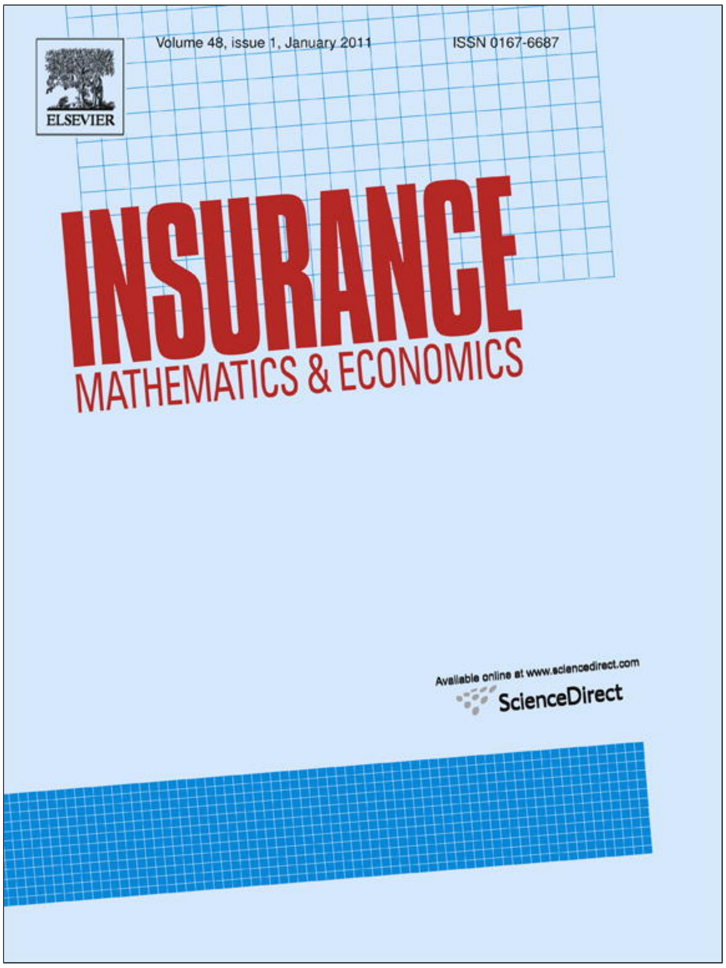
Volume 48, issue 1, January 2011
ISSN 0167-6687
ELSEVIER
INSURANCE
MATHEMATICS & ECONOMICS
Available online at www.sciencedirect.com
ScienceDirect

Insurance: Mathematics and Economics 48 (2011) 111–122

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Risk measures in ordered normed linear spaces with non-empty cone-interior

Dimitrios G. Konstantinides *, Christos E. Kountzakis

Department of Statistics and Actuarial -Financial Mathematics, University of the Aegean, Karlovassi, GR-83 200 Samos, Greece

ARTICLE INFO

Article history:
Received September 2009
Received in revised form
July 2010
Accepted 12 October 2010

JEL classification:
G320

MSC:
46B40
91B30

Keywords:
Base of a cone
Well-based cones
Numeraire asset
Coherent and convex risk measures
Representability of risk measures

ABSTRACT

In this paper, we use tools from the theory of partially ordered normed linear spaces, especially the bases of cones. This work extends the well-known results for convex and coherent risk measures. Its linchpin consists in the replacement of the riskless bond by some interior point in the cone of the space of risks, which stands as the alternative numeraire.

1. Introduction

Wedges and cones appear in a natural way in economic theory, since they indicate the way we interpret the less and the more.

This paper and the results mentioned are devoted to *static risk measures*. Two periods of time (0 and 1) are considered and a nature's probability space $(\Omega, \mathcal{F}, \mu)$ in which the set of states of the world Ω is supposed to be an infinite set.

The space of risks is a normed linear space L being a subspace of the linear space of real-valued $(\Omega, \mathcal{F}, \mu)$-measurable random variables. Either $L = L^p(\Omega, \mathcal{F}, \mu)$ for $1 \leq p \leq \infty$ or $L = C(\Omega)$, where in this case we suppose that Ω is a compact topological space. We suppose that the space in which the risks lie is actually a partially ordered linear space. The possible properties of the cone of the partial ordering of L may imply some important results and extensions of well-known theorems and propositions of the risk-measures' theory.

Our results concern risk measures defined on a partially ordered normed linear space L, whose positive cone L_+ has non-empty interior. We suppose that the investors recognize as a source of relative financial certainty a risky asset e instead of the riskless bond **1**. This idea appears both in Stoica (2006) by the notion of the *reference cash stream* and in Jaschke and Küchler (2001) by the notion of the *relatively secure cash stream*. The numeraire used in the well-known results of this theory (see for example Delbaen, 2002, Th. 2.3) is the riskless bond **1**. By using results about the properties of the bases of cones we deduce representation results similar to those for coherent (Delbaen, 2002, Th. 2.3) and convex (Föllmer and Schied, 2002, Th. 5) risk measures, when L is a reflexive space. For example we can take L to be some L^p space, with $p \in (1, \infty)$. The definitions of coherent and convex risk measures we need, are the same with the standard ones (see Artzner et al., 1999; Föllmer and Schied, 2002). The only difference appears in the expression of the translation invariance property, which is reformulated with respect to the numeraire asset and the monotonicity property, which is re-stated with respect to the partial ordering of L.

Relying on these representability results, we derived the uniform continuity of convex risk measures in this class of partially ordered normed linear spaces, jointly with other continuity results for coherent and convex risk measures, related to additional properties of the partial ordering of the space L. We also prove that Fatou property and other continuity properties of convex risk measures, summarized in Kaina and Rüschendorf (2009, Def. 3.1) for the L^p-spaces, hold for the new convex risk measures.

In recent papers (Biagini and Fritelli, 2009; Kaina and Rüschendorf, 2009) some interest about the continuity of convex risk measures on Banach lattices, ordered by their usual lattice cone, was

* Corresponding author. Tel.: +30 22730 82333; fax: +30 22730 82309.
E-mail addresses: konstant@aegean.gr (D.G. Konstantinides), chr_koun@aegean.gr (C.E. Kountzakis).

0167-6687/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2010.10.003

expressed. For example, in Biagini and Fritelli (2009) the continuity of the coherent risk measures defined on such a space is proved, as a corollary of the Extended Namioka–Klee theorem (Biagini and Fritelli, 2009, Th. 1). A reference to the same result is also made in Kaina and Rüschendorf (2009, Lem. 2.5, Cor. 2.6). Here we indicate that an equivalent result (see Theorem 4.5) holds in the case where the partial ordering of the space L is not a lattice ordering. If an infinite-dimensional space is partially ordered by a cone which has non-empty interior and it is also a well-based cone, then this is not a lattice cone (see Jameson, 1970, Th. 4.4.4). Such paradigms of cones in a normed linear space X are the Bishop–Phelps cones $K(f, a)$, where $f \in X^*$, $\|f\| = 1$, $a \in (0, 1)$ and

$$K(f, a) = \{x \in X \mid f(x) \geq a\|x\|\}. \tag{1.1}$$

Here, we indicate 'nice' continuity properties for a class of risk measures, which generalizes coherent and convex risk measures on normed linear spaces according to the chosen numeraire in the case where the space is not partially ordered by a lattice cone. More specifically, for $L_+ = K(f, a)$, as in (1.1) and if we select an interior point $e \in K(f, a)$ as the numeraire asset, then the mentioned continuity results hold for convex and coherent risk measures in which the translation invariance property and the monotonicity property are established with respect to e and $K(f, a)$ in the way indicated for example in Definition 2.1. Since $K(f, a)$ contains interior points, its dual wedge $(K(f, a))^0$ is a cone and because of the fact that L is reflexive, while Jameson (1970, Th. 3.8.12) holds, we find that e is a uniformly monotonic functional of $(K(f, a))^0$. This is the idea in the proof of uniform continuity for the risk measures we define. Under similar conditions for the cone of L, we suggest the representability of these risk measures (see Theorems 3.1 and 3.6).

The freedom in the choice of a cone for the partial ordering of our payoff space L is due to the fact that the acceptance set may include a variety of derivatives and portfolios of them. In the case where L is some of the well-known spaces (for example reflexive L^p spaces), a natural question arises about altering the partial ordering and moving to another one, which is different from the usual (component-wise) one, is whether sets of individuals may agree to the fact that a coherent set (a wedge) of financial positions contains the "pretty good" ones. If $L = L^p(\Omega, \mathcal{F}, \mu)$ with $p \in (1, \infty)$, we consider that the cone L_+ may contain financial positions for which the probability of negative outcomes is non-zero with respect to μ, because according to the personal beliefs of the individuals of this set, the probability of the states which correspond to the negative outcomes to occur is negligible.

Thus, the positive wedge L_+ represents a collective acceptance set. The acceptance set of a sole individual is rather naturally a superset of this collective acceptance set, since it expresses the specific preference of her/his to certain financial positions. A natural way to obtain L_+ from a set of individualistic acceptance sets being actually coherent (e.g. wedges of L) is the intersection of them.

The coherent and the convex risk measures we define, are supposed to be real-valued. We do not consider the case of the existence of risks X with $\rho(X) = \pm\infty$. Namely, we suppose that the capital requirements faced by the regulator are smooth enough to exclude the case of completely catastrophic and the completely riskless undertaking of insurance.

In Fritelli and Gianin (2002) the usual (component-wise) partial ordering of the L^p spaces was used in order to establish representability results, while on p. 1474 of this paper there appears a hint about the generalization of the representability results in partially ordered topological vector spaces.

A broader level of generality is met in Jaschke and Küchler (2001), where coherent risk measures in partially ordered vector spaces are studied. Jaschke and Küchler (2001, Th. 3) is pretty close to the representability result (Theorem 3.1) about coherent risk measures, but in our result for reflexive spaces the boundedness of the representing spot-price functionals' set B is established, like in the case of a finite state space.

On the other hand, the assumption of L_+ containing interior points, is a geometric concept which assures the compatibility of our results with the ones which hold either for Euclidean spaces (for example Artzner et al., 1999, Th. 4.1, Föllmer and Schied, 2002, Th. 5) or for L^∞, whose positive cone under the usual (component-wise) partial ordering contains interior points with respect to the norm topology (for example Delbaen, 2002, Th. 2.3). Also, the assumption that L_+ contains interior points is connected to the restriction that the risk measures we consider take finite values (see for example Proposition 3.3).

The representability results (Theorems 3.1 and 3.6) can be perceived as particular cases of Hamel (2006, Th. 4) for convex functionals from a topological linear space to the extended real line. However it is worth mentioning that our proofs take into account the partial ordering of the space L, while the statement in theorem Hamel (2006, Th. 4) does not make any reference to the partial ordering of the space which is however related to the monotonicity property included in both the classical definitions of both the coherent and convex risk measures, although Hamel (2006, Pr. 8) deals with monotonicity. Thus, at the end of Hamel (2006, Sec. 4), the representability of convex risk measures in reflexive L^p-spaces is derived only for the case where the numeraire is the riskless bond and the partial ordering of the space is the usual (component-wise) one. On the other hand, Hamel (2006, Cor. 6) refers to the continuity properties of the coherent and convex risk measures defined in the previously mentioned frame with a generalized numeraire in $L_+ \setminus \{0\}$. The continuity results we prove in this paper (for example Theorem 4.5) can be viewed as a partial case of this, but the proof of our result is essentially connected to the geometry of the cone L_+.

The continuity (but not the uniform continuity) of the convex risk measures we define can be also established by Borwein (1987, Cor. 2.4), where $X = L = L^p$, $1 < p < \infty$, $Y = \mathbb{R}$ and $S = -L_+$ which is a generating cone of L since it contains norm-interior points. Also $K = \mathbb{R}_+$ is a normal cone. Consider an e-convex risk measure $\rho : L^p \to \mathbb{R}$. Then ρ is $\mathbb{R}_+$-convex, because the relation $H_\rho : L^p \to \mathbb{R}$ with $H_\rho(x) = \rho(x) + \mathbb{R}_+$, $x \in L^p$ is $\mathbb{R}_+$-convex. This holds due to the convexity property of ρ. According to the monotonicity property of ρ, ρ is S-isotone, hence it is continuous at every point $x \in \mathrm{core}(L) = L$, since its domain is L. But again, our continuity Theorem 4.5 refers to the uniform continuity of the convex risk measures we define, which implies their continuity.

Finally, in Cheridito and Li (2009) coherent and convex risk measures in Orlicz spaces are studied. As it is well known, Orlicz spaces are a class of spaces which includes the L^p-spaces for $1 < p < \infty$. Hence a quote on the relation between the results contained in Cheridito and Li (2009) and ours is considered. The result Cheridito and Li (2009, Th. 4.3) provides the representability of convex risk measures on Orlicz spaces, where the monotonicity property of such a risk measure is taken with respect to the usual (component-wise) partial ordering of random variables and the numeraire asset considered is the riskless bond. Our representation Theorem 3.6 refers to a class of ordering cones different from the ones indicated in Cheridito and Li (2009, Th. 4.3). The continuity result Cheridito and Li (2009, Cor. 4.1) builds a connection between the Lipschitz continuity for a convex risk measure and the boundedness of the set of the representing functionals $\mathcal{Q}$. Our equivalent Theorem 4.5 indicates a case where the boundedness of the set of representing functionals is assured and this property implies the Lipschitz continuity of the convex risk measures we define.

An assumption of Kaina and Rüschendorf (2009, Th. 2.7, Cor. 2.6) is the norm-lower semi-continuity of ρ, which implies that the acceptance set $\mathcal{A}_\rho$ is norm-closed. The corresponding assumption in our Theorem 3.1 is that $\mathcal{A}_\rho$ is weakly closed but this implies that $\mathcal{A}_\rho$ is norm-closed, too.

Throughout the paper, we suppose that the interest rate between the time-period 0 and the time-period 1 is equal to zero. Also, since the spaces we use are also normed linear spaces, when we refer to positive or strictly positive functionals we mean continuous linear functionals (elements of the norm dual space L^*). Finally, for any $d \in L$ by $\hat{d}$ we denote d itself as a functional in L^{**}.

2. Coherent risk measures in ordered normed linear spaces with non-empty cone-interior

A natural generalization of the notion of the coherent risk measure in a partially ordered normed linear space L is the following (we denote by L_+ the positive cone of L which is supposed to be $\sigma(L, L^*)$-closed, containing interior points with respect to the norm topology of L):

Definition 2.1. A real-valued function $\rho : L \to \mathbb{R}$ which satisfies the properties

(i) $\rho(X + ae) = \rho(X) - a$ (Translation Invariance)
(ii) $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (Sub-additivity)
(iii) $\rho(\lambda X) = \lambda\rho(X), \lambda \in \mathbb{R}_+$ (Positive Homogeneity) and
(iv) $X \leq Y$ in terms of the partial ordering of L implies $\rho(Y) \leq \rho(X)$ (Monotonicity)

for any $X, Y \in L$, where e is an interior point of the positive cone L_+ considered to be a numeraire asset, is called *e-coherent risk measure*.

The selection of e as a numeraire asset will be discussed in the next section.

Proposition 2.2. *Consider a set of wedges $\{\mathcal{A}_i,\ i \in I\}$ of L indexed by I. The intersection*

$$\bigcap_{i\in I} \mathcal{A}_i$$

is also a wedge of L.

The above proposition indicates that if I represents a set of individuals whose coherent acceptance sets are the wedges $\mathcal{A}_i,\ i \in I$ of L and their intersection $\bigcap_{i\in I} \mathcal{A}_i$ is non-empty, then we may suppose that the partial ordering relation induced on L by all the individuals of the set I is the one defined by $L_+ = \bigcap_{i\in I} \mathcal{A}_i$. L_+ is in this case the set of the financial positions recognized as "pretty good" by all individuals of the set I. Note that such an approach about the formulation of acceptance sets also appears in Artzner et al. (1999, p. 206).

Suppose that ρ is a coherent risk measure on some partially ordered normed linear space L. Under the assumption that e is an interior point of L_+, L_+ is a generating cone, which implies that L_+^0 is a cone of L^*. Then we have $\mathcal{A}_\rho^0 \subseteq L_+^0$. Also, according to the monotonicity property of an e-coherent risk measure on L, the positive cone L_+ is a subset of the acceptance set $\mathcal{A}_\rho$.

Let

$$\rho_C(X) = \inf\{m \in \mathbb{R} \mid X + me \in C\}, \tag{2.1}$$

be the risk measure being associated with the wedge C of financial positions and e be an interior point of C, while C is actually a cone.

Proposition 2.3. *If C is a wedge of a linear space L and $e \in L$, then:*

(i) *If $e \in C$, then $\rho_C(X) < +\infty$ for any $X \in L$, if and only if e is a radial interior point of C.*
(ii) *$\rho_C(X) > -\infty$ for any $X \in L$, if and only if $-e$ is a radial interior point of the complement of C, if the complement of C is a radial set.*
(iii) *If $-e \notin C$ and $e \in C$, this is equivalent to the fact that ρ_C is Positive Homogeneous.*
(iv) *If ρ_C is Positive Homogeneous and finite-valued, then it is Convex.*
(v) *If $e \in C$ then Translation Invariance with respect to e for ρ_C holds.*
(vi) *ρ_C is Monotone with respect to C.*

About this proposition, we have to note that in Jaschke and Küchler (2001) the authors assert that its conclusions are true ((i)–(v)). Although we cope with normed linear spaces and not with linear spaces without a topology, we give the proof of this proposition with an additional assumption in (ii), where we suppose that the complement of C is a *radial set*. Also in (iv) for the sake of brevity, we suppose that ρ_C is finite-valued and Positive Homogeneous. We remind that if in a normed linear space L, C has norm-interior points then these interior points are also radial interior points, where L is taken as a linear space, too. The definitions of the radial interior point and of some relevant notions are contained in the Section 5.

Proposition 2.4. *If ρ is a e-coherent risk measure on L, then*

$$\rho_{\mathcal{A}_\rho} = \rho.$$

Another result is that the closedness of the acceptance set is implied by the closedness of L_+ in case where the norm of L is an order-unit norm with respect to e, i.e.

$$\|X\| = \inf\{\lambda > 0 \mid X \in [-\lambda e, \lambda e]\}, \tag{2.2}$$

and $\mathcal{A}_\rho$ is a cone.

Proposition 2.5. *Suppose that $\rho : L \to \mathbb{R}$ is an e-coherent risk measure. Then if $\|.\|$ is an order-unit norm with respect to e, L_+ is a $\sigma(L, L^*)$-closed cone and the acceptance set $\mathcal{A}_\rho$ is a cone, then ρ is Lipschitz.*

We note that this is a generalization of what is well known about the case of L^∞ whose norm (the $\|.\|_\infty$ norm) is an order-unit norm with respect to **1**. Another case of such a space is the space of continuous functions $C(\Omega)$ endowed with the $\|.\|_\infty$ norm if the state space Ω is supposed to be a compact topological space.

3. Representability of coherent and convex risk measures in ordered reflexive spaces with non-empty cone-interior

In the rest of the paper we suppose that L is reflexive and infinite-dimensional. As we have quoted, this corresponds to the case of the $L^p(\Omega, \mathcal{F}, \mu)$ spaces with $1 < p < \infty$. An economic explanation about adopting this assumption is that if a risk X belongs to such an L^p space, then the existence of the moments $E(X^k)$, where $k \in \mathbb{N}$ and $1 \leq k \leq p$ provides some information about the nature of uncertainty contained in X. Let us denote by

$$B_e = \{\pi \in L_+^0 \mid \hat{e}(\pi) = 1\} \tag{3.1}$$

the base of the cone L_+^0 defined by e.

Theorem 3.1. *If $\rho : L \to \mathbb{R}$ is an e-coherent risk measure whose acceptance set $\mathcal{A}_\rho$ is $\sigma(L, L^*)$-closed, then*

$$\rho(X) = \sup\{\pi(-X) \mid \pi \in B\}, \tag{3.2}$$

for any $X \in L$, where $B = \{y \in \mathcal{A}_\rho^0 \mid \hat{e}(y) = 1\} = B_e \cap \mathcal{A}_\rho^0$.

Remark 3.2. The steps of the next proof are parallel to these followed in Artzner et al. (1999, Th. 4.1). However the consideration of the two proofs draws the following comparisons:

(i) While in the case of the finite-dimensional spaces (Artzner et al., 1999, Th. 4.1) the closedness of the acceptance set $\mathcal{A}_\rho$

is implied by the continuity of ρ, in the case where L is an infinite-dimensional reflexive space, the closedness of $\mathcal{A}_\rho$ is included in the assumptions.

(ii) The argument in the current theorem is the same either we suppose that L is endowed with the norm topology or it is endowed with the weak topology $\sigma(L, L^*)$. This similarity can be explained by the fact that locally convex topologies, compatible with a given dual pair (which is $\langle L, L^*\rangle$ in our case), have the same closed, convex sets. Therefore the application of the Separation theorem in this proof is the same with the one in theorem Artzner et al. (1999, Th. 4.1).

(iii) A point which has to be verified so that the first step of the proof is valid — and this point also concerns the case of the finite-dimensional spaces (in this case $B = \mathcal{A}_\rho^0 \cap \Delta_{S-1}$ if the number of the states of the world is equal to $S \in \mathbb{N}$) — is whether the set $B = \mathcal{A}_\rho^0 \cap B_e$ is non-empty. But as the Lemma 3.4 indicates, the condition $B \neq \emptyset$ is equivalent to the condition $\mathcal{A}_\rho^0 \neq \{0\}$ which is true in the case of ρ being an e-coherent risk measure, since $\mathcal{A}_\rho$ is a closed wedge of L and by Lemma 5.3 if $\mathcal{A}_\rho^0$ is equal to $\{0\}$, then $\mathcal{A}_\rho$ would be equal to the whole space L, which is not true since for example $\rho(-e) = \rho(0+(-1)e) = \rho(0)+1 = 1 > 0$ from the Positive Homogeneity and the Translation Invariance of ρ.

In order to show that the supremum in the relation (3.2) is finite, we show the following proposition.

Proposition 3.3. *For any $X \in L$, the* $\sup\{\pi(-X) \mid \pi \in \mathcal{A}_\rho^0, \hat{e}(y) = 1\}$ *is finite.*

Lemma 3.4. *If $\rho : L \to \mathbb{R}$ is an e-coherent risk measure, then $B_e \cap \mathcal{A}_\rho^0 \neq \emptyset$ is equivalent to $\mathcal{A}_\rho^0 \neq \{0\}$.*

Let us discuss Theorem 3.1. In the case of $\mathbb{R}^S$, the representability of a coherent risk measure ρ is associated with the existence of a set of probability vectors for the states of the world $\mathcal{P}$ such that $\rho(X)$ is the supremum of the expectations of $-X$ under every element of $\mathcal{P}$. Every probability vector in $\mathcal{P}$ may be seen as a (continuous) linear functional which has the property that it is a positive linear functional of the cone $\mathbb{R}^S_+$. The question which arises in the case of a risk measure defined on some infinite-dimensional space L is the following: whether we can consider some non-empty set $\mathcal{P} \subseteq L_+^0$ such that $\pi(e) = 1$ for any $\pi \in \mathcal{P}$ and ρ is representable with respect to $\mathcal{P}$, namely

$$\rho(X) = \sup_{\pi\in\mathcal{P}} \pi(-X) \tag{3.3}$$

for any $X \in L$. We also may require the set $\mathcal{P}$ to be such that the supremum in (3.3) is finite for any X. Theorem 3.1 indicates whether such a situation is met and it is especially applicable when $L = L^p(\Omega, \mathcal{F}, P)$ with $p \in (1, \infty)$ if Ω being the set of states of the world is infinite. Let us remind that $L^p(\Omega, \mathcal{F}, P)$ where $p \in (1, \infty)$ and the partial ordering of it is the usual one implies that the positive cone does not contain interior points.

We remind that a *spot-price* in the two-period model of financial markets is a linear functional of L, preferably in L^*, which indicates the present value at time-period 0 of any bundle consumed or enjoyed as a payoff of some investment at time-period 1. By a general 'no-arbitrage' principle if this bundle lies in L_+, the present value of it at time-period 0 must be positive, hence any spot-price functional is actually an element of the wedge L_+^0.

The subset of probability vectors $\mathcal{P}$ in theorem Artzner et al. (1999, Th. 4.1), may be seen as a set of normalized spot-price vectors with respect to the riskless bond $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^S_+$. In this case, the set of the normalized spot-price vectors $\pi = (\pi_1, \pi_2, \dots, \pi_S)$ is the whole simplex Δ_{S-1}.

This approach can be transfered in an infinite-dimensional framework. In this case, the set of the normalized spot-price functionals with respect to the numeraire e is the set

$$\{\pi \in L_+^0 \mid \pi(e) = 1\},$$

The payoff vector e is called numeraire, because in economic literature a numeraire asset is an asset with positive payoff at any state of the world, whose price at time-period 0 is equal to 1. As an example of this usual definition of a numeraire asset, see Herings and Polemarchakis (2005, p. 137). In the case where L is infinite-dimensional, the fact that the price of e is equal to 1 in general cannot prevent its value from becoming degenerate. The next lines are devoted to the verification of the geometric nature of e inside L_+ so that this requirement is valid.

Since in this paragraph we suppose that L is reflexive, then

$$(L_+^0)^0 = (L_+)^{00} = \overline{L_+} = L_+$$

if L_+ is supposed to be a closed cone. The set of

$$\{\pi \in L_+^0 \mid \pi(e) = 1\}$$

if $\hat{e}$ is supposed to be a strictly positive functional of L_+^0 and L_+^0 is a cone, is actually a base of L_+^0 just like the simplex Δ_{S-1} which is a base for $\mathbb{R}^S_+$ which is defined by $\mathbf{1}$. If e is an interior point of L_+, then L_+^0 is actually a cone because in this case L_+ is a generating cone (see Jameson, 1970, Pr. 3.1.5). If we select such an e as a numeraire, then the requirement that the value of e must be non-degenerate is satisfied. In this case the set B_e, defined in (3.1), is the base of the cone L_+^0 defined by e.

First, for any normalized spot-price vector or else for any $p \in B_e = \{\pi \in L_+^0 \mid \hat{e}(\pi) = 1\}$, $p(e) = 1 \neq 0$. But the fact that in this case the value of e is non-degenerate is also explained in a more extensive way. If we suppose that during the time period 0 and before the investor's action to buy one unit of e spot-prices alter in a way that they remain normalized but they tend to some spot-price p under the weak topology $\sigma(L^*, L) = \sigma(L^*, L^{**})$ then the value of e cannot become zero under the final price, i.e. if there is a net $(p_\lambda)_{\lambda\in\Lambda} \subseteq B_e$ such that

$$p_\lambda \overset{\sigma(L^*,L)}{\to} p,$$

then $p(e) \neq 0$. If this is the case, then $p \in B_e$ and $p(e) = 1$.

Note that it is rational to examine the variation of the prices with respect to the weak-star topology, since the dual pair $\langle L^*, L\rangle$ indicates price–commodity duality, or else the fact that the variation of the spot-price vectors is verified by the prices of the commodities-financial positions. Moreover, the weak-star topology is the weakest topology consistent with the dual pair $\langle L^*, L\rangle$ and if we choose any other locally convex topology τ' consistent with this dual pair for the convergence of the net of prices above, then the convergence under τ' implies the convergence under the weak-star topology (being actually the weak topology since the space in our case is reflexive).

To see this, we have to remind two well-known statements.

(1) (Jameson, 1970, Th. 3.8.6) Let P be a wedge in a locally convex space L. Then,
   (i) If P has an interior point, then P^0 has a $\sigma(L^*, L)$-compact base.
   (ii) If L is a Mackey space, P is closed and P^0 has a $\sigma(L^*, L)$-compact base, then P has an interior point.

(2) (Jameson, 1970, Pr. 3.8.12) Let L be an ordered normed linear space with positive wedge P. Then an element f of L^* is uniformly monotonic if and only if it is an interior point of P^0 with respect to the norm topology of L^*.

If $e \in L = L^{**}$ is an interior point of $P = L_+ = L_+^{00}$, then the base indicated in item (i) of the Jameson (1970, Th. 3.8.6) is the base defined by e as a functional lying in L^{**}, namely the base $B_e = \{Y \in L_+^0 \mid \hat{e}(Y) = 1\}$ of P^0 being the set of the normalized spot-prices with respect to e. Since we supposed that the space L is reflexive, the base B_e defined by e in this case is not simply weak-star compact, but weakly compact. Hence this base is weakly closed and if there is a net $(\pi_\lambda)_{\lambda\in\Lambda} \subseteq B_e$ such that

$$\pi_\lambda \overset{\sigma(L^*,L)}{\to} \pi,$$

then $\pi \in B_e$.

By the Jameson (1970, Pr. 3.8.12), there is some $b > 0$ such that $\hat{e}(\pi) \geq b\|\pi\|$ for any $\pi \in L_+^0$. Finally in order to complete the whole frame, we have to mention the following. When the set of states is the finite set $\{1, 2, \ldots, S\}$ and we use the riskless bond $\mathbf{1}$ as the numeraire asset, then the riskless bond is an interior point of $\mathbb{R}_+^S$.

Of course, apart from the fact that B_e is weakly compact in the case considered above due to the reflexivity of L and the Alaoglou weak compactness theorem (see Aliprantis and Border, 1999, Th. 6.25), the requirement that the value of e has to be non-degenerate is also satisfied if e is a strictly positive functional of L_+^0 (not necessarily uniformly monotonic) and L^* is endowed with the weak-star topology $\sigma(L^*, L)$. This is due to the fact that L_+^0 is a weak-star closed subset of L^* (see 5.1) and the base B_e is the intersection of L_+^0 and the hyperplane $\{p \in L^* | \hat{e}(p) = 1\}$ (which is also a $\sigma(L^*, L)$-closed set). Hence if we consider a net $(\pi_\lambda)_{\lambda\in\Lambda} \subseteq B_e$ such that

$$\pi_\lambda \overset{\sigma(L^*,L)}{\to} \pi,$$

then $\pi \in B_e$. Hence the case of e being an interior point of L_+ indicates a specific case of e being a strictly positive functional of L_+^0 in the case where L is reflexive.

The fact that the value of the numeraire e is non-degenerate in any case under the frame we posed, can be viewed as a 'dual analog' of the 'absence of free-lunches' analyzed in the seminal work Kreps (1981) as an extension for the notion of 'absence of arbitrage' in an infinite-dimensional commodity space.

If we consider a two-period model in which the commodity space in which the payoffs of the assets or else the financial positions lie in, is a locally convex space (L, τ) partially ordered by a cone C whose origin 0 is deleted and the subspace of the marketed assets is M, then a *free-lunch* is a net $(m_a, x_a)_{a\in A} \subseteq M \times X$ and a position $k \in C$ such that under spot-price functional π, $m_a \geq x_a$,

$$\lim_a x_a = k$$

in the topology τ and

$$\liminf_a \pi(m_a) \leq 0.$$

An example of a dual analog of a free lunch in which the consumption bundle (being e in our case) is given and the prices vary and for this reason we call it 'free-lunch under price variation' for e is the following. In the dual space L^* and for a certain consumption bundle $e \in L_+$ there exists a net $(\pi_a)_{a\in A} \subseteq L_+^0$ such that

$$\pi_a \overset{\sigma(L^*,L)}{\to} \pi,$$

where $\pi \in L_+^0$, $\pi \neq 0$ and

$$\liminf_a \pi_a(e) \leq 0.$$

If e is the numeraire asset, then the existence of such 'free-lunches under price variation' for e, would mean that there exist nets $(\pi_a)_{a\in A}$ in the base B_e of L_+^0 (since e is a numeraire, the prices are normalized with respect to e) such that

$$\pi_a \overset{\sigma(L^*,L)}{\to} \pi$$

where $\pi \in L_+^0$, $\pi \neq 0$, such that $\pi(e) = 0$. This is implied by

$$\liminf_a \pi_a(e) = \lim_a \pi_a(e) \leq 0,$$

while $\pi_a(e) \geq 0$ which implies $\lim_a \pi_a(e) \geq 0$. Note that in this case by the weak-star convergence of the net $(\pi_a)_{a\in A}$, the limit $\lim_a \pi_a(e)$ exists and it is equal to $\pi(e)$.

The assumption we pose that e is an interior point of L_+ prevents the existence of such free-lunches under price variation.

We have to note however, a question about the selection of the numeraire e arises here and for the case of reflexive spaces we work, due to an essential dichotomy result for bases of cones in reflexive spaces proved in Polyrakis (2008, Th. 4): *Suppose that $\langle X, Y\rangle$ is a dual system. If X, Y are normed spaces, P is a $\sigma(X, Y)$-closed cone of X so that the positive part*

$$U_X^+ = U_X \cap P$$

of the closed unit ball U_X of X is $\sigma(X, Y)$-compact, we have: every base for P defined by a vector $y \in Y$ is bounded or every such base for P is unbounded. In our case L is reflexive and that L_+^0 is $\sigma(L^*, L)$-closed holds, while $X = L^*$, $Y = L = L^{**}$, $P = L_+^0$, and

$$U_{L^*}^+ = L_+^0 \cap U_{L^*}$$

is $\sigma(L^*, L)$-compact due to Alaoglou's theorem. According to Polyrakis (2008, Th. 4), if e is an interior point of L_+ and $h \in L_+$ as a linear functional of L^* being a strictly positive linear functional of L_+^0, then the base

$$B_h = \{\pi \in L_+^0 \mid \hat{h}(\pi) = 1\}$$

defined by it is also bounded and hence weakly compact, since h is a uniformly monotonic functional of L^* ordered by L_+^0 (see in Polyrakis, 2008, Pr. 2) and Jameson (1970, Th. 3.8.12) holds (apply it for $X = L^*$, $P = L_+^0$, $P^0 = L_+^{00} = L_+$, $h = f$). In this case, h can be also considered to be a numeraire asset too, since $\hat{h}$ is a uniformly monotonic functional of L_+^0 and h is an interior point of L_+ from the Jameson (1970, Th. 3.8.12). But in this paper we will not deal more with the selection of a numeraire asset.

The bases of cones may be still used for results of representability for convex risk measures. In a way similar to the one of the previous paragraph and having in mind the definition of a convex risk measure in Föllmer and Schied (2002, Def. 1), the natural generalization of the notion of the convex risk measure in a partially ordered normed linear space L whose positive cone L_+ has non-empty interior with respect to the norm topology of L is the following (we denote by L_+ the positive cone of L and by L^* the dual space of L):

Definition 3.5. A real-valued function $\rho : L \to \mathbb{R}$ which satisfies the properties $(X, Y \in L)$

(i) $\rho(X + ae) = \rho(X) - a$ (Translation Invariance)
(ii) $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for any $\lambda \in [0, 1]$ (Convexity) and
(iii) $X \leq Y$ in terms of the partial ordering of L implies $\rho(Y) \leq \rho(X)$ (Monotonicity)

where e is an interior point of the positive cone L_+ considered to be a numeraire asset, is called *e-convex risk measure.*

Following the lines of the theorem Föllmer and Schied (2002, Th. 5) being proved for Euclidean spaces, we may prove the following representability result. If ρ is an e-convex risk measure, the acceptance set of $\mathcal{A}_\rho$ is not a wedge, but a convex subset of L, which contains L_+ due to the monotonicity property of such a ρ. In this case we remark that the set of representing functionals is the whole base defined by e on L_+^0.

Theorem 3.6. *If $\rho : L \to \mathbb{R}$ is an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set, then*

$$\rho(X) = \sup\{\pi(-X) - a(\pi) \mid \pi \in B\}, \tag{3.4}$$

for any $X \in L$, where $B_e = \{y \in L^0_+ \mid \hat{e}(y) = 1\}$ and $a : B_e \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_e$. On the other hand, every ρ defined through (3.4), is an e-convex risk measure.

As we showed above, every e-convex risk measure is associated with a 'penalty function' $a : B_e \to \mathbb{R}$ given in (6.8), where B_e is the base of L^0_+ defined by e. Note that the definition of this penalty function is the same with the one in Föllmer and Schied (2002) in the case where the space in which the risks lie, is a Euclidean space.

Also, if we want to establish a convex duality relationship between the penalty function a and the convex conjugate ρ^* of ρ we remark that

$$a(\pi) = \sup_{X \in L}\{(-\pi)(X) - \rho(X)\} = \rho^*(-\pi)$$

for any $\pi \in B_e$.

The economic meaning of the penalty function of ρ is the following, being associated with the general structure of the two-period model of financial markets, being the frame in which we study the measurement of risk entailed in financial positions: An investor desires to receive the financial position $X \in L$ at time-period 1. For this investment to X, the investor is willing to pay the amount of money $\pi(X)$ at time-period 0, where π is the vector of the spot-prices at time-period 0. In order to face the risk exposure, the investor is willing to invest in $\rho(X)$ shares of the numeraire asset e so that the joint investment is acceptable in terms of the risk measure ρ, or else the joint investment's payoff to lie in $\mathcal{A}_\rho$, which is actually the case since this payoff is $X + \rho(X)e$, which lies in the acceptance set $\mathcal{A}_\rho$. The total loss of money at time-period 0 for 'investment and securitization' of the position $X \in L$ under spot-prices π is $\pi(-X) - \rho(X)$ or else $\pi(X) + \rho(X)$ in net terms. Hence $a(\pi)$ is the minimum relevant loss that an investor may face for any possible investment-financial position in L if the spot-prices are given by π.

Another important remark on representability of e-convex risk measures is that the set of the representing spot-price functionals in (3.4) is the whole base B_e defined by e on L^0_+, while in the equivalent representation of the e-coherent risk measures, the set of the representing spot-price functionals is the set $B_e \cap \mathcal{A}^0_\rho$. The fact that in the case of e-coherent risk measures the set of representing spot-price functionals is smaller, is due to the fact that an e-coherent risk measure ρ satisfies the Positive Homogeneity property and this implies that the representing functionals take positive values on $\mathcal{A}_\rho$, as it was discussed in the proof of the Theorem 3.1. Note that the acceptance set of an e-convex risk measure is in general an unbounded convex set in L which contains L_+. This is equivalent to the fact that ρ in this case does not satisfy the Positive Homogeneity Property. But since in the case of the e-convex risk measures the Positive Homogeneity does not hold, we cannot conclude that the functional $\pi_0 \in L^*$, $\pi_0 \neq 0$ which strongly separates $\mathcal{A}_\rho$ (if it is $\sigma(L, L^*)$-closed) and $\{X + (\rho(X) - \epsilon)e\}$ for any $X \in L$ and any $\epsilon > 0$ takes positive values on $\mathcal{A}_\rho$, which is the key of the proof of Theorem 3.1. That is why the whole base B_e is the set of the representing spot-prices of any e-convex risk measure ρ and that is why the 'penalty function' appears, according to the proof of Theorem 3.6.

4. Continuity results arising from representability of coherent and convex risk measures with non-empty cone-interior

The following result gives a first taste on continuity of e-convex risk measures.

Proposition 4.1. *If $\rho : L \to \mathbb{R}$ is an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set, then*

$$\rho(X) \leq \frac{1}{b}\|X_2\| \tag{4.1}$$

for any $X \in L$ and some $b > 0$, where $X_1, X_2 \in L_+$ such that $X = X_1 - X_2$.

The Proposition 4.1 implies the following result.

Corollary 4.2. *If L is a vector lattice and $\rho : L \to \mathbb{R}$ is an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then*

$$\rho(X) \leq \frac{1}{b}\|X^-\|, \tag{4.2}$$

for any $X \in L$, where X^- is the negative part of X.

The Corollary 4.2 enables the next statement.

Proposition 4.3. *If L is a normed lattice and $\rho : L \to \mathbb{R}$ is an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then*

$$\rho(X) \leq \frac{1}{b}\|X\| \tag{4.3}$$

for any $X \in L$.

The Proposition 4.3 implies the following

Proposition 4.4. *If L is a normed lattice and $\rho : L \to \mathbb{R}$ is an e-coherent risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is a Lipschitz function.*

Relying on the representation Theorem 3.6 for e-convex risk measures and more specifically on the properties of the set B since e is a *uniformly monotonic functional* of L^0_+ as an interior point of L_+ (see Jameson, 1970), i.e. there is some $b \in \mathbb{R}$, $b > 0$ such that

$$\hat{e}(f) \geq b\|f\|$$

where $f \in L^0_+$, we can derive a genuine continuity theorem for e-convex risk measures, which also holds for e-coherent risk measures:

Theorem 4.5. *If $\rho : L \to \mathbb{R}$ is an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is Lipschitz.*

Corollary 4.6. *If $\rho : L \to \mathbb{R}$ is an e-coherent risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is a Lipschitz function.*

In Kaina and Rüschendorf (2009, Def. 3.1) a list of the well-known continuity properties for risk measures in L^p-spaces can be found. The next corollary of Theorem 4.5 is directly related to these properties, which can be deduced by following the lines of the proof of Kaina and Rüschendorf (2009, Th. 3.1):

Corollary 4.7. *If L is a reflexive L^p-space, $\rho : L \to \mathbb{R}$ is either an e-coherent or an e-convex risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is continuous from above, continuous from below, Fatou continuous and Lebesgue continuous.*

Finally, we have to remind the comment made in Remark 3.2 that the requirement of the closedness of $\mathcal{A}_\rho$ under the weak topology on L in all the above proofs may be replaced by the closedness with respect to the norm topology and vice versa without altering the strength of the results, since locally convex topologies compatible with a given dual pair have the same closed, convex sets (see Aliprantis and Border, 1999, Th. 5.86).

5. Appendix: notions from the theory of partially ordered linear spaces

In order to make the paper as self-consistent as possible, we follow the terminology for partially ordered linear spaces mainly listed in Jameson (1970) and we present some results concerning the properties of dual wedges.

A *partial ordering relation* $\geq$ on some vector space L is a binary relation being

(1) *reflexive* ($x \geq x$ for any $x \in L$),
(2) *antisymmetric* ($x \geq y$ and $y \geq x$ implies $y = x$, where $x, y \in L$)
(3) *transitive* ($x \geq y$ and $y \geq z$ implies $x \geq z$, $x, y, z \in L$).

The partial ordering relation is *compatible* with the linear structure of L if

(1) $x + z \geq y + z \Leftrightarrow x \geq y$
(2) $x \geq y \Leftrightarrow \lambda x \geq \lambda y$, if $\lambda \in \mathbb{R}_+ \setminus \{0\}$.

Then $(L, \geq)$ is called a *partially ordered linear space*.

A non-empty subset K of a vector space L is called *wedge* if it satisfies

(1) $K + K \subseteq K$
(2) $\lambda K \subseteq K$ for any real number $\lambda \geq 0$.

A non-empty subset R of a vector space L which satisfies the second of the two above properties is called *radial set*.

A wedge K is a cone, if it satisfies the condition

$$K \cap (-K) = \{0\}. \quad (5.1)$$

Any wedge K in a vector space L implies a *partial ordering relation* $\geq_K$ on it defined as follows:

$$x \geq_K y \Leftrightarrow x - y \in K. \quad (5.2)$$

When $x \geq_K y$ and $x - y \notin K \cap (-K)$ then we write $x >_K y$. An alternative way of denoting $x \geq_K y$ is to write $y \leq_K x$. In this case the partially ordered linear space $(L, \geq_K)$ may be denoted by (L, K). On the other hand, if $(L, \geq)$ then

$$P = \{x \in L \mid x \geq 0\}$$

is a wedge and the two orderings $\geq, \geq_P$ coincide.

A linear functional f of a vector space L is called *positive functional* of the partially ordered linear space (L, K) (or a *positive functional of* K) if

$$f(x) \geq 0, \quad \forall x \in K,$$

and *strictly positive functional* of the ordered linear space (L, K) (or a *strictly positive functional of* K) if it is positive and

$$f(x) > 0, \quad \forall x \in K \setminus (K \cap (-K))$$

if K is a wedge. If L is a normed linear space and K is a cone, a linear functional of L is *uniformly monotonic* if $f(x) \geq a\|x\|$ for any $x \in K \setminus \{0\}$ for some $a > 0$.

If K is a wedge of the normed linear space L, the *dual wedge* of K, denoted by K^0, represents the following subset of L^*:

$$K^0 = \{f \in L^* \mid f(x) \geq 0, \ \forall x \in K\}.$$

If K, C are wedges with $K \subseteq C$, then $C^0 \subseteq K^0$. In the rest of this paper we will denote L_+^0 as the dual wedge of L_+.

If (L, K) is a partially ordered linear space and K is a cone, then a *base* of K is any convex subset B of K such that for any $x \in K \setminus \{0\}$ there exists a unique $\lambda_x \in \mathbb{R}_+ \setminus \{0\}$ such that $\lambda_x x \in B$. If f is a strictly positive functional of L_+, the set

$$B_f = \{x \in K \mid f(x) = 1\}, \quad (5.3)$$

is the *base of* K *defined by* f. The base B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of K such that $0 \notin \overline{B}$ then K is called *well-based*. If K is well-based, then a bounded base of K defined by a $g \in L^*$ exists.

Let us denote by $[x, y]$ the order interval of L defined by x, y, i.e.

$$[x, y] = (x + K) \cap (y - K).$$

An element $e \in L$ is called *order-unit* of (L, K) if

$$L = \cup_{n=1}^{\infty} [-ne, ne].$$

If L is also a normed linear space, then according to Jameson (1970, Pr. 3.1.3), e is an interior point of K in terms of the norm topology if and only if $[-e, e]$ is a neighborhood of zero. Hence an interior point of K is an order-unit of (L, K).

(L, K) is a *vector lattice* if for any $x, y \in L$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by K exists in L. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so,

$$|x| = \sup\{x, -x\}$$

is the *absolute value* of x and if L is also a normed space such that

$$\| |x| \| = \|x\|$$

for any $x \in L$, then (L, K) is called *normed lattice*. About vector lattices, the reader may also see Aliprantis and Burkinshaw (1985).

A wedge K of L is called *generating* if the subspace $K - K$ it generates in L is equal to L itself. If K is a generating wedge in L, then K^0 is a cone of L^*.

Let us note that the usual partial ordering relation on $\mathbb{R}^S$ is defined as follows: $x \geq y$ if and only if $x_s \geq y_s$ for $s = 1, 2, \ldots, S$, where $x, y \in \mathbb{R}^S$. The positive cone of this ordering is denoted by $\mathbb{R}^S_+$. Also, the usual partial ordering of an $L^p(\Omega, \mathcal{F}, \mu)$ space, where $(\Omega, \mathcal{F}, \mu)$ is a probability space is the following: $X \geq Y$ if and only if the set $X \geq Y$, μ-a.s.

Also, if L is a linear space, a subset D of it is *absorbing* if for every $X \in L$, there is a real number $a > 0$ such that $\frac{1}{a}X \in D$. If V is a subset of L, then $X_0 \in V$ is a *radial interior point* of V if $-X_0 + V$ is an absorbing set.

We remind the following essential result about wedges in Euclidean spaces: Consider some wedge K of the space $\mathbb{R}^S$. Then $K^{00} = (K^0)^0 = \overline{K}$. We suppose that the cone L_+ of L is $\sigma(L, L^*)$-closed and we need the following results.

Proposition 5.1. *L_+^0 is a $\sigma(L^*, L)$-closed subset of L^*.*

Proposition 5.2. *If the cone L_+ of L is $\sigma(L, L^*)$-closed and L is reflexive, then*

$$(L_+^0)^0 := L_+^{00} = L_+. \quad (5.4)$$

Lemma 5.3. *If K is a closed wedge of some normed linear space L then the condition $K^0 = \{0\}$ is equivalent to $K = L$.*

6. Proofs

Proof of Proposition 2.2. If we denote by A the intersection $\bigcap_{i \in I} \mathcal{A}_i$, it suffices to prove that if $x, y \in A$ then $x + y \in A$ and $\lambda x \in A$ for any $\lambda \in \mathbb{R}_+$. Since $x \in A$, $x \in A_i$ for any $i \in I$ and since $y \in A$, $y \in A_i$ for any $i \in I$. Since A_i is a wedge for any $i \in I$, $x+y \in A_i$ for any $i \in I$, but this implies $x + y \in A$. Also since A_i is a wedge for any $i \in I$, $\lambda x \in A_i$ for any $\lambda \in \mathbb{R}_+$ and for any $i \in I$. Hence $\lambda x \in A$ for any $x \in A$ and any $\lambda \in \mathbb{R}_+$. Then A is actually a wedge of L. □

Proof of Proposition 2.3.

(i) If we suppose that e is a radial interior point of C, then if $X \in C$, we get $0 \in \{m \in \mathbb{R} | X + me \in C\}$. If $X \notin C$, since e is a radial interior point of e, $-e + C$ is an absorbing set of L. Then there is some real number $a > 0$ such that $\frac{1}{a}X \in -e + C$. This is equivalent to $X \in -ae + aC \subseteq ae + C$, since C is a wedge. Hence, $X + ae \in C$ and $a \in \{m \in \mathbb{R} | X + me \in C\}$, namely the set $\{m \in \mathbb{R} | X + me \in C\}$ is in any case non-empty. This implies $\rho_C(X) < +\infty$ for any $X \in L$. For the converse direction, if $\rho_C(X) < +\infty$ then $\{m \in \mathbb{R} | X + me \in C\} \neq \emptyset$ for any $X \in L$. Namely, for any $X \in L$ there is some $a \in \mathbb{R}$ such that $X + ae \in C$. If $a > 0$, then $\frac{1}{a}X \in -e + C$. If $a = 0$, then $X + e \in C$ due to the properties of C as a wedge. Then $X \in -e + C$. If $a < 0$, since $e + (-a)e \in C$ due to the same property, $X + e \in C$ and finally $X \in -e + C$ which implies that e is a radial interior point of C.

(ii) If $-e$ is a radial interior point of the complement of C, then $e + C^c$ is an absorbing set, where C^c denotes the complement of C. This is equivalent to the existence for any $X \in L$ of a real number $m_0(X) > 0$ such that $\frac{1}{m_0(X)}X \in e + C^c$. This is equivalent to $X \in m_0(X)e + m_0(X)C^c \subseteq m_0(X)e + C^c$ because C^c is a radial set. We have that $X - m_0(X)e \in C^c$ which means that $X - m_0(X)e \notin C$. But this is equivalent to $\rho_C(X) > -\infty$ for any $X \in L$, because if there was some $X_0 \in L$ such that $\rho_C(X_0) = -\infty$ this would mean that we could find some k with $k < -m_0(X_0)$, where $X_0 + ke \in C$. Then we come to a contradiction since then we would have $X - m_0(X_0)e = X + ke + (-m_0(X) - k)e \in C$ from the well-known properties of a wedge, since $X + ke \in C$ by assumption and $(-m_0(X) - k)e \in C$ since $e \in C$ and $(-m_0(X) - k) > 0$. The proof of the converse implication follows the lines of this proof, but with the opposite direction.

(iii) We have to verify that $\rho_C(0) = 0$ ($\lambda = 0$) if and only if $e \in C, -e \notin C$. The proof about the validity of Positive Homogeneity if $\lambda > 0$ is contained in the next paragraph. If $e \in C, -e \notin C$ and $\rho_C(0) > 0$, we put this number equal to ϵ. 2ϵ is not a lower bound of $\{m \in \mathbb{R} | 0 + me \in C\}$. Hence, there is some m_ϵ lying in this set such that $0 < m_\epsilon < 2\epsilon$. This implies $\rho_C(0) \leq 0$. If we suppose that $\rho_C(0) < 0$, then we put $-\delta = \rho_C(0) < 0$ with $\delta > 0$. Then $0 = \rho_C(0) + \delta$ is not a lower bound of the set $\{m \in \mathbb{R} | 0 + me \in C\}$. Then, there is some $m_\delta < 0$ such that $m_\delta e \in C$, a contradiction since $-e \notin C$. About the converse implication, we have that since $\rho_C(0) = 0$, $e \in C$ because $0 < \rho_C(0) + \epsilon = \epsilon$ for any $\epsilon > 0$. Then there is some $0 < m_\epsilon < \epsilon$ such that $m_\epsilon e \in C$, since $\rho_C(0) + \epsilon$ is not a lower bound of the set $\{m \in \mathbb{R} | 0 + me \in C\}$. Then $e \in \frac{1}{m_\epsilon}C \subseteq C$. But $e \in C$ implies $\rho_C(0) \leq 0$ due to the fact that ϵ can be arbitrarily small. We showed that $\rho_C(0) < 0$ implies $-e \in C$, a contradiction since $\rho_C(0)$ is equal to zero. Hence the converse implication is also true.

If $\lambda > 0$ we remark that

$$\{\lambda m \in \mathbb{R} \mid m \in \mathbb{R} \text{ is such that } X + me \in C\} \subseteq \{k \in \mathbb{R} \mid \lambda X + ke \in C\},$$

which implies the inequality

$$\rho_C(\lambda X) \leq \lambda \rho_C(X). \tag{6.1}$$

We can also see that

$$\left\{\frac{k}{\lambda} \in \mathbb{R} \mid k \in \mathbb{R} \text{ is such that } \lambda X + ke \in C\right\} \subseteq \{m \in \mathbb{R} \mid X + me \in C\}.$$

This last remark implies

$$\rho_C(X) \leq \frac{\rho_C(\lambda X)}{\lambda},$$

and from the above inequality, together with the relation (6.1), the required property is established for any $X \in L$ and any $\lambda \in \mathbb{R}_+$.

(iv) If $m_1 \in \{m \in \mathbb{R} \mid X + me \in C\}$ and $m_2 \in \{m \in \mathbb{R} \mid Y + me \in C\}$ then

$$\lambda m_1 + (1-\lambda)m_2 \in \{k \in \mathbb{R} \mid \lambda X + (1-\lambda)Y + ke \in C\}, \quad \lambda \in [0, 1].$$

This implies

$$\rho_C(\lambda X + (1-\lambda)Y) \leq \lambda m_1 + (1-\lambda)m_2, \quad \lambda \in [0, 1].$$

Hence $\rho_C(\lambda X + (1-\lambda)Y) - \lambda m_1 \leq (1-\lambda)m_2$ which implies

$$\rho_C(\lambda X + (1-\lambda)Y) - \lambda m_1 \leq \rho_C((1-\lambda)Y)$$

for any such m_1. In the same way, by $\rho_C(\lambda X + (1-\lambda)Y) - \rho_C((1-\lambda)Y) \leq \lambda m_1$ we obtain

$$\rho_C(\lambda X + (1-\lambda)Y) - \rho_C((1-\lambda)Y) \leq \rho_C(\lambda X)$$

and the required property holds for any $X, Y \in L$, since ρ_C satisfies the Positive Homogeneity and takes finite values.

(v) If $e \in C$ then the proof of the Translation Invariance of ρ_C with respect to e is the following:

$$\begin{aligned}\rho_C(X + ae) &= \inf\{m \in \mathbb{R} \mid (X + ae) + me \in C\} \\ &= \inf\{(m + a) - a \mid X + (m + a)e \in C\} \\ &= \inf\{k \in \mathbb{R} \mid X + ke \in C\} - a = \rho_C(X) - a,\end{aligned}$$

for any $X \in L$ and any $a \in \mathbb{R}$.

(vi) If $Y \geq X$ with respect to the partial ordering induced on L by C we notice that

$$\{m \in \mathbb{R} \mid X + me \in C\} \subseteq \{m \in \mathbb{R} \mid Y + me \in C\}. \tag{6.2}$$

Indeed, if $m_1 \in \{m \in \mathbb{R} \mid X + me \in C\}$, then

$$X + m_1 e \in C, \tag{6.3}$$

and since $Y \geq X$,

$$Y - X \in C. \tag{6.4}$$

By the relations (6.3) and (6.4) we find that

$$Y + m_1 e = (Y - X) + (X + m_1 e) \in C$$

taking into account the properties of a wedge. Hence $m_1 \in \{m \in \mathbb{R} \mid Y + me \in C\}$ and (6.2) is true. Therefore

$$\rho_C(Y) \leq \rho_C(X),$$

and the required property holds for any $X, Y \in L$. □

Proof of Proposition 2.4. By following the lines of the proof of Proposition 2 in Föllmer and Schied (2002), we have that

$$\begin{aligned}\rho_{\mathcal{A}_\rho}(X) &= \inf\{m \in \mathbb{R} | X + me \in \mathcal{A}_\rho\} \\ &= \inf\{m \in \mathbb{R} | \rho(X) - m \leq 0\} \\ &= \inf\{m \in \mathbb{R} | \rho(X) \leq m\} = \rho(X). \quad \square\end{aligned}$$

Proof of Proposition 2.5. If $\|.\|$ is an order-unit norm, then

$$X + \|X\| e \in L_+. \tag{6.5}$$

Indeed, from the characterization of the infimum of a subset of real numbers, there is a sequence $(m_n)_{n\in\mathbb{N}}$ of real numbers such that $m_n \in \left(\|X\|, \|X\| + \frac{1}{n}\right)$ and

$$X + m_n e \in L_+.$$

Then the sequence $\{X_n = X + m_n e, \ n \in \mathbb{N}\}$ converges weakly to $X + \|X\| e$ and since L_+ is a weakly closed set, (6.5) is true. But since $L_+ \subseteq \mathcal{A}_\rho$,

$$\rho(X) \leq \|X\|,$$

holds, because $\rho = \rho_{\mathcal{A}_\rho}$ from the Proposition 2.4. Hence

$$\rho(X) = \rho(X - Y + Y) \leq \rho(X - Y) + \rho(Y),$$

and

$$\rho(Y) = \rho(Y - X + X) \leq \rho(Y - X) + \rho(X),$$

which implies $|\rho(X) - \rho(Y)| \leq \|X - Y\|$. Therefore ρ is a Lipschitz function being norm-continuous and the acceptance set is closed, since it is the inverse map of a closed set of the real numbers through a continuous function. □

Proof of Theorem 3.1. We see that

$$\rho(X + \rho(X)e) = 0,$$

for any $X \in L$ by the translation invariance property. Hence $X + \rho(X)e \in \mathcal{A}_\rho$, and consequently

$$\pi(X + \rho(X)e) \geq 0$$

for any $\pi \in B$. Then we see for any $X \in L$,

$$\pi(X) + \rho(X) \geq 0$$

for any $\pi \in B$ which implies

$$\rho(X) \geq \pi(-X)$$

for any $\pi \in B$. Thus, we get

$$\rho(X) \geq \sup\{\pi(-X) \mid \pi \in B\}.$$

We are going to show that the inverse inequality is also true. Consider some $\varepsilon > 0$. Then

$$\rho(X + (\rho(X) - \varepsilon)e) = \varepsilon > 0,$$

by the translation invariance property of the coherent risk measures. Hence $X + (\rho(X) - \varepsilon)e$ does not belong to the acceptance set $\mathcal{A}_\rho$ for any X. From the fact that $0 \in \mathcal{A}_\rho$, for any X there is some $\pi_0 \in B$, depending on X and ε such that

$$\pi_0(X + (\rho(X) - \varepsilon)e) = \pi_0(X) + \rho(X) - \varepsilon < 0.$$

Indeed, the existence of this $\pi_0 \neq 0$ is enabled by the Strong Separation theorem for disjoint convex sets in locally convex spaces. The two sets separated are the singleton $\{X + (\rho(X) - \varepsilon)e\}$, being weakly compact, and the acceptance set $\mathcal{A}_\rho$, which is a weakly closed subset of L. Hence, there is some $\pi_0 \in L^*$ with $\pi_0 \neq 0$, some $\delta > 0$ and some $\alpha \in \mathbb{R}$ such that

$$\pi_0(X + (\rho(X) - \varepsilon)e) \leq \alpha < \alpha + \delta \leq \pi_0(Y), \tag{6.6}$$

for any $Y \in \mathcal{A}_\rho$. Then we can observe that π_0 takes positive values on $\mathcal{A}_\rho$.

Indeed, if we suppose that there is some $Y \in \mathcal{A}_\rho$ such that $\pi_0(Y) < 0$ then since $\lambda Y \in \mathcal{A}_\rho$ for any $\lambda > 0$, then for $\lambda \to +\infty$,

$$\pi_0(\lambda Y) \to -\infty,$$

a contradiction since the separation inequalities (6.6) hold. Then $\pi_0 \in \mathcal{A}^0_\rho \subseteq L^0_+$ and $\pi_0 \neq 0$. Hence for this X and this $\varepsilon > 0$, we take that

$$\rho(X) - \varepsilon < \pi_0(-X) \leq \sup\{\pi(-X) \mid \pi \in B\}. \tag{6.7}$$

Now from the fact that the set $\{\pi \in L^0_+ \mid \hat{e}(\pi) = 1\}$ is a base of the cone L^0_+, there is some $\lambda > 0$ such that $\lambda\pi_0 \in B$ and we denote the last vector also by π_0. We remark that the supremum in (6.7) is finite. From the inequalities in (6.7) and for this X if we put

$$\varepsilon_n = \frac{1}{n},$$

for any $n \in \mathbb{N}$, we get

$$\rho(X) - \frac{1}{n} \leq \sup\{\pi(-X) \mid \pi \in B\}$$

and by taking the limit of the sequence

$$\left\{\rho(X) - \frac{1}{n},\ n \in \mathbb{N}\right\}$$

we get the inequality

$$\rho(X) \leq \sup\{\pi(-X) \mid \pi \in B\}. \quad \square$$

Proof of Proposition 3.3. Let us see that $B = \{y \in \mathcal{A}^0_\rho \mid \hat{e}(y) = 1\}$ is a subset of the base B_e of L^0_+ defined by e. Since e is uniformly monotonic, there exists some $b > 0$ such that $e(y) \geq b\|y\|$ for any $y \in L^0_+$. Hence

$$\pi(-X) \leq \|\pi\| \cdot \|-X\| \leq \frac{1}{b}\|-X\| = \frac{1}{b}\|X\|$$

for any $\pi \in B$. Hence

$$\sup\{\pi(-X) \mid \pi \in B\} \leq \frac{1}{b}\|X\|. \quad \square$$

Proof of Lemma 3.4. If $\mathcal{A}^0_\rho \neq \{0\}$ then there is some $\pi_0 \in \mathcal{A}^0_\rho \setminus \{0\}$. Since this π_0 belongs to the cone L^0_+ of L^* there is a unique $\lambda_{\pi_0} > 0$ such that $\lambda_{\pi_0}\pi_0 \in B_e$ (we actually have $\lambda_{\pi_0} = \frac{1}{\pi_0(e)}$). Hence, since $\lambda_{\pi_0}\pi_0 \in \mathcal{A}^0_\rho$ too, because $\mathcal{A}^0_\rho$ is a wedge of L^*, we get that $B_e \cap \mathcal{A}^0_\rho \neq \emptyset$. On the other hand if $B_e \cap \mathcal{A}^0_\rho \neq \emptyset$, then $\mathcal{A}^0_\rho \neq \{0\}$ because if we select some $\pi_1 \in B_e \cap \mathcal{A}^0_\rho$, then $\pi_1 \neq 0$. That is because if $\pi_1 = 0$ we would have $\pi_1(e) = 0$, which is not true since $\pi_1(e) = 1$. $\square$

Proof of Theorem 3.6. If we consider an e-convex risk measure ρ, there exists a penalty function a such that ρ has a representation like the one indicated in (3.4). To see this, we remark that for any $\pi \in B$ we define

$$a(\pi) = \sup\{\pi(-X) - \rho(X) \mid X \in L\}. \tag{6.8}$$

Then as in the proof of Föllmer and Schied (2002, Th. 5), we denote

$$\hat{a}(\pi) = \sup\{\pi(-X) \mid X \in \mathcal{A}_\rho\}. \tag{6.9}$$

We will prove that $a(\pi) = \hat{a}(\pi)$ for any $\pi \in B$. We remark that $a(\pi) \geq \hat{a}(\pi)$. This holds because for any $X \in \mathcal{A}_\rho$, $\pi(-X) - \rho(X) \geq \pi(-X)$. Hence

$$\sup\{\pi(-X) - \rho(X) | X \in L\} \geq \sup\{\pi(-X) - \rho(X) | X \in \mathcal{A}_\rho\} \geq \sup\{\pi(-X) | X \in \mathcal{A}_\rho\}.$$

To prove the inverse, we take $X \in L$ and we consider $X' = X + \rho(X)e \in \mathcal{A}_\rho$. Hence

$$\hat{a}(\pi) \geq \pi(-X') = \pi(-X) - \rho(X),$$

so $a(\pi) = \hat{a}(\pi)$ by taking suprema over all $X \in L$. We remark that $a(\pi) \in (-\infty, +\infty]$ for any $\pi \in B$. Next, we remark that for any $Y \in L$ and by the expression (6.8) of a, we have

$$\rho(Y) \geq \sup\{\pi(-Y) - a(\pi) \mid \pi \in B\}$$

for any $Y \in L$. In order to prove the desired equality, we have the following: Suppose that there is some $Y_0 \in L$, such that

$$\rho(Y_0) > \sup\{\pi(-Y_0) - a(\pi) \mid \pi \in B\}.$$

Hence there exists some $m \in \mathbb{R}$ such that

$$\rho(Y_0) > m > \sup_{\pi \in B}\{\pi(-Y_0) - a(\pi)\}.$$

From the last remark we take that

$$\rho(Y_0 + me) = \rho(Y_0) - m > 0$$

and that $Y_0 + me \notin \mathcal{A}_\rho$. The singleton $\{Y_0 + me\}$ is a convex, weakly compact set and $\mathcal{A}_\rho$ is by assumption a weakly closed set of L which is also convex, since ρ is an e-convex risk measure. Since these two sets are disjoint, from the Strong Separation theorem for convex sets in locally convex spaces there is some $\ell \in L^*$, $\ell \neq 0$, an $\alpha \in \mathbb{R}$ and a $\delta > 0$ such that

$$\ell(Y_0 + me) \geq \alpha + \delta > \alpha \geq \ell(X)$$

for any $X \in \mathcal{A}_\rho$. Hence we take that

$$\ell(Y_0 + me) > \sup\{\ell(X) \mid X \in \mathcal{A}_\rho\}.$$

The functional ℓ takes negative values on L_+ since if there is some $X_0 \in L_+$ such that $\ell(X_0) > 0$, then for any $\lambda \in \mathbb{R}_+$ we take $\lambda X_0 \in L_+ \subseteq \mathcal{A}_\rho$. Then if $\lambda \to +\infty$ then

$$\ell(\lambda X_0) > \ell(Y_0 + me)$$

for λ big enough, being a contradiction according to the previous separation argument. Then since we have that $-\ell \in L_+^0$, $\ell \neq 0$, we may suppose that

$$-\ell(e) = 1$$

holds, or else $-\ell \in B$. Hence the separation of the sets $\{Y_0 + me\}$ and $\mathcal{A}_\rho$ implies that

$$(-\ell)(-Y_0) - m > \sup_{X \in \mathcal{A}_\rho} (-\ell)(-X) = a(-\ell).$$

Denote $-\ell$ by π_0 and we get

$$\pi_0(-Y_0) - a(\pi_0) > m,$$

which is a contradiction, since in this case

$$m > \sup\{\pi(-Y_0) - a(\pi) \mid \pi \in B\} \geq \pi_0(-Y_0) - a(\pi_0) > m.$$

The contradiction was due to the assumption that some $Y_0 \in L$ exists, such that

$$\rho(Y_0) > \sup\{\pi(-Y_0) - a(\pi) \mid \pi \in B\}.$$

Hence for any $Y \in L$ we get (3.4).

For the opposite direction, it suffices to show that any $\rho : L \to \mathbb{R}$, defined through (3.4), is an e-convex risk measure. For this, we have to verify that ρ satisfies the properties of an e-convex risk measure:

(i) (Translation Invariance):

$$\begin{aligned}\rho(X + ke) &= \sup\{\pi(-X - ke) - a(\pi) \mid \pi \in B\} \\ &= \sup\{\pi(-X) - a(\pi) - k\pi(e) \mid \pi \in B\} \\ &= \sup\{\pi(-X) - a(\pi) - k \mid \pi \in B\} \\ &= \sup\{\pi(-X) - a(\pi) \mid \pi \in B\} - k = \rho(X) - k\end{aligned}$$

for any $X \in L$ and any $k \in \mathbb{R}$.

(ii) (Convexity): The function which maps every X to $\pi(-X) - a(\pi)$ for some $\pi \in B$ is a convex real-valued function on L, hence ρ is a convex function on L as the supremum of convex functions defined on L.

(iii) (Monotonicity): If $Y \geq X$ in terms of the partial ordering of L, then $-X \geq -Y$, hence

$$\pi(-X) \geq \pi(-Y)$$

for any $\pi \in B$ since $B \subseteq L_+^0$. Hence

$$\pi(-X) - a(\pi) \geq \pi(-Y) - a(\pi)$$

for any $\pi \in B$ and by taking suprema over the elements of B we get that

$$\begin{aligned}\rho(X) &= \sup_{\pi \in B}\{\pi(-X) - a(\pi)\} \geq \\ \rho(Y) &= \sup_{\pi \in B}\{\pi(-Y) - a(\pi)\}. \quad \square\end{aligned}$$

Proof of Proposition 4.1. Since e is a interior point of L_+, then the cone L_+ is generating. Then for any $X \in L$ there are $X_1, X_2 \in L_+$ such that $X = X_1 - X_2$. Then

$$\begin{aligned}\rho(X) &= \rho(X_1 - X_2) = \sup\{\pi(-X) - a(\pi) \mid \pi \in B\} \\ &= \sup\{\pi(X_2 - X_1) - a(\pi) \mid \pi \in B\}\end{aligned}$$

by the Theorem 3.6. By the properties of suprema (see also in the proof of Theorem 4.5), we get

$$\begin{aligned}\rho(X) &\leq \sup\{\pi(X_2) \mid \pi \in B\} + \sup\{\pi(-X_1) - a(\pi) \mid \pi \in B\} \\ &= \sup\{\pi(X_2) \mid \pi \in B\} + \rho(X_1) \\ &\leq \sup\{\pi(X_2) \mid \pi \in B\}\end{aligned}$$

because $X_1 \in \mathcal{A}_\rho$ since $L_+ \subseteq \mathcal{A}_\rho$, hence $\rho(X_1) \leq \rho(0) \leq 0$. But

$$\pi(X_2) \leq \frac{1}{b}\|X_2\|$$

where $b > 0$ is the real number indicated by the fact that e is a uniformly monotonic functional of L_+^0 and $\hat{e}(f) \geq b\|f\|$ for any $f \in L_+^0$. That is because $\pi(X_2) \leq \|\pi\| \cdot \|X_2\|$ and since $\pi(e) = \hat{e}(\pi) = 1 \geq b\|\pi\|$,

$$\|\pi\| \leq \frac{1}{b}. \quad \square$$

Proof of Corollary 4.2. Directly from Proposition 4.1 and the fact that $X = X^+ - X^-$ for any $X \in L$. $\square$

Proof of Proposition 4.3. By the Corollary 4.2, we have that

$$\rho(X) \leq \frac{1}{b}\|X^-\|$$

for any $X \in L$. Since $X = X^+ - X^-$, $|X| = X^+ + X^-$ for any X, we get as it well known that

$$X^- = \frac{1}{2}(|X| - X)$$

for any X. From well-known properties of the norm,

$$\|X^-\| \leq \frac{1}{2}(\| |X| \| + \|X\|) = \|X\|$$

since $\| |X| \| = \|X\|$ (L is a normed lattice) and the conclusion follows. $\square$

Proof of Proposition 4.4. By the Proposition 4.3 and the subadditivity property of ρ we have

$$\rho(X) - \rho(Y) \leq \rho(X - Y) \leq \frac{1}{b}\|X - Y\|$$

and

$$\rho(Y) - \rho(X) \leq \rho(Y - X) \leq \frac{1}{b}\|X - Y\|$$

which gives

$$|\rho(X) - \rho(Y)| \leq \frac{1}{b}\|X - Y\|. \quad \square$$

Proof of Theorem 4.5. If $\{f_i : L \to \mathbb{R},\ i \in I\}$ is a family of real-valued functions defined on L, then we observe that

$$\sup_{i \in I} f_i(X) - \sup_{i \in I} f_i(Y) \leq \sup_{i \in I}\{f_i(X) - f_i(Y)\},$$

where $X, Y \in L$ (for better interpretation, we may suppose that the family of functions $\{f_i : L \to \overline{\mathbb{R}},\ i \in I\}$ is such that $\sup_{i \in I} f_i(X) \neq \infty$ for any X). Indeed, this holds because if we denote by A the set $\{f_i(X) - f_i(Y) \mid i \in I\}$ and by D the set $\{f_i(Y) \mid i \in I\}$, then

$$\{f_i(X) \mid i \in I\} \subseteq A + D.$$

Hence

$$\sup_{i \in I} f_i(X) \leq \sup_{i \in I}\{f_i(X) - f_i(Y)\} + \sup_{i \in I} f_i(Y).$$

We have seen that ρ has the representation (3.4). Then

$$\rho(X)-\rho(Y)=\sup\{\pi(-X)-a(\pi)\mid \pi\in B\}-\sup\{\pi(-Y)-a(\pi)\mid \pi\in B\}.$$

By the above remark we take that

$$\rho(X)-\rho(Y)\le\sup\{\pi(-X)-\pi(-Y)\mid \pi\in B\}=\sup\{\pi(Y-X)\mid \pi\in B\}.$$

We actually have that $I=B$ and $f_i=f_\pi$ for any $\pi\in B$ where $f_\pi : L\to\overline{\mathbb{R}}$ is such that $f_\pi(X)=\pi(-X)-a(\pi)$ for any $X\in L$. Also, we suppose that $\pi(-X)-a(\pi)-(\pi(-Y)-a(\pi))=\pi(-X)-\pi(-Y)$ for any $X, Y\in L$ and any $\pi\in B$, namely that $a(\pi)-a(\pi)=0$. This is a simple subtraction in the case where $a(\pi)\in\mathbb{R}$, but in the case where $a(\pi)=\infty$ we have the subtraction of two infinite values. But we may suppose that their difference is equal to zero, since we subtract infinities 'of the same form'. On the other hand we may say that if $a(\pi)=\infty$ then $-a(\pi)=-\infty$, hence $\pi(-X)-a(\pi)=\pi(-X)$ because if we add a real number to $-\infty$ we take the real number itself.

In order to complete the proof, we note that

$$\pi(Y-X)\le|\pi(Y-X)|\le\|\pi\|\cdot\|X-Y\|\le\frac{1}{b}\|X-Y\|$$

for any $\pi\in B$, from the definition of B and the fact that $\hat{e}$ is a uniformly monotonic linear functional of L^0_+. Hence

$$\rho(X)-\rho(Y)\le\frac{1}{b}\|X-Y\|$$

and in the same way we may show

$$\rho(Y)-\rho(X)\le\frac{1}{b}\|X-Y\|.$$

The last two inequalities imply that

$$|\rho(X)-\rho(Y)|\le\frac{1}{b}\|X-Y\|$$

and the conclusion is ready. □

Proof of Corollary 4.6. Directly from the Theorem 4.5 and the representability of e-coherent risk measures provided in Theorem 3.1, since in this case the set of representing spot-price functionals is a subset of the base defined by e on L^0_+. □

Proof of Corollary 4.7. About the Lebesgue continuity of ρ, since $X_n\to X$, μ-almost everywhere we deduce that $|X_n-X|^p\to 0$, μ-almost everywhere while since $|X_n|\le Y$, μ-almost everywhere and $|X|\le Y$, μ-almost everywhere and $Y\in L^p$, from the Lebesgue dominated convergence theorem (Aliprantis and Burkinshaw, 1998, Th. 22.11), $X_n\to X$ in terms of the L^p-norm. Since 4.6 indicates that ρ is continuous with respect to the L^p-norm, $\lim_n \rho(X_n)=\rho(X)$ holds. By the same remark, the Fatou continuity of ρ can be established and since $\lim_n\rho(X_n)=\rho(X)$ holds if $X_n\to X$, μ-almost everywhere, the desired property

$$\rho(X)\le\liminf_n\rho(X_n)$$

holds under equality since $\liminf_n\rho(X_n)=\lim_n\rho(X_n)$. For the validity of the continuity from above, we remark that if

$$(X_n)_{n\in\mathbb{N}}\subseteq L^p,\quad p\in(1,\infty)$$

is such that $X_n\downarrow X$, μ-almost everywhere, then $|X_n|\le|X_1|\vee|X|\in L^p$ and by the same argument arising from the Lebesgue dominated convergence theorem, we take that $\lim_n\rho(X_n)=\rho(X)$ holds. Finally, for the validity of the continuity from below, we remark that if

$$(X_n)_{n\in\mathbb{N}}\subseteq L^p,\quad p\in(1,\infty)$$

is such that $X_n\uparrow X$, μ-almost everywhere, we take that $-X_n\downarrow -X$, P-almost everywhere and since $|-X_n|=|X_n|\le|X|\vee|X_1|$ and we repeat the previous argument. Note that L^p ordered by its usual partial ordering is a lattice and this is the property which allows us to deduce the continuity from above and the continuity from below of ρ. □

Proof of Proposition 5.1. Consider a net $(p_\lambda)_{\lambda\in\Lambda}\subseteq L^0_+$ such that

$$p_\lambda\overset{\sigma(L^*,L)}{\to}p.$$

Then $p_\lambda(X)\ge 0$ for any $X\in L_+$. From the weak-star convergence of $(p_\lambda)_{\lambda\in\Lambda}$ we get $p_\lambda(Y)\to p(Y)$ for any $Y\in L$, hence the same holds for any $Y\in L_+$. Hence for any $Y\in L_+$, $p(Y)$ is the limit of a convergent net consisted by positive real numbers. Finally, we get that $p(Y)\ge 0$ for any $Y\in L_+$ which implies $p\in L^0_+$. □

Proof of Proposition 5.2. It suffices to show that $L_+\subseteq L^{00}_+$ and $L^{00}_+\subseteq L_+$.

Let $X\in L_+$. Then $X\in L^{00}_+$ since $L^{00}_+=\{Y\in L^{**}\mid Y(\pi)\ge 0$, for any $\pi\in L^0_+\}$. Since L is reflexive, $L^{00}_+=\{Y\in L\mid\hat{Y}(\pi)\ge 0$, for any $\pi\in L^0_+\}$. As far as $X\in L_+$, then $\hat{X}(\pi)=\pi(X)\ge 0$ for any $\pi\in L^0_+$, and the inclusion $L_+\subseteq L^{00}_+$ is deduced.

For the converse inclusion let us suppose that there were some $Y_0\in L^{00}_+$ for which $Y_0\notin L_+$. Since L_+ is a weakly closed and convex set, while the singleton $\{Y_0\}$ is a weakly compact set, which is also convex. Then, from the Separation theorem for disjoint convex sets in locally convex spaces (see for example Aliprantis and Border, 1999, Th. 5.58), there is some $\pi_0\in L^*$, $\pi\neq 0$ which strongly separates them. In other words, there is some $\alpha\in\mathbb{R}$ and some $\delta>0$ such that

$$\pi_0(Y_0)\le\alpha<\alpha+\delta\le\pi_0(Y)\tag{6.10}$$

for any $Y\in L_+$. Since L_+ is a wedge, π_0 takes positive values on the elements of it, since if we suppose that there is some $Y_1\in L_+$ such that $\pi_0(Y_1)<0$, then if $\lambda\to+\infty$, $\lambda Y_1\in L_+$ and

$$\pi_0(\lambda Y_1)=\lambda\pi_0(Y_1)\to-\infty,$$

a contradiction from the separation argument. Hence, π_0 takes positive values on L_+, thus $\pi_0\in L^0_+$. For $Y=0$ we obtain from (6.10), $\pi_0(Y_0)<0$, while we supposed that $Y_0\in L^{00}_+$, which implies $\hat{Y}_0(\pi_0)=\pi_0(Y_0)\ge 0$. This is a contradiction deduced from the assumption that such a Y_0 exists. Namely, the inverse inclusion $L^{00}_+\subseteq L$ is also true. □

Proof of Lemma 5.3. If $K=L$ then if $y_0\in K^0$, $y_0(x)\ge 0$ for any $x\in L$, which implies $y_0(-x)=-y_0(x)\ge 0$. Then for any $x\in L$, we get $y_0(x)=0$ and by the properties of $\langle L, L^*\rangle$ as a dual pair (see for example Definition 5.79 in Aliprantis and Burkinshaw, 1998) we get $y_0=0$. On the other hand, if K is such that $K^0=\{0\}$, then if we suppose that there is some $x_0\in L\setminus K$, then by applying the Strong Separation theorem for disjoint convex sets in locally convex spaces, we have that $\{x_0\}$ is a compact convex set and K is a closed convex set in L which are disjoint. This implies the existence of some $f\in L^*$, $f\neq 0$ and of some $a\in\mathbb{R}$, $\delta>0$ such that

$$f(x_0)\le a<a+\delta\le f(c),$$

for any $c\in K$. But f takes positive values on K since if there is some c_0 with $f(c_0)<0$ then for $\lambda\to\infty$ we would take that $\lim_{\lambda\to\infty}f(\lambda c_0)=-\infty$, while $\lambda c_0\in K$ since K is a wedge. But in this case the separation condition would be violated and this is a contradiction. Hence $f\in K^0=\{0\}$, a contradiction since $f\neq 0$ by the separation argument. We were led to a contradiction by supposing that a $x_0\in L\setminus K$ exists. Hence $K=L$ and the conclusion is ready. □

7. Conclusions

In this paper we focus on the study of scalar risk measures in infinite-dimensional normed linear spaces. The property of monotonicity which is usually needed in defining such a measure establishes a connection between the study of risk measures and the geometric properties of the positive cone of the partial ordering assumed for the space of risks. This paper establishes some consequences of the existence of norm-interior points in this cone into the relevant theory when the numeraire asset is supposed to be such a point. The origins of those assumptions are two. The first is that such a point defines a base in the dual cone if the space is reflexive and the second is that base is norm-bounded. The first origin implies representation results similar to well-known ones for convex and coherent risk measures, while the second one implies the Lipschitz and other forms of continuity for these risk measures.

References

Aliprantis, C.D., Border, K.C., 1999. Infinite Dimensional Analysis, A Hitchhiker's Guide, second ed. Springer.
Aliprantis, C.D., Burkinshaw, O., 1985. Positive Operators. Academic Press, San Diego, New York.
Aliprantis, C.D., Burkinshaw, O., 1998. Principles of Real Analysis, fifth ed. Academic Press.
Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203–228.
Biagini, S., Fritelli, M., 2009. On the extension of the Namioka–Klee theorem and on the Fatou property for risk measures. In: Optimality and Risk: Modern Trends in Mathematical Finance. Springer, Berlin, Heidelberg, pp. 1–28.
Borwein, J.M., 1987. Automatic continuity of convex relations. Proceedings of the American Mathematical Society 99, 49–55.
Cheridito, P., Li, T., 2009. Risk measures on Orlicz hearts. Mathematical Finance 19, 189–214.
Delbaen, F., 2002. Coherent risk measures on general probability spaces. In: Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann. Springer-Verlag, Berlin, New York, pp. 1–38.
Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance and Stochastics 6, 429–447.
Fritelli, M., Gianin, E., 2002. Putting order in risk measures. Journal of Banking & Finance 26, 1473–1486.
Hamel, A.H., 2006. Translative sets and functions and their applications to risk measure theory and nonlinear separation. IMPA. Preprint D 21/2006.
Herings, P.J.-J., Polemarchakis, H., 2005. Pareto improving price regulation when the asset market is incomplete. Economic Theory 25, 135–154.
Jameson, G., 1970. Ordered Linear Spaces. In: Lecture Notes in Mathematics, vol. 141. Springer-Verlag.
Jaschke, S., Küchler, U., 2001. Coherent risk measures and good-deal bounds. Finance and Stochastics 5, 181–200.
Kaina, M., Rüschendorf, L., 2009. On convex risk measures on L^p-spaces. Mathematical Methods of Operations Research 69, 475–495.
Kreps, D.M., 1981. Arbitrage and equilibrium in economies with infinitely many commodities. Journal of Mathematical Economics 8, 15–35.
Polyrakis, I.A., 2008. Demand functions and reflexivity. Journal of Mathematical Analysis and Applications 338, 695–704.
Stoica, G., 2006. Relevant coherent measures of risk. Journal of Mathematical Economics 42, 794–806.