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The restricted convex risk measures in actuarial solvency

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Abstract In this article, we propose a class of convex risk measures defined on appropriate wedges of a space of financial positions which denote the cumulative surplus variables created by undertaking risks by either an insurance or a reinsurance company. The form of the wedge which is the domain of such a risk measure expresses the form of the company, and it is a subspace in the case of reinsurance companies and a cone in the case of the insurance companies. The value of such a risk measure on an insurance position denotes the capital that the corresponding company has to receive or to keep in advance so that it will not be exposed to risk due to this position. We prove some dual representation and continuity results being similar to the unrestricted case. Finally, we contribute to a decision theory related to the choice of a numeraire asset when the space in which the positions lie in is reflexive.

Keywords Incomplete asset markets · Insurance financial positions · Acceptance set of (re)insurance company · Base of cone · Dual representation of convex risk measures

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1 A financial approach in solvency capital calculation

In this article, we study the application of convex risk measures in insurance practice. We consider a closed interval $[0, T]$ for some constant $T > 0$ representing the time horizon in our model and an infinite set of states of the world Ω. We suppose that all the transactions take place at time 0. The state $\omega \in \Omega$ faced by the investors is contained in some event $A \in \mathcal{F}$, where $\mathcal{F}$ represents some σ-algebra of subsets of Ω, which provides information about the states being available at time T. We denoted by μ the objective probability measure on the measurable space $(\Omega, \mathcal{F})$, and we consider the probability space $(\Omega, \mathcal{F}, \mu)$. The information evolution along time horizon $[0, T]$ is represented by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ of $(\Omega, \mathcal{F}, \mu)$.

We consider an extension of the classical Lundberg risk model of claim arrivals. We may consider an insurance company whose $\mathbb{F}$-adapted capital process $(C_t)_{t \in [0,T]}$ evolves according to the following stochastic differential equation

$$\mathrm{d}C_t = (m(t, C_t) + \phi_t r_t B_t)\mathrm{d}t - \theta_t s(t, C_t)\mathrm{d}W_t^{(1)}, \quad \mu - a.e.,$$

where $C_0 = c$ is some constant, which denotes the initial capital of it, and $W^{(1)}$ is some one-dimensional Brownian motion with respect to $\mathbb{F}$. It is usual in recent literature the capital process $(C_t)_{t \in [0,T]}$ to have a diffusion form (see for example Grandits et al. 2007; Hipp and Schmidli 2004). The set of admissible trading strategies for the investment of the capital of the insurance company to the risky asset is denoted by $\Theta = \{(\theta_t)_{t \in [0,T]}\}$. These processes are adapted to the filtration $\mathbb{F}$ of $(\Omega, \mathcal{F}, \mu)$. Θ includes those processes for which $\theta_t(\omega) = a_t, a_t \in \mathbb{R}, t \in [0, T]$ for any $\omega \in \Omega$. We may suppose that the interest-rate process $(r_t)_{t \in [0,T]}$ of the riskless bond is uniformly bounded or else that $r_t(\omega) \leq M$ for any $(t, \omega) \in [0, T] \times \Omega$. Φ is a set of admissible trading strategies for the riskless bond, while Θ and Φ have the same properties. The integrability properties that the investment strategies $\phi \in \Phi, \theta \in \Theta$ have to satisfy are the following:

$$\int_0^T |\theta_t r_t| \mathrm{d}t < \infty, \mu - a.e.,$$

$$\int_0^T |\phi_t r_t| \mathrm{d}t < \infty, \mu - a.e.,$$

$$\int_0^T \theta_t^2 s^2(t, C_t) \mathrm{d}t < \infty, \mu - a.e.$$

These integrability conditions are implied from Karatzas and Shreve (1998, Def. 2.1, p. 7).

We suppose that the risk model we consider relies on the classical *Lundberg risk model* of claim arrivals. Specifically, the present model is an extension of the model described in Hipp and Schmidli (2004). The process $p_t = \int_0^t m(s, C_s)\mathrm{d}s$ may be interpreted as the *total premium process*, or else it indicates the total premium payments that the insurance company receives till time period t. The process $\{m(t, C_t), t \in [0, T]\}$ is the *premium payment intensity*, and we may suppose that the pure diffusion term $\int_0^t \theta_u s(u, C_u) \mathrm{d}W_u^{(1)}$ corresponds to the payoff of the investment of these received premiums to some risky asset whose volatility term is $s(t, C_t), t \in [0, T]$ with respect to an one-dimensional Brownian motion $W^{(1)}$. The premiums are invested to this asset as soon as they are received by the company. Moreover, the claim process is supposed to be a compound Poisson process $S_t = \sum_{i=1}^{N_t} Y_i, t \in [0, T]$, where $N_t, t \in [0, T]$ is a Poisson process with rate $\lambda > 0$, and claim size random variables $Y_i, i = 1, 2, \ldots$ are positive, i.i.d. with common distribution function M and generic element denoted by Y. Specifically, the generic interoccurrence random time θ is exponentially distributed with mean $\frac{1}{\lambda}$, and the random times $T_n = \sum_{k=1}^{n} \theta_k, , n = 1, 2, \ldots$ of the claims arrival form a renewal counting process $N_t = \mathrm{Card}\{T_n \leq t, n = 1, 2, \ldots\}, t \in [0, T]$. Namely, the random variables θ_k are i.i.d. with an exponential distribution of mean $\frac{1}{\lambda}$. We also suppose that the sequence $\{Y_i, i = 1, 2, \ldots\}$ is independent of $\{N_t, t \in [0, T]\}$. In a more general model, we suppose that the claim process is a sum of an $\mathbb{F}$-adapted diffusion process $Z_t, t \in [0, T]$ and a compound Poisson process $S_t, t \in [0, T]$, described as above. This is the case where the claim process is $A_t = Z_t + S_t, t \in [0, T]$, where the dynamics of Z_t are also described by the stochastic differential equation 1.1, where $W^{(2)}$ is an one-dimensional Brownian motion with respect to an appropriate filtration $\mathbb{F}$ of $(\Omega, \mathcal{F}, \mu)$. If we pose Z to be the zero process, we take the classical Lundberg risk model case and the process equation $A = S$ holds.

Then,

$$\mathcal{X} = \left\{ U \in L^2(\Omega, \mathcal{F}, \mu) | U = \int_0^T (m(t, C_t) + \phi_t r_t B_t) \mathrm{d}t - \int_0^T \theta_t s(t, C_t) \mathrm{d}W_t^{(1)} \right.$$
$$\left. - S_T, \theta \in \Theta, \phi \in \Phi \right\},$$

is the set of the attainable surplus positions for the insurance company at the time period T.

Actually, in this case, $\mathcal{X}$ is the transition of the subspace $\mathcal{U} = \{U \in L^2(\Omega, \mathcal{F}, \mu) | U = \int_0^T \phi_t \mathrm{d}B_t - \int_0^T \theta_t s(t, C_t) \mathrm{d}W_t^{(1)}, \theta \in \Theta, \phi \in \Phi\}$ at the point $\int_0^T m(t, C_t) \mathrm{d}t - S_T$. Namely, the addition on the affine subspace $\mathcal{X}$ is defined as follows: if $\theta_1, \theta_2 \in \Theta, \phi_1, \phi_2 \in \Phi$ then $\theta_1 + \theta_2 \in \Theta, \phi_1 + \phi_2 \in \Phi$. Then, $U_1 = \int_0^T \phi_{1,t} r_t B_t \mathrm{d}t - \int_0^T \theta_{1,t} s(t, C_t) \mathrm{d}W_t^{(1)} + \int_0^T m(t, C_t) \mathrm{d}t - S_T \in \mathcal{X}$ and $U_2 = \int_0^T \phi_{2,t} r_t B_t \mathrm{d}t - \int_0^T \theta_{2,t} s(t, C_t) \mathrm{d}W_t^{(1)} + \int_0^T m(t, C_t) \mathrm{d}t - S_T \in \mathcal{X}$. Hence,

$$U_1 + U_2 = \int_0^T (\phi_{1,t} + \phi_{2,t}) r_t B_t \mathrm{d}t - \int_0^T (\theta_{1,t} + \theta_{2,t}) s(t, C_t) \mathrm{d}W_t^{(1)}$$
$$+ \int_0^T m(t, C_t) \mathrm{d}t - S_T \in \mathcal{X}.$$

Also for the scalar multiplication,

$$a \cdot U_1 = \int_0^T (a \cdot \phi_{1,t}) r_t B_t \mathrm{d}t - \int_0^T (a \cdot \theta_{1,t}) s(t, C_t) \mathrm{d}W_t^{(1)} + \int_0^T m(t, C_t)\mathrm{d}t - S_T \in \mathcal{X},$$

since $a \cdot \theta \in \Theta$, $a \cdot \phi \in \Phi$, for any $a \in \mathbb{R}$. The economic meaning of the definition of the addition and the scalar multiplication over $\mathcal{X}$ also expresses that the surplus is affected by the changes in the trading strategies. Note that since S_T comes from a compound Poisson process, we have that $\mathbb{E}_\mu(S_T) = \lambda T \cdot \mathbb{E}_\mu(Y)$, $V(S_T) = \lambda T \cdot \mathbb{E}_\mu(Y^2)$. Hence, the existence or the non-existence of the moment $\mathbb{E}_\mu(Y^2)$ implies whether $\mathcal{X}$ is a subset of $L^2(\Omega, \mathcal{F}, \mu)$ or not. This relies on the form of the distribution M.

Hence, the reflexive spaces L under which the theorems and propositions of the present article are valid can be set to be equal to $L^2(\Omega, \mathcal{F}, \mu)$ if the moments $\mathbb{E}_\mu(Y^2)$, $\mathbb{E}_\mu(Y)$ of the generic claim size element exist, except those which need a closed wedge of surplus positions N or $\mathcal{U}$, like Proposition 4.4, Theorem 4.5, Theorem 4.7.

In the present article, we mainly consider a model in which the capital of the insurance company represents an Itô process, evolving according to the following stochastic differential equation

$$\mathrm{d}C_t = m(t, C_t)\mathrm{d}t + s(t, C_t)\mathrm{d}W_t^{(1)}\,, \quad \mu - a.e.,$$

where $C_0 = c$ is some constant and $W^{(1)}$ is some one-dimensional Brownian motion with respect to an appropriate filtration $\mathbb{F}$ of $(\Omega, \mathcal{F}, \mu)$. We also may assume the total claim process to be approximated by an Itô process

$$\mathrm{d}Z_t = q(t, Z_t)\mathrm{d}t + \sigma(t, Z_t)\mathrm{d}W_t^{(2)}\,, \quad \mu - a.e., \tag{1.1}$$

where $Z_0 = z$ is the initial total claim and $W^{(2)}$ is some one-dimensional Brownian motion with respect to the same filtration $\mathbb{F}$, being independent from $W^{(1)}$.

Let us assume two periods, the time period 0 when the company subtracts a solvency capital in order to anticipate the future potential claims and the time period T when the total claim should be subtracted from the previous surplus of the company to give the new surplus on this moment. We suppose that during the time period $[0, T]$, the insurance company receives the premiums paid by its clients. At the time period T, the net contribution (surplus) U_T of the insurance company is equal to U_T,

$$U_T = C_T - Z_T,$$

where Z_T represents the claims paid at the period T. Let $u = c - z$ represents the initial surplus at the time period 0. The stochastic process $\{U_t\,,\ t \in [0, T]\}$ of the surplus of the insurance company is an Itô process (by consideration of the Itô Lemma), and its dynamics are demonstrated in the form of a stochastic differential equation:

$$\mathrm{d}U_t = (m(t, U_t + Z_t) - q(t, Z_t))\mathrm{d}t + [s(t, U_t + Z_t), -\sigma(t, Z_t)] \cdot \begin{bmatrix} \mathrm{d}W_t^{(1)} \\ \mathrm{d}W_t^{(2)} \end{bmatrix},$$

$$\mu - a.e.,$$

where $U_0 = u$. With notation of $\mathrm{d}W_t = \begin{bmatrix} \mathrm{d}W_t^{(1)} \\ \mathrm{d}W_t^{(2)} \end{bmatrix}$ for the stochastic differential of the 2-dimensional Brownian motion W ($W^{(1)}, W^{(2)}$ are independent) and $l(t, U_t, Z_t) = m(t, C_t) - q(t, Z_t)$, $t \in [0, T]$, while $v(t, U_t, Z_t) = [s(t, C_t)\,,\ -\sigma(t, Z_t)]\,,\ t \in [0, T]$, the stochastic differential equation for $\{U_t,\ t \in [0, T]\}$ is obtained in the form

$$\mathrm{d}U_t = l(t, U_t, Z_t)\mathrm{d}t + v(t, U_t, Z_t)\mathrm{d}W_t, \quad \mu - a.e.,$$

where $U_0 = u$. For the risk model we propose, we may refer to Norberg (1999) in which both the capital of the insurance company and the total liabilities of the company are modeled as simple diffusion processes. The stochastic process $W = (W^{(1)}, W^{(2)})$ is a 2-dimensional Brownian motion with respect to the filtration $\mathbb{F}^W$ generated by it, which means $\mathbb{F}^W \subseteq \mathbb{F}$. Hence, the surplus process can be considered as a financial asset whose drift is $l(t, Z_t)$ and whose volatility process with respect to the 2-dimensional Brownian motion W is $v(t, Z_t)\,,\ t \in [0, T]$, with realizations as vectors of $\mathbb{R}^2$. It is not restrictive to require

$$v_t(U_t(\omega), Z_t(\omega)) \neq 0\,, \quad \lambda_{[0,T]} \otimes \mu - a.e.,$$

where $\lambda_{[0,T]}$ denotes the Lebesgue measure on $[0, T]$. It implies that

$$\operatorname{rank} v_t(U_t(\omega), Z_t(\omega)) = 1\,, \quad \lambda_{[0,T]} \otimes \mu - a.e.$$

Let us consider a financial market consisted of the surplus asset and a ‘bank account’ (namely, the riskless bond) whose dynamics are driven by the stochastic differential equation

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \quad B_0 = 1, \quad \mu - a.e.,$$

where r denotes the short-term rate process, to conclude (see Øksendal 2000, Th. 12.1.8, Th. 12.2.5) that the market is incomplete. This criterion of incompleteness was established in Karatzas and Shreve (1998, Th. 6.6, p. 24). Hence, the subspace of the attainable European Contingent Claims $\mathcal{U}$ of the specific market is not equal to $L = L^2(\Omega, \mathcal{F}_T^W, Q)$ for any equivalent martingale measure Q of it. Hence, we consider as the space in which the positions $U = U_T$ lie in, any of the spaces $L^2(\Omega, \mathcal{F}_T^W, Q)$ for any equivalent martingale measure Q of the market consisted of the surplus asset and the riskless bond. These spaces are reflexive; therefore, we can refer to the same frame of ordered normed linear spaces, which appear in Konstantinides and Kountzakis (2001), where the main results are proved under the assumption that L is an ordered linear space with a non-empty cone interior. As in Konstantinides and Kountzakis (2001), the order structure of L is given by a closed wedge L_+, which is not necessarily the positive cone L_+^p of the pointwise partial ordering if $L = L^p$. It is in general a wedge whose elements may be financial positions whose outcomes are negative at some of the states of the world. The geometry of the positive wedge reflects the joint beliefs of the investors about the payoffs of these positions. The beliefs of the investors determine the set of the positions which are ‘jointly considered to be the nice investments’. Actually, we mean that the positive cone L_+ is itself an acceptance set in the sense of the definition contained in

Artzner et al. (1999). Moreover, we mention two families of cones with non-empty interior, since the results of the present paper will join this frame. An element of these families may represent the positive cone L_+ of L.

Example 1.1 A family of cones in normed linear spaces having non-empty cone interior are the Bishop–Phelps cones (see in Konstantinides and Kountzakis 2001). The family of these cones in a normed linear space L is the following:

$$K(f, a) = \{x \in L | f(x) \geq a\|x\|\}, \ f \in L^*, \|f\| = 1, a \in (0, 1).$$

An argument for the existence of interior points in these cones is found in Jameson (1970, p. 127).

Example 1.2 Another family of cones with non-empty interior is the family of Henig Dilating cones. These cones are defined as follows: Consider a closed, well-based cone C in the normed linear space L, which has a base B, such that $0 \notin \overline{B + \delta B(0, 1)}$. Let $\delta \in (0, 1)$ be such that

$$2\delta B(0, 1) \cap B = \emptyset,$$

where $B(0, 1)$ denotes the closed unit ball in L. If

$$K_n = \overline{cone}\left(B + \frac{\delta}{n} B(0, 1)\right), \quad n \in \mathbb{N},$$

then $C \subseteq K_{n+1} \subseteq K_n$, $n \in \mathbb{N}$, K_n is a cone for any $n \geq 2$, $C \backslash \{0\} \subseteq int(K_n)$, $n \geq 1$. About these cones, see for example Gong (1994, Lem. 2.1). For example, a Bishop–Phelps cone $C = K(f, a)$ in a reflexive space which is a well-based cone as the construction of the $K_n, n \in \mathbb{N}$ requires, provides a set of interior points $C \backslash \{0\}$ of the cone $K_n, n \geq 1$. If we consider the base $B_f = \{x \in C | f(x) = 1\}$ defined by f, this base is a closed set where $0 \notin B_f$. Hence there is a $g \neq 0, g \in L^*$ such that $g(y) \geq \delta' > 0$ for any $y \in B_f$. g can be selected to be such that $\|g\| = 1$, hence $\|y\| \geq g(y) \geq \delta' > 0$. By setting $\delta' = 2\delta$, we may construct a sequence of approximating cones $K_n, n \in \mathbb{N}$, since we can set $g = f, \delta' = a \in (0, 1)$. We remind that if D is a convex set, then the set $cone(D) = \{x \in L | x = \lambda d, d \in D, \lambda \in \mathbb{R}_+\}$ is a wedge and by $\overline{cone}(D)$ we denote its norm (or weak) closure.

The importance of the order structure of the space also arises by the following remarks, which are related to the financial model and the type of the insurance company. In classical stochastic finance, the debt that the typical investor can tolerate is supposed to be finite, since the *admissible portfolios* that are allowed are such that for the payoff $V^\theta(t, \omega)$ of such a portfolio θ at time period T,

$$V^\theta(t, \omega) \geq -K(\theta),$$

almost surely in terms of the product measure $\lambda \otimes Q$, holds in terms of the usual partial ordering of $L^2(\Omega, \mathcal{F}_T^W, Q)$, where Q is a EMM with $K(\theta) > 0$ (see Øksendal 2000, Def. 12.1.2). This definition comes from the one of *tame* portfolio, given in Karatzas and Shreve (1998, Def. 2.4, p. 9) in which a $\mathbb{F}$-adapted, $\mathbb{R}^N$-valued process θ (where N denotes the number of the assets in the market) is a tame portfolio, if the discounted gains semi-martingale is almost surely bounded from below by a real constant $K(\theta)$. We may pose a similar condition in the case of insurance, if we depart from the usual partial ordering of the spaces L. The debt constraint expresses an acceptable deficit level in terms of both shares of the numeraire position E and the partial ordering of L. We may put a lower debt bound by considering the cone

$$G = \{U \in \mathcal{U} | U \geq_{L_+} -kE\},$$

in L, directly related to the partial ordering defined on L by L_+ and by the numeraire asset E. We suppose that $k > 0$ if such a constraint is active. Hence, we may suppose that the reinsurance companies are those whose restricted risk measure is defined on the entire subspace $\mathcal{U}$, while the primal insurance companies are those whose restricted risk measure is defined on a cone of the form G for a numeraire position E. The numeraire is an interior point of L_+.

We will give an example of a non-tame portfolio whose gains become lower-bounded at $t = T$ when the cone of partial ordering for the space L is substituted by a Bishop–Phelps ordering.

Example 1.3 Let us consider the Bishop–Phelps cone

$$L_+ = K\left(\mathbf{1}, \frac{1}{2}\right),$$

as an ordering cone for L. This cone contains interior points and an interior point is $E = \mathbf{1}$. According to what is mentioned in Jameson (1970, p. 127), the radius of the ball centered at E contained in the cone is $r = 4/3$. To explain it more, $E = \mathbf{1}$ is an element in L such that $\|E\| = 1$ and

$$f(\mathbf{1}) = \langle \mathbf{1}, \mathbf{1} \rangle = 1 > \frac{1}{2}(1 + 2\epsilon),$$

where $\epsilon > 0$. Select $\epsilon = 1/3$ and we get that

$$1 > \frac{1}{2}(1 + 2\epsilon).$$

If $\|y\| \leq 1/6$, then $\|\mathbf{1} + y\| \leq 4/3$ and $f(\mathbf{1} + y) \geq (1 + \epsilon)/2$. This is true, since

$$|f(y)| = |\mathbb{E}_\mu(y)| \leq \|y\|_2 \|\mathbf{1}\|_2 \leq \frac{1}{6},$$

due to Hölder inequality. Then, $1 + \mathbb{E}_\mu(y) \geq 2/3$. Hence,

$$\mathbf{1} + y \in K\left(f, \frac{1}{2}\right).$$

According to the Example in Karatzas and Shreve (1998, p. 9) we may construct a self-financing portfolio θ whose gains process satisfies $V_T^\theta = a$, Q almost surely (we use the probability measure of the space L). We suppose that the market is normalized, namely the interest rate is equal to zero. Then, if there was a constant $k \in \mathbb{R}$ such that

$V_T^\theta \geq_{L_+} kE$, this would mean that $V_T^\theta - kE \in L_+$ or else that

$$\mathbb{E}_Q(V_T^\theta - kE) \geq \frac{1}{2}\sqrt{\mathbb{E}_Q((V_T^\theta - kE)^2)}.$$

The last inequality means that for any a, there is k such that

$$a - k \geq \frac{1}{2}\sqrt{a^2 - 2ka + k^2}.$$

What we specified is that for a non-tame portfolio under a specific partial ordering, there is a constant such that the condition of a tame portfolio for the terminal period $t = T$ is satisfied under another partial ordering.

In the rest of the article, we suppress the notation of the time period T in the notation of the surplus positions U_T.

2 Previous results, origins of the main ideas and content of the article

First, we remind two general definitions in which A is a wedge of L.

Definition 2.1 A real-valued function $\rho : L \to \mathbb{R} \cup \{+\infty\}$ which satisfies the properties

(i) $\rho(x + ae) = \rho(x) - a$ (translation invariance)
(ii) $\rho(\lambda x + (1 - \lambda)x) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for any $\lambda \in [0, 1]$ (convexity) and
(iii) $y \geq_A x$ implies $\rho(y) \leq \rho(x)$ (A-monotonicity)

where $x, y \in L$ is called (A, e)-*convex risk measure*.

Definition 2.2 A real-valued function $\rho : L \to \mathbb{R} \cup \{+\infty\}$ is a (A, e)-*coherent risk measure* if it is an (A, e)-convex risk measure and it satisfies the following property: $\rho(\lambda x) = \lambda\rho(x)$ for any $x \in E$ and any $\lambda \in \mathbb{R}_+$ (positive homogeneity).

In Kaina and Rüschendorf (2009, Th. 2.12), the coherent or convex (see Definition 3.8) risk measures which are defined on a domain which is a proper subset of L^p, $1 \leq p \leq \infty$ are called *restricted*. We have to remind that in Kaina and Rüschendorf (2009), the restricted convex risk measures are studied on cash-invariant subsets of L^p-spaces, where $1 \leq p \leq \infty$. A dual representation theorem is proved there (Kaina and Rüschendorf 2009, Th. 2.12) for $(L_+^p, \mathbf{1})$-convex risk measures. A primal reference for the study of restricted convex risk measures is Filipović and Kupper (2007), in which such risk measures are studied in L^∞ and mainly by means of theory of conjugate convex functions defined on normed linear spaces. A similarity between the frame of Filipović and Kupper (2007) and ours is that authors consider a convex cone $\mathcal{P}$, which implies a partial ordering for a locally convex topological vector space E and a numeraire $\Pi \in E\backslash\{0\}$. The examples of Filipović and Kupper (2007) focus on normed linear spaces and moreover on L^p spaces ordered by their usual (componentwise) partial ordering. In Filipović and Kupper (2007, Lem. 3.5), Filipović and Kupper (2007, Lem. 3.6), the conditions are directly related with the boundedness of the base that the numeraire asset, (see Jameson 1970, Th. 4.4.4). It is valid only in the case where $p = \infty$, because $\mathbf{1}$ is an interior point of L^{∞}_{+}. We define the restricted risk measures on wedges of ordered normed linear spaces containing the numeraire asset E, which may be proper subspaces of reflexive spaces—like the spaces L^p, $1 < p < \infty$. We define the restricted measures on such spaces, because this arises from our financial model (Itô processes correspond to L^2 spaces according to Itô Isometry). As is well known, the $L^p(\Omega, \mathcal{F}, \mu)$-spaces are associated with the existence of the moments $\mathbb{E}_{\mu}(X^p)$ for $1 \leq p < \infty$, where $X : \Omega \to \mathbb{R}$ is a $\mathcal{F}$-measurable random variable. If $X \in L^{\infty}$, all these moments are real numbers. The model of risk measurement would be set in a different mode in which the insurance company examines the effects of its decisions continuously, and it calculates the equivalent solvency capital according to the evolution of its investment portfolio, by using dynamic risk measures (see for example Detlefsen and Scandolo 2005; Roorda et al. 2005; Riedel 2004). But the intention of the present article is to provide a general theory of static, restricted convex risk measures and emphasizing in their application in actuarial cases. However, the time moments 0 and T cannot be very far, because the control of the decision effects of an insurer even if it is set in a static place cannot refers to a time interval which is great enough. Moreover in practice, the insurance companies do not pay their clients as far as the claims arrive, but after a certain time in which they can anticipate the total risk that they adopted by those claims partially by their investment portfolio and partially by the solvency capital they shaped. This however fits the definition of the SCR capital or else the solvency capital requirement as it is defined in Solvency II. This mechanism partially excuses a static frame than a dynamic one, but a dynamic one is however accurate, too.

In general, there is a similarity of the proved results to the ones of the classic papers on unrestricted risk measures [Proposition 3.14 is equivalent to Artzner et al. (1999, Pr. 3.1), while Propositions 3.12, 3.13 follow Föllmer and Schied (2002, Pr. 2), Föllmer and Schied (2002, Pr. 4) in the case of L^{∞}; moreover, in a general partially ordered-space setting the conclusions of Propositions 3.11, 3.12, 3.13 are found in Jaschke and Küchler (2001, pp. 186–187), respectively]. But we have to focus on some interesting points. The duality representation theorem for restricted convex risk measures on wedges (see 3.20) relies both on the strong separation theorem for convex sets in locally convex spaces and on the Hahn–Banach extension theorem for linear functionals, by which we reach the primal Delbaen (2002, Th. 2.3), which refers to the case of coherent risk measures defined on $L^{\infty}(\Omega, \mathcal{F}, \mu)$, which is ordered by its usual partial ordering and the numeraire asset is $\mathbf{1}$, which is an interior point of L^{∞}_{+}. The important is that without the assumption of the existence of interior points in the positive cone L_{+}, which was given from the idea of the application of the Krein–Rutman theorem (Jameson 1970, Cor. 1.6.2) and the extension result (Jameson 1970, Pr. 3.1.8), we would not prove this theorem. Of course, any other proposal is acceptable, and we are waiting for it. Another important point which is also discussed in Konstantinides and Kountzakis (2001) is that the continuity theorem 3.21 follows a direction of proof, which arises from Jameson (1970, Th. 3.8.12) and implies Lipschitz continuity. The choice of the solvency capital scaling corresponds to the choice of an interior point among the variety of the interior points in the cone L_{+}. The condition that at least one interior point exists in L_{+} together with the reflexivity of the spaces $L = L^2(\Omega, \mathcal{F}^W_T, Q)$ implies—according to the Polyrakis dichotomy theorem for the bases of cones (see Polyrakis 2008, Th. 4)—that every strictly positive functional of L^0_+ being defined by an element of L defines a bounded base on L^0_+. Hence, every such point is an interior point of L_+, according to Jameson (1970, Th. 3.8.12). Namely, Polyrakis (2008, Th. 4) indicates an answer to the problem of what kind of cone with non-empty interior arises if we work in some reflexive space and more specifically in some L^p space.

A third important remark is the fact that the reflexive L is ordered by a closed cone L_+ with non-empty interior; then by Polyrakis (2008, Th. 4), this cone has a plenty of interior points, which are the elements of L_+ which define bounded bases on the dual cone L^0_+. In Sect. 6, we develop a numeraire selection theory, which directly relies on the fact that the interior point E we may select for the solvency capital scaling is not unique.

The insurance company determines a reference financial position E, which corresponds to an insurable claim and belongs to an acceptance set $\mathcal{A}$. This set $\mathcal{A}$, namely it demonstrates the exposure limits of risk for this insurance company. If $U \in \mathcal{A}$, then the company can anticipate the risk arising from the variability of U among the states of Ω. $\mathcal{A}$ is a wedge of L, because it should satisfy the main properties of the acceptance sets of the coherent risk measures introduced in Artzner et al. (1999):

(i) $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$
(ii) $\lambda \mathcal{A} \subseteq \mathcal{A}, \quad \lambda \in \mathbb{R}_+$.

We remind that a set $C \subseteq L$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. The coherence of the acceptance set $\mathcal{A}$ is related to the property of monotonicity. In Artzner et al. (1999, Ax. 2.1), a coherent acceptance set is supposed to contain the cone L_+ of the space in which the financial positions lie in. In the case where the risk measure is defined in the whole subspace $\mathcal{U}$, this property can be expressed in the following way:

$$\mathcal{U} \cap L_+ \subseteq \mathcal{A},$$

which indicates that the cone of the induced partial ordering of the subspace $\mathcal{U}$ of L is contained in the acceptance set. The acceptance set $\mathcal{A}$ is not necessarily a subset of $\mathcal{U}$ because the risk exposure limits are rather independent from the specific net contribution process $(U_t, t \in [0, T])$. Also, if we talk about convex risk measures which are not coherent, $\mathcal{A}$ is an unbounded convex subset of L containing L_+.

If D is a subspace of L, this subspace becomes a partially ordered linear space itself by using the order structure of $(L, \geq)$. In this case, we call D an *ordered subspace* of $(L, \geq)$ whose ordering wedge is $C \cap D$. For the rest notions related to the partially ordered linear spaces, the reader may append to the Appendix of Konstantinides and Kountzakis (2001).

If $\rho : \mathcal{U} \to \mathbb{R}$, then the value $\rho(U)$, $U \in \mathcal{U}$ can be understood as the solvency capital for the financial position U under ρ or else as the minimum amount of shares of E that allows U jointly with these shares of E to belong to the acceptance set of ρ,

$$\mathcal{A}_\rho = \{U \in \mathcal{U} | \rho(U) \leq 0\}.$$

The solvency position that the insurance company needs in order to be insured itself is $\rho(U)E$. But since the solvency capital of the numeraire position E can be normalized according to a price-functional $\pi \in L^*$, which is normalized at E ($\pi(E) = 1$). Then the solvency capital of E with respect to ρ is equal to -1. By the same way, the number of shares $\rho(U)$ also indicates also the amount of money that the company which is characterized by risk exposure limit set $\mathcal{A}_\rho$ has to keep in advance as a premium for the insurance toward U. The premium calculation for the surplus position U is more meaningful when $U \notin \mathcal{A}$. In this case, the insurance company, in order to insure its surplus U, needs a positive number of shares $\rho(U)$ of the numeraire asset E in order to be induced with respect to ρ.

We notice that according to the description of the model, the initial surplus u of the insurance company becomes $u - \rho(U)$, since the company keeps the solvency capital in advance, so that it can anticipate the risk against the surplus position U. For these reasons, in relevant optimization problems, the constraint $\rho(U) \leq u$ is active. But such problems are out of the content of this article. $\rho(U)$ is the optimal payment of the company for this scope, as it is indicated by Propositions 3.12, 3.13, where $N = \mathcal{U}$.

A point of discussion is whether the value ∞ must be taken to be a possible value of a ρ for the actuarial case. In general, this is realistic, since there are catastrophic claims. But the results that follow are valid without this assumption, which does not fit the form of the diffusion model we described. Moreover, the implications of the Propositions 3.11, 3.12 verify this fact.

3 Restricted convex risk measures defined on wedges

Restrictions in cash invariance or actually in numeraire invariance may appear in the attempt of the insurance company to draw the limits of its risk exposure. The sorts of invariance are described in the following definitions.

Definition 3.1 The subset $N \subseteq L$ is **E-translation invariant** (**cash translation invariant**) if $E \in N$ ($\mathbf{1} \in N$) and for any $U \in N$ and any $n \in \mathbb{R}$, $U + nE \in N$ ($U + n\mathbf{1} \in N$).

Definition 3.2 The subset $N \subseteq L$ is **weak E-translation invariant** (**weak cash translation invariant**) if there is some $t \in \mathbb{R}$ such that for any $U \in N$, $U + \lambda E \in N$ ($U + \lambda\mathbf{1} \in N$), where $\lambda \geq t$.

The sets of particular interest are the wedges of L, since the domain of a risk measure used for insurance purposes is a wedge (it is either a subspace in the case of a reinsurance company or a cone in the case of a primal insurance company), and the equivalent terminology **restricted coherent** or **restricted convex** will be used in the case of a wedge N containing $\mathbf{1}$ or E respectively, since a wedge N satisfying this property is weak $\mathbf{1}$ (E)-translation invariant.

Lemma 3.3 *A wedge N of L is weak E-translation invariant with respect to any element E of it.*

Proof If $U \in N$, then for any $t \geq 0$, $U + tE \in N$. Hence, N is weak E-translation invariant. □

The definition of translation invariance for a risk measure defined on a wedge N takes a more general form

Definition 3.4 Let N be a wedge of L and ρ be a function $\rho : N \to \mathbb{R}$. ρ is E-translation invariant if

$$\rho(U + aE) = \rho(U) - a,$$

for any $U \in N$ and any $a \in \mathbb{R}$ such that $U + aE \in N$.

Definition 3.5 Let N be a wedge of L, ρ be a function $\rho : N \to \mathbb{R}$, and L_+ be a cone of L. Then, ρ is L_+-monotone (or equivalently it satisfies the L_+ -monotonicity), if the inclusion

$$N \cap L_+ \subseteq \mathcal{A}_\rho,$$

holds.

Definition 3.6 Let N be a wedge of L and the function $\rho : N \to \mathbb{R}$ satisfy the properties of (L_+, E)-translation invariance, sub-additivity, positive homogeneity and L_+-monotonicity. Then, we call it (L_+, E)-**restricted coherent**.

Definition 3.7 A risk measure $\rho : N \to \mathbb{R}$ satisfies the convexity property if it is a convex function

$$\rho(\lambda U_1 + (1 - \lambda)U_2) \leq \lambda\rho(U_1) + (1 - \lambda)\rho(U_2),$$

for any $\lambda \in [0, 1]$ and $U_1, U_2 \in N$.

Definition 3.8 If N is a wedge of the ordered linear space L and a function $\rho : N \to \mathbb{R}$ satisfies the properties of (L_+, E)-translation invariance, convexity and L_+-monotonicity, then it is called (L_+, E)-**restricted convex**.

The rest results concerning the primal insurance and the reinsurance companies' risk measures are simple implications of the following results.

Remark 3.9 The main condition under which the following results hold is that N is a (closed) wedge of an ordered normed space L (which is reflexive if it is required and equal to some appropriately defined L^2-space, too) such that $N \cap L_+$ is a non-empty wedge of L containing the numeraire E.

Definition 3.10 The risk measure $\rho^N_{(\mathcal{A},E)} : N \to \overline{\mathbb{R}}$ defined on a wedge N of L is defined as follows:

$$\rho^N_{(\mathcal{A},E)}(U) = \inf\{m \in \mathbb{R} | mE + U \in \mathcal{A}\},$$

where E is supposed to be an interior point of the positive cone L_+ of L. We call the above risk measure N-**restricted risk measure associated with the pair** $(\mathcal{A}, E)$.

Proposition 3.11 *If E is an interior point of the cone L_+, then $\rho^N_{(\mathcal{A},E)}$ is a (L_+, E)-restricted coherent risk measure.*

Proof We have to verify that the properties of a coherent risk measure hold for $\rho^N_{(\mathcal{A},E)}$. Note that according to Jameson (1970, Pr. 3.1.3), E is an order unit of L. Further, for any $U \in N$, there is some $m \in \mathbb{R}$ such that $U + mE \in L_+$. That is because

$$L = \cup_{n=1}^{\infty}[-nE\ ,\ nE],$$

where $[\,.\,,\,.\,]$ denotes the corresponding order interval with respect to the partial ordering induced on L by L_+. Hence, for some $n(U) \in \mathbb{N}$,

$$U \in [-n(U)E\ ,\ n(U)E],$$

which implies $U + n(U)E \in L_+$ for any $U \in L$. The set $\{m \in \mathbb{R} \,|\, U + mE \in \mathcal{A}\}$ is non-empty for any $U \in L$ and hence for any $U \in N$. This is true because if $U \in N$,

$$U + n(U)E \in N \cap L_+ \subseteq \mathcal{A},$$

which indicates that $\rho^N_{(\mathcal{A},E)}(U) \neq \infty$ for any $U \in N$.

In order to show that

$$\rho^N_{(\mathcal{A},E)}(U) \neq -\infty,$$

for every $U \in N$, we have to show that for any $U \in N$, the set of real numbers $\{m \in \mathbb{R} \,|\, mE + U \in \mathcal{A}\}$ is lower-bounded. Since $E \in \text{int}L_+$, this implies that E is an order unit of L. As we mentioned before, this implies that for any $U \in L$ there is some $n(U) \in \mathbb{N}$ such that

$$U \in [-n(U)E\ ,\ n(U)E],$$

where $[\,.\,,\,.\,]$ denotes an order interval with respect to the partial ordering induced by L_+ on L. But, on the other hand, for any $U \in L$ there is some nonzero $m_0(U) \in \mathbb{R}_+$ such that

$$U - m_0(U)E \notin L_+.$$

If $U = 0$ this m_0 is any $\varepsilon > 0$. If $U \neq 0$ we suppose that such a positive real number does not exist, so we would have for any $\lambda > 0$, that

$$U - \lambda E \in L_+.$$

Since $E \in \text{int}L_+$, then

$$L = \cup_{n=1}^{\infty}[-nE\ ,\ nE].$$

Suppose that $n(U) \in \mathbb{N}$ is some natural number such that

$$U \in [-n(U)E\ ,\ n(U)E].$$

Since $U \geq n(U)E$ by assumption, we get that

$$n(U)E \leq U \leq n(U)E,$$

in terms of the partial ordering induced by L_+ on L and because L_+ is a cone, it implies

$$U = n(U)E.$$

But $U \in [-(n(U)+1)E\ ,\ (n(U)+1)E]$ and by the same assumption

$$U \geq (n(U)+1)E,$$

which implies by the same way that

$$U = (n(U)+1)E,$$

which is a contradiction because if this is the case, we find $U = 0$. Hence, for any $U \neq 0$, there exists some nonzero $m_0(U) \in \mathbb{R}_+$ such that

$$U - m_0(U)E \notin L_+.$$

After all, we remark that $-m_0(U)$ is a lower bound for the set $\{m \in \mathbb{R} \mid mE+U \in L_+\}$, for any $U \in N$, too. $-m_0(U)$ is a lower bound of the set $\{m \in \mathbb{R} | U + mE \in \mathcal{A}\}$ because if we suppose that there are some $Z_0 \in N$ and $k < -m_0(Z_0)$ such that

$$kE + Z_0 \in \mathcal{A},$$

then we come to a contradiction since then we would have

$$Z_0 - m_0(Z_0)E = Z_0 + kE + (-m_0(Z_0) - k)E \in \mathcal{A}$$

from the well-known properties of a wedge, since $Z_0 + kE \in \mathcal{A}$ by assumption and

$$(-m_0(U) - k)E \in N \cap L_+ \subseteq \mathcal{A},$$

(notice that $(-m_0(U) - k) > 0$). Finally, we found that $\rho^N_{(\mathcal{A},E)}$ cannot take the value $-\infty$.

About the properties of a coherent risk measure, we have the following:

(i) (E-translation invariance):

$$\begin{aligned}\rho^N_{(\mathcal{A},E)}(U + aE) &= \inf\{m \in \mathbb{R} \mid (U + aE) + mE \in \mathcal{A}\}\\ &= \inf\{(m + a) - a \mid U + (m + a)E \in \mathcal{A}\}\\ &= \inf\{k \in \mathbb{R} \mid U + kE \in \mathcal{A}\} - a = \rho^N_{(\mathcal{A},E)}(U) - a,\end{aligned}$$

for any $U \in N$ and any $a \in \mathbb{R}$ such that $U + aE \in N$.

(ii) (Sub-additivity):
If $m_1 \in \{m \in \mathbb{R} \mid U + mE \in \mathcal{A}\}$ and $m_2 \in \{m \in \mathbb{R} \mid U' + mE \in \mathcal{A}\}$ then

$$m_1 + m_2 \in \{k \in \mathbb{R} \mid (U + U') + kE \in \mathcal{A}\}.$$

It means

$$\rho^N_{(\mathcal{A},E)}(U + U') \leq m_1 + m_2.$$

Hence $\rho^N_{(\mathcal{A},E)}(U+U')-m_1 \le m_2$ which implies

$$\rho^N_{(\mathcal{A},E)}(U+U')-m_1 \le \rho^N_{(\mathcal{A},E)}(U')$$

for any such m_1. In the same way, by $\rho^N_{(\mathcal{A},E)}(U+U')-\rho^N_{(\mathcal{A},E)}(U') \le m_1$ we obtain

$$\rho^N_{(\mathcal{A},E)}(U+U')-\rho^N_{(\mathcal{A},E)}(U') \le \rho^N_{(\mathcal{A},E)}(U)$$

and the required property holds for any $U, U' \in N$.
The above proof of sub-additivity holds in case where $U, U' \in N$ are such that $\rho^N_{(\mathcal{A},E)}(U+U'),\ \rho^N_{(\mathcal{A},E)}(U),\ \rho^N_{(\mathcal{A},E)}(U') \in \mathbb{R}$. If

$$\rho^N_{(\mathcal{A},E)}(U+U') = -\infty,$$

and at least one of $\rho^N_{(\mathcal{A},E)}(U)$, $\rho^N_{(\mathcal{A},E)}(U')$ is equal to $-\infty$ the sub-additivity holds. Note that if $\rho^N_{(\mathcal{A},E)}(U+U') = -\infty$ and $\rho^N_{(\mathcal{A},E)}(U), \rho^N_{(\mathcal{A},E)}(U') \in \mathbb{R}$ the sub-additivity property is true.
Finally, we note that if for example

$$\rho^N_{(\mathcal{A},E)}(U) = -\infty,$$

and $\rho^N_{(\mathcal{A},E)}(U') \in \mathbb{R}$, then $\rho^N_{(\mathcal{A},E)}(U+U') = -\infty$. This is true since $\rho^N_{(\mathcal{A},E)}(U) = -\infty$ there is a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, such that $a_n \to -\infty$ and

$$U + a_n E \in \mathcal{A}.$$

Also, since $\rho^N_{(\mathcal{A},E)}(U') \in \mathbb{R}$ consider some $a \in \mathbb{R}$ with $U' + aE \in \mathcal{A}$. Then since for the sequence $(d_n)_{n\in\mathbb{N}}$ with $d_n = a_n + a$ for any $n \in \mathbb{N}$

$$(U+U') + (a_n + a)E \in \mathcal{A},$$

from the property $\mathcal{A}+\mathcal{A} \subseteq \mathcal{A}$ of $\mathcal{A}$, while $\lim_n (a_n + a) = -\infty$, we get $\rho^N_{(\mathcal{A},E)}(U+U') = -\infty$. In case where $\mathcal{A}$ is a cone, this point may be omitted since $\rho^N_{(\mathcal{A},E)}$ takes only finite values.

(iii) (Positive Homogeneity):
For $\lambda = 0$ we have that

$$\rho^N_{(\mathcal{A},E)}(0) \le 0,$$

since $0 \in \{m \in \mathbb{R} \mid 0 + mE \in \mathcal{A}\}$. If we suppose that $\rho^N_{(\mathcal{A},E)}(0) < 0$, then for

$$\delta = -\rho^N_{(\mathcal{A},E)}(0),$$

and by the definition of $\rho^N_{(\mathcal{A},E)}(0)$ as the infimum of a subset of the real numbers, there is some $m_\delta \in \{m \in \mathbb{R} \mid 0 + mE \in \mathcal{A}\}$ such that

$$m_\delta < \rho^N_{(\mathcal{A},E)}(0) + \delta = 0.$$

Hence, $m_\delta E \in \mathcal{A}$ with $m_\delta < 0$. But from the properties of a wedge, we have that $m_\delta E \in -\mathcal{A}$. Then,

$$E \in \mathcal{A} \cap (-\mathcal{A}) = \{0\}$$

because $\mathcal{A}$ is a cone, a contradiction since E is an interior point of L_+. The contradiction is implied by the assumption that $\rho^N_{(\mathcal{A},E)}(0) < 0$. This implies

$$\rho^N_{(\mathcal{A},E)}(0) = 0,$$

and the positive homogeneity if $\lambda = 0$ holds.
If $\lambda > 0$ we remark that

$$\{\lambda m \in \mathbb{R} \,|\, m \in \mathbb{R} \text{ is such that } U + mE \in \mathcal{A}\} \subseteq \{k \in \mathbb{R} \,|\, \lambda U + kE \in \mathcal{A}\},$$

which implies the inequality

$$\rho^N_{(\mathcal{A},E)}(\lambda U) \leq \lambda \rho^N_{(\mathcal{A},E)}(U). \tag{3.1}$$

We can see that

$$\left\{\frac{k}{\lambda} \in \mathbb{R} \,|\, k \in \mathbb{R} \text{ is such that } \lambda U + kE \in \mathcal{A}\right\} \subseteq \{m \in \mathbb{R} \,|\, U + mE \in \mathcal{A}\}.$$

This last remark implies

$$\rho^N_{(\mathcal{A},E)}(U) \leq \frac{\rho^N_{(\mathcal{A},E)}(\lambda U)}{\lambda},$$

and from the above inequality, together with the relation (3.1), the required property is established for any $U \in N$ and any $\lambda \in \mathbb{R}_+$.

(iv) (L_+-Monotonicity):
If $U, U' \in N$ and $U' \geq U$ with respect to the partial ordering induced on the elements of N by L_+, we notice that

$$\{m \in \mathbb{R} \,|\, U + mE \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \,|\, U' + mE \in \mathcal{A}\}. \tag{3.2}$$

Indeed, if $m_1 \in \{m \in \mathbb{R} \,|\, U + mE \in \mathcal{A}\}$, then

$$U + m_1 E \in \mathcal{A}, \tag{3.3}$$

and since $U' \geq U$,

$$U' - U \in N \cap L_+ \subseteq \mathcal{A}. \tag{3.4}$$

By the relations (3.3) and (3.4), we find that

$$U' + m_1 E = (U' - U) + (U + m_1 E) \in \mathcal{A}$$

by the properties of a wedge. Hence, $m_1 \in \{m \in \mathbb{R} \,|\, U' + mE \in \mathcal{A}\}$ and the (3.2) is true. Therefore

$$\rho^N_{(\mathcal{A},E)}(U') \leq \rho^N_{(\mathcal{A},E)}(U),$$

and the required property holds for any $U, U' \in N$. $\square$

Proposition 3.12 *If N is a wedge of L and $\rho : N \rightarrow \mathbb{R}$ is a (L_+, E)-restricted coherent risk measure, then*

$$\rho^N_{(\mathcal{A}_\rho, E)} = \rho.$$

Proof By following the lines of the proof of Föllmer and Schied (2002, Pr. 2), we have that

$$\begin{aligned}\rho^N_{(\mathcal{A}_\rho, E)}(U) &= \inf\{m \in \mathbb{R} | U + mE \in \mathcal{A}_\rho\} \\ &= \inf\{m \in \mathbb{R} | \rho(U) - m \leq 0\} = \inf\{m \in \mathbb{R} | \rho(U) \leq m\} \\ &= \rho(U), U \in N.\end{aligned}$$

□

Proposition 3.13 *If N is a wedge of L and the wedge $\mathcal{A}$ is a closed sub-wedge of N, then $\mathcal{A}_{\rho^N_{(\mathcal{A},E)}} = \mathcal{A}$.*

Proof Obviously, $\mathcal{A} \subseteq \mathcal{A}_{\rho^N_{(\mathcal{A},E)}}$ because $0 \in \{m \in \mathbb{R} | mE + U \in \mathcal{A}\}$ if $U \in \mathcal{A}$, hence

$$\rho^N_{(\mathcal{A},E)}(U) \leq 0,$$

and $U \in \mathcal{A}_{\rho^N_{(\mathcal{A},E)}}$. Suppose there is some $Z_0 \in \mathcal{A}_{\rho^N_{(\mathcal{A},E)}} \setminus \mathcal{A}$. Then, $\rho^N_{(\mathcal{A},E)}(Z_0) \leq 0$. If $\rho^N_{(\mathcal{A},E)}(Z_0) = 0$, then for any $\epsilon > 0$, there is some $m_\epsilon > 0$ with $m_\epsilon < 0 + \epsilon = \epsilon$ such that $m_\epsilon E + Z_0 \in \mathcal{A}$. If we put $\epsilon_n = 1/n$ for any $n \in \mathbb{N}$, we take a sequence $(m_n)_{n \in \mathbb{N}}$ of real numbers, such that $m_n E + Z_0 \in \mathcal{A}$ and $0 < m_n < 1/n$. The limit of the last sequence of elements of $\mathcal{A}$ is Z_0 and since $\mathcal{A}$ is closed, $Z_0 \in \mathcal{A}$. But this is a contradiction, since we supposed that Z_0 is not an element of $\mathcal{A}$. Then, we conclude that such a Z_0 does not exist in case where $\rho^N_{(\mathcal{A},E)}(Z_0) = 0$. If $\rho^N_{(\mathcal{A},E)}(Z_0) < 0$, then for $\epsilon_0 = -\rho^N_{(\mathcal{A},E)}(Z_0)$ we have that there is some

$$m_{\epsilon_0} < \rho^N_{(\mathcal{A},E)}(Z_0) + \epsilon_0 = 0,$$

such that $m_{\epsilon_0} E + Z_0 \in \mathcal{A}$. Hence, $Z_0 \in \mathcal{A} - m_{\epsilon_0} E \subseteq \mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$. But this is a contradiction, since we supposed that Z_0 is not an element of $\mathcal{A}$. Then, we conclude that such a Z_0 does not exist in case where $\rho^N_{(\mathcal{A},E)}(Z_0) < 0$. Hence, $\mathcal{A}_{\rho^N_{(\mathcal{A},E)}} = \mathcal{A}$ is true. □

A last proposition that can be proved for restricted (L_+, E)-convex risk measures defined on a wedge N is the following:

Proposition 3.14 *If N is a wedge of some ordered normed space L with non-empty cone interior and*

$$\rho_P(U) = \sup\{\pi(-U) | \pi \in P\},$$

where $P \subseteq \{\pi \in \mathcal{A}^0_\rho | \pi(E) = 1\}$ then ρ_P is a (L_+, E)-restricted coherent risk measure.

Proof Let us check the four properties of the coherent measure.

(i) (E-translation invariance):

$$\begin{aligned}\rho_P(U+aE) &= \sup\{\pi(-U-aE)\,|\,\pi\in P\} = \sup\{\pi(-U)-a\pi(E)\,|\,\pi\in P\}\\ &= \sup\{\pi(-U)-a\,|\,\pi\in P\} = \sup\{\pi(-U)\,|\,\pi\in P\}-a\\ &= \rho_P(U)-a,\end{aligned}$$

for any $U\in N$ and any $a\in\mathbb{R}$ such that $U+aE\in N$.

(ii) (Sub-additivity):

$$\begin{aligned}\rho_P(U+U') &= \sup\{\pi(-U-U')\,|\,\pi\in P\} = \sup\{\pi(-U)+\pi(-U')\,|\,\pi\in P\}\\ &\le \sup\{\pi(-U)\,|\,\pi\in P\}+\sup\{\pi(-U')\,|\,\pi\in P\}\\ &= \rho_P(U)+\rho_P(U'),\end{aligned}$$

for any $U, U'\in\mathcal{U}$. $U+U'\in N$ in this case, because N is a wedge.

(iii) (Positive homogeneity):

$$\begin{aligned}\rho_P(\lambda U) &= \sup\{\pi(-\lambda U)\,|\,\pi\in P\} = \sup\{\lambda\pi(-U)\,|\,\pi\in P\}\\ &= \lambda\sup\{\pi(-U)\,|\,\pi\in P\} = \lambda\rho_P(U)\end{aligned}$$

for any $U\in N$ and any $\lambda\in\mathbb{R}_+$. In this case, $\lambda U\in N$ because N is a wedge.

(iv) (L_+-monotonicity): If $U'\ge U$ in the partial ordering of L, then $-U\ge -U'$, hence

$$\pi(-U)\ge\pi(-U')$$

for any $\pi\in P$. Taking supremums over P we get that

$$\rho_P(U)=\sup_{\pi\in P}\pi(-U)\ge\sup_{\pi\in P}\pi(-U')=\rho_P(U').$$

□

3.1 Dual representation and Lipschitz continuity

The following theorem relies on Hahn–Banach extension theorem for linear functionals defined on subspaces of linear spaces.

Theorem 3.15 (Krein–Rutman) (Jameson 1970, Cor. 1.6.2) *Let L be an ordered linear space and J be a linear subspace of it containing an order unit of L. Then a positive linear functional defined on J has a positive extension to L.*

The following proposition assures the continuity of this extension.

Proposition 3.16 (Jameson 1970, Pr. 3.1.8) *Let L be an ordered locally convex space with positive wedge P and suppose that $E\in \mathrm{int}P$. Let J be a linear subspace containing E and f a positive linear functional defined on J. Then f has a continuous, positive extension to L.*

The above theorem and proposition are crucial for the similarity between the proof of the theorem of dual representation for convex risk measures in case where the domain is the whole reflexive space L and the cone interior $\text{int}L_+$ is non-empty (like the ones studied in Konstantinides and Kountzakis 2001) and the restricted ones studied in this article. The results in Konstantinides and Kountzakis (2001) and other related articles cannot be proved in the restricted case, because the separation theorem for convex sets cannot be applied in this case. Thus, we assure that every positive linear functional has a continuous extension all over L, by the previous theorem and proposition.

For the application of the above theorem and the above proposition, we need the following:

Lemma 3.17 *If D is a wedge of L then $D - D$ is a subspace of L.*

Proof It suffices to verify that the set $D - D$ is closed under the addition of vectors and the scalar multiplication. If $y_1, y_2 \in D - D$, then $y_1 = x_1 - x_2$, $y_2 = x_3 - x_4$, where $x_1, x_2, x_3, x_4 \in D$. Then

$$y_1 + y_2 = (x_1 + x_3) - (x_2 + x_4),$$

and by the properties of D as a wedge, $x_1 + x_3 \in D$, $x_2 + x_4 \in D$. Also, if $\lambda \geq 0$, $\lambda y_1 = \lambda x_1 - \lambda x_2$ and by the properties of D as a wedge, $\lambda x_1 \in D$, $\lambda x_2 \in D$. Finally, if $\lambda < 0$, $\lambda y_1 = (-\lambda)x_2 - (-\lambda)x_1$ and by the properties of D as a wedge, $(-\lambda)x_1 \in D$, $(-\lambda)x_2 \in D$. □

Lemma 3.18 *The set $D = N \cap L_+$ is a cone of L.*

Proof Obviously, $D = N \cap L_+$ is a wedge. In order to prove that it is a cone, it suffices to prove that

$$D \cap (-D) = \{0\}.$$

Suppose that $n \in D \cap (-D)$. Then $n \in L_+$, $n \in (-L_+)$. Hence, $n \in L_+ \cap (-L_+) = \{0\}$. □

Corollary 3.19 *The set $N \cap L_+ - N \cap L_+$ is a subspace of L.*

Theorem 3.20 *If N is a closed wedge in a reflexive ordered normed space L with non-empty cone interior containing the L_+-interior point E, $\rho : N \to \mathbb{R}$ is a (L_+, E)-restricted convex risk measure with $\sigma(L, L^*)$-closed acceptance set, then*

$$\rho(U) = \sup\{\pi(-U) - a(\pi) \mid \pi \in B_E\}, \tag{3.5}$$

for any $U \in N$, where $B_E = \{y^* \in L_+^0 \mid \hat{E}(y^*) = 1\}$ and $a : B_E \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_E$. On the other hand, every ρ defined through (3.5) is a (L_+, E)-restricted convex risk measure.

Proof If we consider a (L_+, E)-restricted convex risk measure ρ, there exists a penalty function a such that ρ has a representation like the one indicated in (3.5). To see this, we remark that for any $\pi \in B_E$ we define

$$a(\pi) = \sup\{\pi(-U) - \rho(U) \mid U \in N\}. \tag{3.6}$$

Then as in the proof of Föllmer and Schied (2002, Th. 5), we denote

$$\hat{a}(\pi) = \sup\{\pi(-U) \mid U \in \mathcal{A}_\rho\}. \tag{3.7}$$

We will prove that $a(\pi) = \hat{a}(\pi)$ for any $\pi \in B_E$. We remark that $a(\pi) \geq \hat{a}(\pi)$. This holds because for any $U \in \mathcal{A}_\rho$, $\pi(-U) - \rho(U) \geq \pi(-U)$. Hence,

$$\begin{aligned}\sup\{\pi(-U) - \rho(U) | U \in \mathcal{U}\} &\geq \sup\{\pi(-U) - \rho(U) | U \in \mathcal{A}_\rho\} \\ &\geq \sup\{\pi(-U) | U \in \mathcal{A}_\rho\}.\end{aligned}$$

To prove the inverse, we take $U \in N$ and we consider $U' = U + \rho(U)E \in \mathcal{A}_\rho$. Hence,

$$\hat{a}(\pi) \geq \pi(-U') = \pi(-U) - \rho(U),$$

so $a(\pi) = \hat{a}(\pi)$ by taking supremums over all $U \in N$. We remark that $a(\pi) \in (-\infty, +\infty]$ for any $\pi \in B_E$. Next, we remark that for any $U \in N$ and by the expression (3.6) of a, we have

$$\rho(U) \geq \sup\{\pi(-U) - a(\pi) \mid \pi \in B_E\}$$

for any $U \in N$. In order to prove the desired equality, we have the following: Suppose that there is some $U_0 \in N$, such that

$$\rho(U_0) > \sup\{\pi(-U_0) - a(\pi) \mid \pi \in B_E\}.$$

Hence, there exists some $m \in \mathbb{R}$ such that

$$\rho(U_0) > m > \sup_{\pi \in B_E} \{\pi(-U_0) - a(\pi)\}.$$

From the last remark, we take that

$$\rho(U_0 + mE) = \rho(U_0) - m > 0$$

and that $U_0 + mE \notin \mathcal{A}_\rho$.

The singleton $\{U_0 + mE\}$ is a convex, weakly compact set and $\mathcal{A}_\rho$ is by assumption a weakly closed set of L which is also convex, since ρ is an (L_+, E)-restricted convex risk measure. Since these two sets are disjoint, from the strong separation theorem for convex sets in locally convex spaces there is some $\ell \in L^*$, $\ell \neq 0$, an $\alpha \in \mathbb{R}$ and a $\delta > 0$ such that

$$\ell(U_0 + mE) \geq \alpha + \delta > \alpha \geq \ell(U)$$

for any $U \in \mathcal{A}_\rho$. Hence, we take that

$$\ell(U_0 + mE) > \sup\{\ell(U) \mid U \in \mathcal{A}_\rho\}.$$

The functional ℓ takes negative values on $N \cap L_+$ since if there is some $Z_0 \in N \cap L_+ \setminus \{0\}$ such that $\ell(Z_0) > 0$, then for any $\lambda \in \mathbb{R}_+$ we take $\lambda Z_0 \in N \cap L_+$. Then if $\lambda \to +\infty$

$$\ell(\lambda Z_0) > \ell(U_0 + mE),$$

being a contradiction according to the previous separation argument. Then, since we have that $-\ell$ is a positive linear functional on the subspace, $J = N \cap L_+ - N \cap L_+$. But from Theorem (Jameson 1970, Cor. 1.6.2) and Proposition (Jameson 1970, Pr. 3.1.8) since E is an order unit of L which belongs to $N \cap L_+$, $-\ell$ has a continuous, linear extension all over L. For the sake of brevity, we denote again this functional by $-\ell$ and we have that $-\ell \in L_+^0$. If we suppose that $-\ell(E) = 0$, then we would take $-\ell(U) = 0$ for any $U \in L$. This is true since $U \in [-n_0(U)E, n_0 E]$, for appropriate $n_0(U) \in \mathbb{N}$, for any $U \in L$, because E is an order unit of L. Hence, $n_0(U)\ell(E) \leq -\ell(U) \leq -n_0(U)\ell(E)$. But since $\ell(E) = 0$ by assumption, this implies $-\ell(U) = 0$ for any $U \in L$. Hence $-\ell = 0$ because $L = \cup_{n=1}^{\infty}[-nE, nE]$. This implies that the initial assumption that $-\ell(E) = 0$ is not true and $-\ell(E) > 0$ holds. Hence, we may suppose that

$$-\ell(E) = 1$$

holds, or else $-\ell \in B_E$.

Hence, the separation of the sets $\{U_0 + mE\}$ and $\mathcal{A}_\rho$ implies that

$$(-\ell)(-U_0) - m > \sup_{U \in \mathcal{A}_\rho} (-\ell)(-U) = a(-\ell).$$

Denote $-\ell$ by π_0 and we get

$$\pi_0(-U_0) - a(\pi_0) > m,$$

which is a contradiction, since in this case

$$m > \sup\{\pi(-U_0) - a(\pi) \,|\, \pi \in B_E\} \geq \pi_0(-U_0) - a(\pi_0) > m.$$

The contradiction was due to the assumption that some $U_0 \in N$ exists, such that

$$\rho(U_0) > \sup\{\pi(-U_0) - a(\pi) \,|\, \pi \in B_E\}.$$

Hence, for any $U \in N$ we get (4.1). For the opposite direction, it suffices to show that any $\rho : N \to \mathbb{R}$, defined through (4.1), is an (L_+, E)-restricted convex risk measure. For this, we have to verify that ρ satisfies the properties of a (L_+, E)-restricted convex risk measure:

(i) (Translation invariance):

$$\begin{aligned}\rho(U + kE) &= \sup\{\pi(-U - kE) - a(\pi) \,|\, \pi \in B_E\} \\ &= \sup\{\pi(-U) - a(\pi) - k\pi(E) \,|\, \pi \in B_E\} \\ &= \sup\{\pi(-U) - a(\pi) - k \,|\, \pi \in B_E\} \\ &= \sup\{\pi(-U) - a(\pi) \,|\, \pi \in B_E\} - k = \rho(U) - k\end{aligned}$$

for any $U \in N$ and any $k \in \mathbb{R}$ such that $U + kE \in N$.

(ii) (Convexity): The function which maps every U to $\pi(-U) - a(\pi)$ for some $\pi \in B_E$ is a convex real-valued function on N; hence, ρ is a convex function on N as the supremum of convex functions defined on N.

(iii) (L_+-monotonicity): If $U' \geq U$ where $U, U' \in N$ in terms of the partial ordering of L induced by L_+, then $-U \geq -U'$, hence

$$\pi(-U) \geq \pi(-U')$$

for any $\pi \in B_E$ since $B_E \subseteq L_+^0$. Hence

$$\pi(-U) - a(\pi) \geq \pi(-U') - a(\pi)$$

for any $\pi \in B_E$ and by taking supremums over the elements of B_E we get that

$$\rho(U) = \sup_{\pi \in B_E} \{\pi(-U) - a(\pi)\} \geq \rho(U') = \sup_{\pi \in B_E} \{\pi(-U') - a(\pi)\}.$$

□

Theorem 3.21 *If N is a closed wedge in a reflexive ordered normed space L with non-empty cone interior, $\rho : N \to \mathbb{R}$ is a (L_+, E)-restricted convex risk measure with $\sigma(L, L^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is Lipschitz-continuous.*

Proof If $\{f_i : N \to \mathbb{R},\ i \in I\}$ is a family of real-valued functions defined on N, then we observe that

$$\sup_{i \in I} f_i(U) - \sup_{i \in I} f_i(U') \leq \sup_{i \in I} \left\{f_i(U) - f_i(U')\right\},$$

where $U, U' \in L$ (for better interpretation, we may suppose that the family of functions $\{f_i : N \to \overline{\mathbb{R}},\ i \in I\}$ is such that $\sup_{i \in I} f_i(U) \neq \infty$ for any U). Indeed, this holds because if we denote by A the set $\{f_i(U) - f_i(U') \mid i \in I\}$ and by D the set $\{f_i(U') \mid i \in I\}$, then

$$\{f_i(U) \mid i \in I\} \subseteq A + D.$$

Hence

$$\sup_{i \in I} f_i(U) \leq \sup_{i \in I}\{f_i(U) - f_i(U')\} + \sup_{i \in I} f_i(U').$$

We have seen that ρ has the representation (4.1). Then

$$\rho(U) - \rho(U') = \sup\{\pi(-U) - a(\pi) \mid \pi \in B_E\} - \sup\{\pi(-U') - a(\pi) \mid \pi \in B_E\}.$$

By the above remark, we take that

$$\rho(U) - \rho(U') \leq \sup\{\pi(-U) - \pi(-U') \mid \pi \in B_E\} = \sup\{\pi(U' - U) \mid \pi \in B_E\}.$$

We actually have that $I = B_E$ and $f_i = f_\pi$ for any $\pi \in B_E$, where $f_\pi : N \to \overline{\mathbb{R}}$ is such that $f_\pi(U) = \pi(-U) - a(\pi)$ for any $U \in N$. We suppose that $\pi(-U) - a(\pi) - (\pi(-U') - a(\pi)) = \pi(-U) - \pi(-U')$ for any $U, U' \in N$ and any $\pi \in B_E$, namely that $a(\pi) - a(\pi) = 0$. This is a simple substraction in the case where $a(\pi) \in \mathbb{R}$, but in the case where $a(\pi) = \infty$, we have the substraction of two infinity values. But we may suppose that their difference is equal to zero, since we subtract infinities 'of the same form'. On the other hand, we may say that if $a(\pi) = \infty$ then $-a(\pi) = -\infty$, hence $\pi(-U) - a(\pi) = \pi(-U)$ because if we add a real number to $-\infty$ we take the real number itself.

In order to complete the proof, we note that

$$\pi(U' - U) \leq |\pi(U' - U)| \leq \|\pi\| \cdot \|U - U'\| \leq \frac{1}{b}\|U - U'\|$$

for any $\pi \in B_E$, from the definition of B_E and the fact that $\hat{E}$ is a uniformly monotonic linear functional of L^0_+. Hence

$$\rho(U) - \rho(U') \leq \frac{1}{b}\|U - U'\|$$

and in the same way we may show

$$\rho(U') - \rho(U) \leq \frac{1}{b}\|U - U'\|.$$

The last two inequalities imply that

$$|\rho(U) - \rho(U')| \leq \frac{1}{b}\|U - U'\|$$

and the conclusion is ready. □

4 Implications to solvency capital calculation

4.1 Restricted convex risk measures for reinsurance companies

The reinsurance companies' restricted coherent and restricted convex risk measures are defined on the entire subspace $\mathcal{U}$ of the attainable European Contingent Claims which are actually the set of all the surplus positions of this company. Since a reinsurance company may adopt the insurance of any surplus in a short-time environment, its own restricted risk measure ρ by which it calculates the solvency capital is defined on $\mathcal{U}$, which is a wedge. The proofs of the equivalent results are omitted since they are similar, except some useful remarks and additions. In this case, the wedge N is the subspace $\mathcal{U}$ and the intersection $N \cap L_+$ is the cone $\mathcal{U} \cap L_+$ of the induced partial ordering of $\mathcal{U}$. $\mathcal{U}$ is a weak E-translation invariant subset of L, since it contains $\mathcal{U} \cap L_+$, which is a weak E-translation invariant set of L. This holds, because for any $U \in \mathcal{U} \cap L_+$ and any $t \geq 0$, $U + tE \in \mathcal{U} \cap L_+$.

These restricted risk measures are adequate for the solvency capital calculation in insurance applications under the frame we presented, since the value $\rho(H)$ for some $H \in \mathcal{U}$, indicates the minimum amount of shares of the 'numeraire risk' E needed for H to belong in the acceptance set $\mathcal{A}_\rho$ jointly with H. This 'functionality' of the solvency capital in this case is the same like in the case of $\rho^{\mathcal{U}}_{(\mathcal{A},E)}$ as it is going to be verified through the following results. The word 'restricted' mentioned before is related to the fact that the domain of ρ is not the whole space L, but a proper subspace of it. These risk measures are mentioned to be appropriate for reinsurance companies, since in these cases, there are not 'debt constraints' and they can adopt any risk appears, namely the mechanism of insurance via its own risk measure by using shares of the numeraire is valid for any attainable surplus of a reinsurance company. On the contrary, this is not always achieved for a simple insurance company.

We remind the definition of the acceptance set of a given risk measure, appropriately stated for this case.

If $\mathcal{U}$ is the subspace of L containing the attainable ECC of the financial market of the surplus asset and the riskless bond and the function $\rho : \mathcal{U} \to \mathbb{R}$ is a risk measure, then the set $\mathcal{A}_\rho = \{U \in \mathcal{U} | \rho(U) \leq 0\}$ is defined to be the *acceptance set* of ρ.

In the case of reinsurance companies, we may repeat the Definitions 3.5, 3.6, 3.7, 3.8 for $N = \mathcal{U}$. Also, Proposition 3.11 could be repeated for $N = \mathcal{U}$ in this case.

Remark 4.1 The only difference between the proof of the proposition being equivalent to 3.11 for $N = \mathcal{U}$ and 3.11 is that $\mathcal{U}$ is (L_+, E)-translation invariant; hence, in the point where the translation invariance of $\rho^{\mathcal{U}}_{(\mathcal{A},E)}$ is proved, this property holds for any $U \in \mathcal{U}$ and $a \in \mathbb{R}$ due to the fact that $\mathcal{U}$ is (L_+, E)-translation invariant.

The same happens with Propositions 3.13 and 3.12.

Remark 4.2 There is not any difference between the proof of the equivalent proposition to 3.12 in the case of $N = \mathcal{U}$ and 3.12, because $\mathcal{U}$ is (L_+, E)-translation invariant.

Lemma 4.3 and Proposition 4.4 are essential in the the study of the restricted convex risk measures that are defined on closed subspaces of L^2-spaces. The subspace which is the domain of such a risk measure may be the subspace $\mathcal{U}$ of the attainable financial positions for a market formulated by the surplus asset of insurance company and a bank account. This is a closed subspace of a $L^2(\Omega, \mathcal{F}_T^W, Q)$ for any equivalent martingale measure Q of this market and this subspace $\mathcal{U}$ will be considered as the subspace of the surplus positions of the reinsurance company.

Lemma 4.3 *Since $\mathcal{U}$ is a subspace of L, then $\mathcal{U}^0$ is the annihilator*

$$\mathcal{U}^\perp = \{y^* \in L^* | y^*(x) = 0\,, \quad \forall x \in \mathcal{U}\}.$$

Proof Let $\lambda > 0$ and $x \in \mathcal{U}$. Then $\lambda x \in \mathcal{U}$ and $(-\lambda)x \in \mathcal{U}$. If $y^* \in \mathcal{U}^0$, then $y^*(\lambda x) \geq 0,\ \forall x \in \mathcal{U}$, while $y^*((-\lambda)x) \geq 0,\ \forall x \in \mathcal{U}$. Then $y^*(x) \geq 0,\ \forall x \in \mathcal{U}$, while $y^*(x) \leq 0,\ \forall x \in \mathcal{U}$ since $x \in \mathcal{U}$ and $\lambda > 0$. Hence $y^*(x) = 0,\ \forall x \in \mathcal{U}$, which means that $y^* \in \mathcal{U}^\perp$. This implies $\mathcal{U}^0 \subseteq \mathcal{U}^\perp$. Obviously, $\mathcal{U}^\perp \subseteq \mathcal{U}^0$. □

Proposition 4.4 *The subspace $\mathcal{U}$ is norm-closed under the norm of $L = L^2(\Omega, \mathcal{F}_T^W, Q)$.*

Proof Without loss of generality, we may consider that the market is normalized; hence the interest-rate process is equal to zero. Let us consider a sequence $(U_n)_{n\in\mathbb{N}}$ of attainable ECC under the market generated by the surplus asset and the bank account. Since these U_n are attainable, there is a sequence of admissible portfolio processes $(\theta_{1,n})_{n\in\mathbb{N}}$ which denotes shares of the surplus asset such that

$$U_n = \int_0^T \theta_{1,n}(t) v(t, U_t, Z_t) \mathrm{d}\hat{W}_t\,,$$

where $\hat{W}$ is the $(Q, \mathbb{F}^W)$-Brownian motion which arises from the change of measure under the Girsanov–Cameron–Martin theorem. Suppose that $U_n \overset{L^2(\Omega, \mathcal{F}_T^W, Q)}{\to} U$.

We actually can assume that $\mathbb{E}_Q(U_n) = 0$, since if this is not true, the attainability of U_n is equivalent to the following representation equation:

$$U_n = \mathbb{E}_Q(U_n) + \int_0^T \theta_{1,n}(t) v(t, U_t, Z_t) \mathrm{d}\hat{W}_t.$$

Note that if $U_n \overset{L^2(\Omega, \mathcal{F}_T^W, Q)}{\rightarrow} U$, then $U_n \overset{L^1(\Omega, \mathcal{F}_T^W, Q)}{\rightarrow} U$, hence $\mathbb{E}_Q(U_n) \longrightarrow \mathbb{E}_Q(U)$. Namely, the norm convergence of the sequence $\{U_n - \mathbb{E}_Q(U_n)\mathbf{1},\ n \in \mathbb{N}\}$ to the random variable $U - \mathbb{E}_Q(U)\mathbf{1}$, where $\mathbf{1}$ denotes the indicator random variable of Ω, implies the equivalent convergence of the sequence $\{U_n - \mathbb{E}_Q(U_n)\mathbf{1},\ n \in \mathbb{N}\}$ to $U - \mathbb{E}_Q(U)\mathbf{1}$ in terms of the $L^1(\Omega, \mathcal{F}_T^W, Q)$-norm. But $\mathbb{E}_Q(U_n - \mathbb{E}_Q(U_n)\mathbf{1}) = 0$ and $\mathbb{E}_Q(U - \mathbb{E}_Q(U)\mathbf{1}) = 0$. This means that it is enough to study the case with $\mathbb{E}_Q(U_n) = 0$ for any $n \in \mathbb{N}$. From Itô isometry, we get that the sequence of processes $(\theta_{1,n}(t)v(t, U_t, Z_t))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^2([0,T] \times \Omega, \mathcal{B} \otimes \mathcal{F}_T^W, \lambda_{[0,T]} \otimes Q)$, where $\mathcal{B}$ is the Borel σ-algebra of sets on $[0,T]$. Since this sequence is a Cauchy sequence, it converges with respect to the norm of the space $L^2([0,T] \times \Omega, \mathcal{B} \otimes \mathcal{F}_T^W, \lambda_{[0,T]} \otimes Q)$ to some process ϕ. Then, by Aliprantis and Border (1999, Th. 12.6) there exists some subsequence $(\theta_{1,n_k}(t)v(t, U_t, Z_t))_{k\in\mathbb{N}}$ of this sequence such that this subsequence converges to ϕ, $\lambda_{[0,T]} \otimes Q - a.e.$, or else $\theta_{1,n_k}(t)(\omega)v(t, U_t(\omega), Z_t(\omega)) \longrightarrow \phi(t, \omega)$ for almost all (t, ω) with respect to the measure $\lambda_{[0,T]} \otimes Q$. By the last convergence, we have that $\phi(t,\omega) = \kappa(t,\omega)\, v(t, U_t(\omega), Z_t(\omega))$, $\lambda_{[0,T]} \otimes Q - a.e.$. Finally, $\mathcal{U}$ is norm-closed because $U = \int_0^T \kappa_t\, v(t, U_t, Z_t)\, \mathrm{d}\hat{W}_t$. □

The next theorem in which the above lemma and the above proposition are actually used is the following dual representation theorem for (L_+, E)-restricted convex risk measures according to the Definition 3.8, which is similar to Konstantinides and Kountzakis (2001, Th. 3.6) and the Theorem 3.20.

The space L in the next duality representation theorem is any of the spaces mentioned in Proposition 4.4. Actually, the spaces which are of particular interest for the previous results of this section as immediate examples are the same.

Theorem 4.5 *If $\mathcal{U}$ is the subspace of the surplus positions of a reinsurance company in the reflexive ordered normed spaces $L = L^2(\Omega, \mathcal{F}_T^W, Q)$ with non-empty cone interior containing the L_+-interior point E, $\rho : \mathcal{U} \to \mathbb{R}$ is a (L_+, E)-restricted convex risk measure with $\sigma(L, L^*)$-closed acceptance set, then*

$$\rho(U) = \sup\{\pi(-U) - a(\pi) \,|\, \pi \in B_E\}, \tag{4.1}$$

for any $U \in \mathcal{U}$, where $B_E = \{y^* \in L_+^0 \,|\, \hat{E}(y^*) = 1\}$ and $a : B_E \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_E$. On the other hand, every ρ defined through (4.1), is a (L_+, E)-restricted convex risk measure.

Remark 4.6 In the proof of the Theorem 4.5, we use that $\mathcal{U}^0 = \mathcal{U}^\perp$ and that $\mathcal{U}$ is norm-closed according to the Proposition 4.4.

Also, Theorem 3.21 can be proved in the same way for $N = \mathcal{U}$.

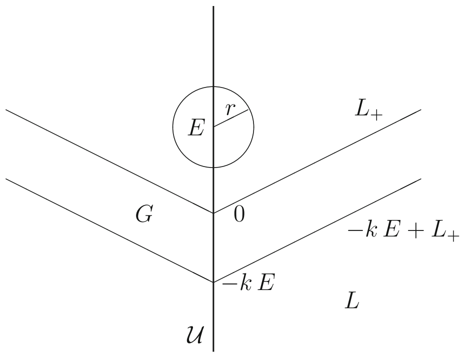

Fig. 1 Representation of domain G

4.2 Restricted convex risk measures for insurance companies

When the insurance company is not a reinsurance one, a debt constraint is present, or else if the domain of the risk measures is the cone of the admissible surplus positions $G = \mathcal{U} \cap (-kE + L_+)$, $k > 0$. This debt constraint denotes that an insurance company adopts the insurance of any surplus that it does not exceed a deficit level. In this case, the wedge N is the cone G and the intersection $G \cap L_+$ is the cone L_+ of the induced partial ordering of L. G is a weak E-translation invariant subset of L, because $L_+ = G \cap L_+$ is such a set.

If G is the cone of the admissible surplus positions of the insurance company and the function $\rho : G \to \mathbb{R}$ is a risk measure, then the set $\mathcal{A}_\rho = \{U \in G | \rho(U) \leq 0\}$ is defined to be the *acceptance set* of ρ.

Then, propositions equivalent to 3.11, 3.12, 3.13 can be proved in the case where $N = G$.

The space L in any of the next duality representation theorem is any of the spaces mentioned in 4.4. Actually, the spaces which are of particular interest for the previous results of this section as immediate examples are the same.

Theorem 4.7 *If G is the cone of the admissible surplus positions of an insurance company lying in one of the reflexive ordered normed spaces $L = L^2(\Omega, \mathcal{F}_T^W, Q)$ with non-empty cone interior containing the L_+-interior point E, $\rho : G \to \mathbb{R}$ is a (L_+, E)-restricted convex risk measure with $\sigma(L, L^*)$-closed acceptance set, then*

$$\rho(U) = \sup\{\pi(-U) - a(\pi) \mid \pi \in B_E\}, \tag{4.2}$$

for any $U \in G$, where $B_E = \{y^* \in L_+^0 \mid \hat{E}(y^*) = 1\}$ and $a : B_E \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_E$. On the other hand, every ρ defined through (4.2), is a (L_+, E)-restricted convex risk measure.

Also, Theorem 3.21 can be proved in the same way for the case of $N = G$.

Figure 1 shows the geometry of the domain G of the equivalent restricted risk measures.

5 Reinsurance and solvency capital calculation via restricted convex risk measures

If $\rho_1 : G \to \mathbb{R}$ is a (L_+, E)-restricted convex risk measure defined on the subspace of L of European Contingent Claims of the market formulated by the surplus asset of an insurance company and a riskless bond(Q is any EMM of this market), then the solvency position which corresponds to the position U is $\rho_1(U)E$. The company has to buy these shares from some reinsurance company.

The reinsurance company has its own (L_+, E)-restricted convex risk measure $\rho_2 : \mathcal{V} \to \mathbb{R}$. The subspace $\mathcal{V}$ is the subspace of the attainable European Contingent Claims for the equivalent market of the surplus asset of the reinsurance company and a bank account.

The condition $\mathcal{V} \cap G \neq \emptyset$ is a required condition for the ability of the insurance company to apply for reinsurance. Since the reinsurance company provides an amount of shares of the numeraire asset E, we suppose that E lies in $\mathcal{V} \cap \mathcal{U}$.

If we denote by $\mathcal{A}_{\rho_1}$ the acceptance set of the insurance company, the reinsurance appears in the case where the surplus U_T does not belong to the acceptance set $\mathcal{A}_{\rho_1}$. Since the reinsurance company provides insurance to the primal insurance company, it seems rational to suppose that this action does not expose the reinsurance company to risk. Hence, the financial positions for which reinsurance applies are the positions lying in the set

$$\mathcal{R}(\rho_1, \rho_2) = \mathcal{A}_{\rho_2} \backslash \mathcal{A}_{\rho_1}.$$

Here, we suppose that the reinsurance company adopts the insurance of the whole surplus U of the primal insurance company and not of a part of it.

The set of the financial positions $\mathcal{A}_{\rho_2} \backslash \mathcal{A}_{\rho_1}$ is a subset of $\mathcal{V} \cap G$.

The question is which is the cost of the shares of the numeraire that the primal insurance company has to pay to the reinsurance company in order to be insured. The **solvency position** for the primal insurance company is

$$\rho_1(U)E.$$

Hence, since the reinsurance adopts the insurance of U,

$$U + \rho_1(U)E \in \mathcal{A}_{\rho_2}.$$

The fact that the reinsurance company adopts the insurance of U means that $\rho_1(U)E$ which is the solvency position corresponding to U together with U formulates a riskless position from the aspect of the reinsurance company. For this reason, this position lies in the relevant acceptance set $\mathcal{A}_{\rho_2}$. This, by translation invariance, implies that

$$\rho_1(U) \geq \rho_2(U).$$

Of course the insurance company may require a partial reinsurance instead of a complete reinsurance. This corresponds to the reinsurance of the position tU, where $t \in (0, 1)$ is the fraction of the surplus that the company desires to be insured. The procedure of the reinsurance is the same in this case. The only remark we may make is that the fractions are not eliminated in the case of non-coherent, restricted convex risk measures due to the fact that they lack the positive homogeneity.

The price of the solvency position would be set equal to $\rho_1(U)$, since if we consider some pricing functional $\pi_0 \in L^0_+$ concerning the time period 0, this functional could be normalized and it could be taken to lie on the base B_E defined by E on L^0_+. Hence, the price of the position $\rho_1(U)E$ under π_0 is equal to $\pi_0(\rho_1(U)E) = \rho_1(U)\pi_0(E) = \rho_1(U)$.

A question is how this price-functional is selected among the functionals in B_E, which arise in the dual representation of the restricted risk measures. A possible answer is that π_0 is the equilibrium price formulated in the market for the financial positions corresponding to the time period T viewed as consumption bundles at this time period. We suppose that insurance company and the reinsurance company are not involved in price shaping, or else our approach is a 'partial equilibrium approach'. Namely, we may suppose that the reinsurance and the insurance company have not market power so that their actions to affect the prices of the commodities (of the asset payoffs). Hence, they face the equilibrium price formulated as an exogenous price and not as an effect of their own selections.

6 Choice of the solvency capital scaling

According to Polyrakis (2008, Th. 4), if L is reflexive and E is an interior point of L_+ and we consider some $H \in L_+$ as a linear functional of L^* being a strictly positive linear functional of L^0_+, then the base

$$B_H = \{\pi \in L^0_+ \mid \hat{H}(\pi) = 1\}$$

defined by it is bounded and hence weakly compact, since H is a uniformly monotonic functional of L^* ordered by L^0_+ (See Polyrakis 2008, Pr. 2 and Jameson 1970, Th. 3.8.12. We may apply it for $L := L^*$, $P = L^0_+$, $P^0 = L^{00}_+ = L_+$, $H = f$). In this case, H can be considered as a price normalization position (numeraire) too, since $\hat{H}$ is a uniformly monotonic functional of L^0_+ and H is an interior point of L_+ from Jameson (1970, Th. 3.8.12). But if the (re)insurance company desires a selection of scaling for the solvency capital for the elements which do not belong to the acceptance set $\mathcal{A}$, how could it select between the scaling provided by E and the scaling provided by H? Obviously, the company has to adopt a decision function which has to compare the scaling functionals. Namely, since every scaling of the solvency capital depends on the normalization position E, it has to compare these positions. First, suppose that the reinsurance company's acceptance set is $\mathcal{A}$. One possible decision function for the company should the following one: The company determines a set of positions $(U_i)_{i\in I} \subseteq \mathcal{U}\backslash\mathcal{A}$, which is the set of surplus positions which are forecast to be more favorable for reinsurance, where I is a non-empty set and $\mathcal{I}$ is a σ-algebra of subsets of this set. $(U_i)_{i\in I} \subseteq \mathcal{U}\backslash\mathcal{A}$ indicates the most probable attainable surplus positions of the reinsurance company, in which case this surplus is exposed to risk. Moreover, the probability space $(I, \mathcal{I}, p)$ is considered, where p is a probability measure which denotes the subjective probability of the company about demand of reinsurance over the elements of I. Hence, the decision function could be defined to be the average solvency capital over the elements of $(U_i)_{i\in I}$ under p:

$$\mathbb{E}_{(p,I,\mathcal{A})}(E) = \int_I \rho_{(\mathcal{A},E)}(U_i)\mathrm{d}p(i),$$

where the function $f : I \to \mathbb{R}$ with $f(i) = \rho_{(\mathcal{A},E)}(U_i)$ is supposed to lie in $L^1(I,\mathcal{I},p)$, while this decision function defines the following binary relation on $\mathrm{Int}L_+$:

$$E_1 \succeq_{(p,I,\mathcal{A})} E_2 \Leftrightarrow \mathbb{E}_{(p,I,\mathcal{A})}(E_2) \geq \mathbb{E}_{(p,I,\mathcal{A})}(E_1).$$

The strong binary relation is defined as follows:

$$E_1 \succ_{(p,I,\mathcal{A})} E_2 \Leftrightarrow \mathbb{E}_{(p,I,\mathcal{A})}(E_2) > \mathbb{E}_{(p,I,\mathcal{A})}(E_1).$$

The above binary relations determine the choice of scaling. Other decision functions which may be defined are of variation type or of deviation type, respectively. If we suppose that $g : I \to \mathbb{R}$ with $g(i) = \rho^2_{(\mathcal{A},E)}(U_i)$ is supposed to lie in $L^1(I,\mathcal{I},p)$ while this decision function defines the following binary relation on $\mathrm{Int}L_+$:

$$E_1 \succeq_{(p,I,\mathcal{A})} E_2 \Leftrightarrow \int_I \rho^2_{(\mathcal{A},E_2)}(U_i)\mathrm{d}p(i) \geq \int_I \rho^2_{(\mathcal{A},E_1)}(U_i)\mathrm{d}p(i).$$

The strong binary relation is defined as follows:

$$E_1 \succ_{(p,I,\mathcal{A})} E_2 \Leftrightarrow \int_I \rho^2_{(\mathcal{A},E_2)}(U_i)\mathrm{d}p(i) > \int_I \rho^2_{(\mathcal{A},E_1)}(U_i)\mathrm{d}p(i).$$

The above comparison can be named comparison of variation type. If for the comparison of the solvency capital scaling (or else of the numeraire assets) the square root of the integral

$$\int_I \rho^2_{(\mathcal{A},E)}(U_i)\mathrm{d}p(i),\ E \in \mathrm{Int}L_+$$

is used, this comparison can be named of deviation type. In the case where $g : I \to \mathbb{R}$ with $g(i) = \rho^2_{(\mathcal{A},E)}(U)$ is supposed to lie in $L^1(I,\mathcal{I},p)$, we may use

$$\frac{1}{2}\int_I \rho^2_{(\mathcal{A},E)}(U_i)\mathrm{d}p(i) + \frac{1}{2}\int_I \rho_{(\mathcal{A},E)}(U_i)\mathrm{d}p(i),\ E \in \mathrm{Int}L_+$$

as a comparison function. $\frac{1}{2}$ can be replaced by any other $\lambda \in (0,1)$.

If we consider the induced norm topology on the cone L_+, then we may consider a relatively compact set of numeraire assets C, being a subset of the interior of the cone L_+. As we explained above, the cone L_+ has plenty of interior points if it has non-empty interior since L is a reflexive space. Hence, the topological space considered is the interior of the cone $\mathrm{int}L_+$ endowed with the relative norm topology. If we consider the binary relation

$$E_1 \succeq_{(p,I,\mathcal{A})} E_2 \Leftrightarrow \mathbb{E}_{(p,I,\mathcal{A})}(E_2) \geq \mathbb{E}_{(p,I,\mathcal{A})}(E_1) \tag{6.1}$$

for the comparison of the numeraire assets ($E_1, E_2 \in \text{int}L_+$), then since C is a relatively compact set, we may use the finite intersection property in order to deduce the existence of maximal elements for $\succeq_{(p,I,\mathcal{A})}$ in C. For brevity, the binary relation 6.1 will be denoted by $\succeq$. The set of maximal elements as it is well known is the intersection $\cap_{E\in C}\{E_1 \in C | E_1 \succeq E\}$ and if the sets $\{E_1 \in C | E_1 \succeq E\}, E \in C$ are relatively closed, the intersection of them is non-empty from the fact that C is a relatively compact subset of $\text{int}L_+$. Instead of L_+, if the acceptance set $\mathcal{A}$ is assumed to be a cone with non-empty interior ($\text{int}\mathcal{A} \neq \emptyset$), the above arguments are transferred in $\text{int}\mathcal{A}$ in the same way. This assumption is also crucial for the following results.

If we suppose that $\mathcal{A}$ satisfies the assumptions of the Konstantinides and Kountzakis (2001, Pr. 2.3), and this is true if for example $\mathcal{A}$ is a cone of L containing L_+ having non-empty interior, then according to the dual representation Theorem (Konstantinides and Kountzakis 2001, Th. 3.1), $\rho_{(\mathcal{A},E)}(X) = \sup\{\pi(-X) | \pi \in \mathcal{A}^0 \cap B_E\}$, where B_E is the base defined on the cone L^0_+ of L^* by E. If we suppose that L is partially ordered by $\mathcal{A}$, this relation becomes $\rho_{(\mathcal{A},E)}(X) = \sup\{\pi(-X) | \pi \in B_E\}, E \in \text{int}\mathcal{A}$, where B_E denotes the base of $\mathcal{A}^0$ defined by E. Since every such base is weakly compact, the supremum is actually a maximum for any $X \in L$. C in this case is equal to $\text{int}\mathcal{A}$. We define the correspondence $\phi : C \to 2^{\mathcal{A}^0}$, as follows: $\phi(E) = B_E$. The graph of ϕ is defined as follows $Gr\phi = \{(E, \pi) \in C \times \mathcal{A}^0 | \pi \in \phi(E)\}$ as well. Consider a certain $X \in L$ as a function $f_X : Gr\phi \to \mathbb{R}$, which acts on the arguments of $Gr\phi$ as follows : $f_X(E, \pi) = \widehat{(-X)}(\pi) = \pi(-X)$ for any $\pi \in \phi(E)$. The *value function* for f_X is

$$m_X(E) = \max_{\pi\in\phi(E)} f_X(E, \pi) = \max_{\pi\in B_E} \pi(-X) = \rho_{(\mathcal{A},E)}(X).$$

Berge's maximum theorem, (Aliprantis and Border 1999, Lem. 16.29; Aliprantis and Border 1999, Lem. 16.30; Aliprantis and Border 1999, Th. 16.31; Berge 1963) indicates that lower (upper)-hemicontinuity of ϕ implies lower (upper)-semicontinuity of m_X.

Proposition 6.1 *ϕ is a lower hemicontinuous correspondence.*

Proof According to Duffie (1987, Lem. 2), we may transfer this proof for lower hemicontinuity of budget correspondences for finite event-tree economies to our setting of infinite-dimensional economies. We use the sequence (nets') characterization of lower hemicontinuity of a correspondence which is indicated by Aliprantis and Border (1999, Th. 16.19).

Let us take $\pi \in \phi(E)$ and a nonzero sequence $(\pi_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}^0$, which converges in π, in terms of the norm topology; hence, in terms of the weak topology (since L^* is reflexive, the weak and the weak-star topology on L^* coincide). Such a sequence exists because of the fact that $\mathcal{A}^0$ is a norm-closed set of L^*.

We suppose that $E_n \to E$ in terms of the norm topology of L. Since $E_n, n \in \mathbb{N}$ (and E, too) are interior points of the cone $\mathcal{A}$, according to Jameson (1970, Th. 3.8.12), $E_n, n \in \mathbb{N}$ (and E, too) are uniformly monotonic functionals of $\mathcal{A}^0$ in this case. Then since $\phi(E_n)$ is the base that E_n defines on $\mathcal{A}^0$, for any π_n there is a unique $\lambda_n > 0$ such that $\pi_n(\lambda_n E_n) = 1$, or else that $\lambda_n \pi_n \in \phi(E_n)$. We have to prove that the sequence $(\lambda_n \pi_n)_{n\in\mathbb{N}}$ converges weakly in π. If we prove that $\lambda_n \to 1$, the proof has finished. This holds because $\lambda_n \pi_n \to \pi$ in the norm topology of L^*, hence in terms of the weak topology. The norm convergence $E_n \to E$ implies that by weak convergence $\pi(E_n) \to \pi(E)$. From the weak convergence $\pi_n \xrightarrow{w} \pi$, we get that $\pi_n(E_n) \to \pi(E)$. Hence $\pi_n(E_n) \to 1$ and since $\lambda_n \pi_n(E_n) = 1$, we get what was needed. □

Proposition 6.2 *The sets of the form*

$$\left\{ E_1 \in C | \int_I \rho_{(\mathcal{A},E_1)}(U_i) \mathrm{d}p(i) \leq c \right\}$$

are relatively closed subsets of C (under the induced norm topology in int$\mathcal{A}$*) for any $c \in \mathbb{R}$.*

Proof $\rho_{(\mathcal{A},E)}(U_i) = m_{U_i}(E)$, $E \in C$, $i \in I$. If ϕ is lower hemicontinuous, then by Berge's maximum theorem, (Aliprantis and Border 1999, Lem. 16.29; Aliprantis and Border 1999, Lem. 16.30; Aliprantis and Border 1999, Th. 16.31; Berge 1963) the value function m_X for a certain X is lower semicontinuous. This implies that for any $i \in I$, m_{U_i} is lower semicontinuous, or else that for any $c \in \mathbb{R}$, the sets $\{E \in C | m_{U_i}(E) \leq c\}$ are relatively closed in C. We are going to use Fatou's lemma (see Aliprantis and Border 1999, Th. 11.19) and the characterization of semicontinuity by nets (sequences) in order to show that the same holds for the sets $\{E \in C | \int_I m_{U_i}(E)\mathrm{d}p(i) \leq c\}$ if p is a probability measure over I. We actually notice that if $E_n \to E$ in the norm of L, we have $m_{U_i}(E) \leq \liminf_{n\to\infty} m_{U_i}(E_n)$ (by lower semicontinuity of the value function m_E, $E \in \mathrm{int}L_+$). Hence if $m_{U_i}(E_n) \leq c$ for any $i \in I$ and for any $n \in \mathbb{N}$, this implies

$$\int_I m_{U_i}(E)\mathrm{d}p(i) \leq \int_I \liminf_{n\to\infty} m_{U_i}(E_n)\mathrm{d}p(i) \leq \liminf_{n\to\infty} \int_I m_{U_i}(E_n)\mathrm{d}p(i) \leq c,$$

implied by Fatou's lemma. □

Theorem 6.3 *If C is a relatively compact set in* int$\mathcal{A}$ *under the induced norm topology, then the set of maximal elements of the binary relation $\succeq$ is non-empty.*

Proof This arises from the fact that the family of sets $\{E_1 \in C | E_1 \succeq E\}$, $E \in C$ has the finite intersection property, the fact that C is relatively compact and the fact that the sets $\{E_1 \in C | E_1 \succeq E\}$, $E \in C$ are relatively closed in terms of the induced norm topology of intL_+, due to the Proposition 6.2. Note that for a certain $E \in C$, the set $\{E_1 \in C | E \succeq E\} = \{E_1 \in C | \int_I m_{U_i}(E_1)\mathrm{d}p(i) \leq \int_I m_{U_i}(E)\mathrm{d}p(i)\}$. If we set $c = \int_I m_{U_i}(E)\mathrm{d}p(i)$, the set $\{E_1 \in C | E \succeq E\}$ takes the form mentioned in 6.2, hence it is relatively closed. □

By the last theorem, we conclude that the optimal selection of a numeraire asset is feasible under the framework we posed.

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