Coherent risk measures under dominated variation

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1 Distributions, wedges and risk measures

We study coherent-like risk measures, defined on certain sets of random variables. We remind that a set $C \subseteq L$ in a linear space L is called wedge if satisfies $C + C \subseteq C$ and $r\,C \subseteq C$ for any $r \geq 0$. A wedge for which $C \cap (-C) = \{0\}$ is called cone.

Definition 1. A class of distributions $\mathcal{J}$ is called *wedge-closed* if it is convolution closed $F_1 * F_2 \in \mathcal{J}$, whenever $F_1, F_2 \in \mathcal{J}$ and nonnegative scalar product closed $F_{rX} \in \mathcal{J}$, whenever $F_X \in \mathcal{J}$, for any scalar $r \geq 0$.

Let us consider the following set of random variables

$$\mathfrak{W}^{\leftarrow}(\mathcal{J}) := \{X \in L^0(\Omega,\, \mathcal{F},\, \mu) \mid F_X \in \mathcal{J}\}\,,$$

where L^0 denotes the linear space of the $\mathcal{F}$-measurable functions $X : \Omega \to \mathbb{R}$ and $\mathcal{J}$ some class of heavy-tailed distributions. The map $\mathfrak{W} : L^0(\Omega, \mathcal{F}, \mu) \to \mathbb{D}$, which is implied by the definition of the above set is $\mathfrak{W}(X) = F_X$, where $\mathbb{D}$ denotes the set of the distribution functions and F_X denotes the (cumulative) distribution function of X.

We look for a risk measure on $L^{1+\varepsilon}(\Omega, \mathcal{F}, \mu)$ for any $\varepsilon \geq 0$. We consider the Lundberg or the renewal risk model, where we face an optimization problem, having in mind the minimization of the solvency capital for an insurance company. The proposed risk measure, represents a modification of the classical Expected Shortfall. The definition relies on its dual (robust) representation.

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The domain of this new risk measure is a wedge in $L^{1+\varepsilon}(\Omega,\, \mathcal{F},\, \mu)$ for any $\varepsilon \geq 0$, which is connected to a class of heavy-tailed distributions. For an actuarial definition of Solvency Capital, we refer to [9], where the *capital requirement functional* was connected with both risk measures and solvency. Further, we show the existence and uniqueness of the solution for the optimization problem of this risk measure over some constraint set $\mathcal{X}$, which represents a convex, closed and bounded set in $L^{1+\varepsilon}$.

So, we start with a classical result (see [5, Th. 25.6] or [6, Th. 6.7]).

Lemma 1 (Skorohod [21]). *Consider a sequence of the random variables $\{X_n\,,\ n \in \mathbb{N}\}$ and X from space $L^0(\Omega, \mathcal{F}, \mu)$ and $\mathfrak{W}(X)$ with separable support, such that holds the weak convergence $X_n \stackrel{d}{\to} X$. Then, there exist random variables $\{Y_n\,,\ n \in \mathbb{N}\}$ and Y from $L^0(A,\, \mathcal{G},\, \nu)$, all defined on a common probability space $(A,\, \mathcal{G},\, \nu)$, such that $X_n \stackrel{d}{=} Y_n$, for any $n \in \mathbb{N}$, $X \stackrel{d}{=} Y$ and holds the convergence $Y_n \to Y$, ν-a.s.*

The $\overline{\mathfrak{W}^{\leftarrow}(\mathcal{J})}^{d}$ denotes the weak closure in $L^0(\Omega,\, \mathcal{F},\, \mu)$, in the sense of the convergence in distribution.

Proposition 1. *If $\mathcal{J}$ is wedge-closed, then $\overline{\mathfrak{W}^{\leftarrow}(\mathcal{J})}^{d}$ is wedge of $L^0(A,\, \mathcal{G},\, \nu)$, where $(A,\, \mathcal{G},\, \nu)$ is an appropriate probability space.*

Remark 1. In order to use the conclusion of Proposition 1 the probability space may be changed from $(\Omega,\, \mathcal{F},\, \mu)$ to $(A,\, \mathcal{G},\, \nu)$.

Thus, we refer to sets of random variables of the form $\mathfrak{W}^{\leftarrow}(\mathcal{J})$ that are closed under addition (if $X_1, X_2 \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$, then $X_1 + X_2 \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$), and closed under non-negative scalar multiplication (if $X \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$, then $r\,X \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$, for any $r \geq 0$).

Definition 2. A family of distributions $\mathcal{J}$ is called *convex-closed*, if for any $F_1 \in \mathcal{J}$ and $F_2 \in \mathcal{J}$, holds $aF_1 + (1-a)F_2 \in \mathcal{J}$ for any $a \in (0,\, 1)$.

Lemma 2. *If a family $\mathcal{J}$ is wedge-closed, then the family $\mathcal{J}$ is convex-closed.*

Definition 3. A risk measure $\rho : C \to \mathbb{R}$, where C is a wedge of an ordered linear space L, is called $(C,\ L_+)$-***wedge-coherent*** if it satisfies the following properties

1. $\rho(X + t\mathbf{1}) = \rho(X) - t$, where $t \in \mathbb{R}$ is such that $X + t\mathbf{1} \in C$ (the C-Cash Invariance property),
2. $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$, where $X_1, X_2 \in C$ (the Sub-additivity property),
3. $\rho(r\,X) = r\,\rho(X)$, where $X \in C$ and $r \geq 0$ (the Positive Homogeneity property),
4. $\rho(X) \geq \rho(Y)$ if $Y \geq X$ under the partial ordering in L and $X, Y \in C$ (the L_+-Monotonicity property).

The wedge C may be either of the form $\overline{\mathfrak{W}^{\leftarrow}(\mathcal{J})}^{d}$ where $\mathcal{J}$ is a wedge-closed class of distributions, or a cone of the form $\mathfrak{W}^{\leftarrow}(\mathcal{J}) \cup \{0\}$ where $\mathcal{J}$ is wedge-closed too. In the last case the risk measure is called **cone-coherent**.

A distribution function $F_X =: \mathfrak{W}(X)$ of a random variable $X : \Omega \to \mathbb{R}$ belongs to the class of *heavy-tailed* distributions $\mathcal{K}$, if does not exist exponential moment (violates the Cramér condition)

$$\mathbb{E}_{\mu}(e^{rX}) = \infty,$$

for any $r > 0$.

Due to the properties of Sub-additivity $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ and the Positive Homogeneity $\rho(r\,X) = r\,\rho(X)$, for any $r \geq 0$, the domain of any coherent risk measure has to be in general a wedge of the space of the financial positions. Since the domain C of such a risk measure contains the subspace $C \cap (-C)$, we have to include zero random variable in our analysis. As the next Example indicates, the degenerated zero random variable may be the weak-limit (in the sense of convergence in distribution), of a sequence of heavy-tailed random variables.

Example 1. Consider the following sequence of Pareto-type random variables $\{X_n\,,\ n \in \mathbb{N}\}$, where their distribution functions are given by

$$F_n(x) = 1 - \frac{1}{n\,(x+1)^n}\,,\ \forall\, x \geq \frac{1}{n^{1/n}} - 1\,,$$

and $F_n(x) = 0\,,\ \forall\, x < 1/n^{1/n} - 1$. The point-wise limit of this sequence is equal to $F(x) = 0\,,\ x < 0$ and $F(x) = 1\,,\ x \geq 0$, but this function represents the defective distribution function of the random variable zero.

We also mention the remark [16, Rem. 2.1] concerning the 'explosion' of the tail for extreme values of the parameters of heavy-tail distributions which contradicts with the typical decreasing form of tails. According to the above Example, the appropriate form of the domain for a coherent risk measure related to a class of heavy-tailed distributions $\mathcal{J}$ is not $\mathfrak{W}^{\leftarrow}(\mathcal{J})$ but its weak closure $\overline{\mathfrak{W}^{\leftarrow}(\mathcal{J})}^{d}$.

We point out that the coherent risk measures are the appropriate functionals for the investigation of heavy-tailed phenomena, since they determine effectively the minimal size of the solvency capital, needed for the exclusion of the risk exposure. We construct coherent risk measures, whose the acceptance sets represent general conic subsets of topological linear spaces. Furthermore, the sub-additivity in coherent risk measures is related to the properties of the *convolution closure* and the *max-sum equivalence* of distributions. We remind the classes of distributions with *long tails*

$$\mathcal{L} = \left\{ F \;\middle|\; \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1,\ \forall\, y \in \mathbb{R} \right\},$$

(or equivalently for some $y \in \mathbb{R}$), where $\overline{F} = 1 - F$ denotes the tail of the distribution, with *subexponentiality*

$$\mathcal{S} = \left\{ F \;\Big|\; \lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n,\ \forall\, n = 2,\, 3, \dots \right\},$$

and with *dominatedly varying tails*

$$\mathcal{D} = \left\{ F \;\Big|\; \limsup_{x\to\infty} \frac{\overline{F}(xu)}{\overline{F}(x)} < \infty,\ \forall\, u \in (0,1) \right\}.$$

By $\mathbf{1}$ is denoted the constant random variable with $\mathbf{1}(\omega) = 1,\ \forall\, \omega \in \Omega$.

Lemma 3. *If $F_X \in \mathcal{D}$, then $F_{X+a\mathbf{1}} \in \mathcal{D}$, for any $a \in \mathbb{R}$.*
If $F_X \in \mathcal{L}$, then $F_{X+a\mathbf{1}} \in \mathcal{L}$, for any $a \in \mathbb{R}$.

Lemma 4. *If $F_X \in \mathcal{D}$, then $F_{rX} \in \mathcal{D}$ for any $r \geq 0$.*
If $F_X \in \mathcal{L}$, then $F_{rX} \in \mathcal{L}$ for any $r \geq 0$.

Remark 2. We notice that the properties indicated by both Lemmas 4 and 3 are related to Positive Homogeneity and Translation Invariance, respectively.

In order to examine whether the class of heavy-tailed distributions $\mathcal{L} \cap \mathcal{D}$, is wedge-closed in the corresponding L^p-space, we check not only the convolution-closure property but the existence of the moments

$$\mathbb{E}_\mu(|X|^p)\,,$$

as well. For example, Pareto distributions belong to the convolution-closed family $\mathcal{L} \cap \mathcal{D}$, but there are members from this class, which do not belong to L^1. In our approach we stay clear from the classical financial setup. This deviation is to be expressed in geometrical terms.

Let us consider the *Lundberg risk model*, namely a sequence $\{Y_k\,,\ k = 1,\, 2,\, \dots\}$ of i.i.d. positive random variables with generic distribution F_Y and an independent homogeneous Poisson process $N(t)$ with constant intensity $\lambda > 0$. Then we have a compound Poisson process in the form

$$S(t) := \sum_{k=1}^{N(t)} Y_k\,,$$

with distribution

$$F_t(x) := \mathbf{P}[S(t) \leq x] = \sum_{n=0}^{\infty} \frac{(\lambda\, t)^n}{n!}\, e^{-\lambda\, t}\, F_Y^{n*}(x)\,, \tag{1}$$

where we denote by

$$F_Y^{n*}(x) := \mathbf{P}[Y_1 + \cdots + Y_n \leq x]\,,$$

the n-th order convolution of F_Y. From [10, Th. 3] we find that the following statements are equivalent

1. $F_t \in \mathcal{S}$.
2. $F_Y \in \mathcal{S}$.
3. $$\lim_{x\to\infty} \frac{\overline{F}_1(x)}{\overline{F_Y}(x)} = \lambda\,.$$

Lemma 5. *If $F_{Y_1},\, F_{Y_2} \in \mathcal{L}\cap\mathcal{D}$, then*

$$a\,F_{Y_1} + (1-a)\,F_{Y_2} \in \mathcal{L}\cap\mathcal{D}\,,$$

for any $a \in (0,\,1)$.

The superposition of the two independent Poisson process $N_1(t)$ and $N_2(t)$ forms another Poisson point process $N(t) = N_1(t) + N_2(t)$, which contains all the points of the two initial point processes. Its intensity $\lambda := \lambda_1 + \lambda_2$ is the sum of the intensities of the two component processes.

Proposition 2. *If $S_1(t),\, S_2(t) \in \mathfrak{W}^{\leftarrow}(\mathcal{L}\cap\mathcal{D})$ then $S_1(t) + S_2(t) \in \mathfrak{W}^{\leftarrow}(\mathcal{L}\cap\mathcal{D})$. Furthermore $\mathcal{L}\cap\mathcal{D}$ is wedge-closed according to Definition 1.*

The **moment index**

$$I(X) = \sup\{v > 0 \mid \mathbb{E}_\mu(|X|^v) < \infty\}\,,$$

of a random variable $X \in L^0(\Omega,\,\mathcal{F},\,\mu)$, determines the minimal locally convex L^p-space which contains X, if $I(X) \geq 1$. If the value of $I(X)$ is equal to ∞ this implies that every moment of it exists X, therefore is not a heavy-tail distributed random variable. In this case, the corresponding minimal L^p-space is L^∞. If $1 \leq I(X) = p < \infty$, then the random variable is heavy-tailed distributed, and the corresponding minimal space is actually $L^{I(X)}$. This is the way by which the clases of heavy-tailed distributions are combined with the L^p spaces.

2 Adjusted Expected Shortfall

Definition 4. The risk measure

$$AES_{a,\,b,\,\varepsilon}(X) := \sup_{Q\in\mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-X)\,,$$

defined on the wedge $C \cap L^{1+\varepsilon}$ for any $\varepsilon \geq 0$, where $a < b$ with $(a,\,b) \in (0,\,1] \times [0,\,\infty)$ and

$$\mathcal{Z}_{a,\,b,\,\varepsilon} = \left\{ Q \ll \mu \,\Big|\, \frac{1}{b} \leq \frac{dQ}{d\mu} \leq \frac{1}{a},\ \mu - \text{a.}s.\,,\ \frac{dQ}{d\mu} \in L^{1+1/\varepsilon}(\Omega, \mathcal{F}, \mu) \right\}\,,$$

is called Adjusted Expected Shortfall.

Due to the result [15, Th. 4.1] about the dual representation for the usual Expected Shortfall on L^1, the financial solvency capital under $AES_{a,\,b,\,\varepsilon}(X)$ is less than $ES_a(X)$ for a position $X \in C \cap L^{1+\varepsilon}$, where $C \subseteq \mathfrak{W}^{\leftarrow}(\mathcal{K})$,

$$ES_a(X) \geq AES_{a,\,b,\,\varepsilon}(X)\,.$$

This fact indicates that the regulator under $AES_{a,\,b,\,\varepsilon}$ is more risk averse, underestimating the needed capital for the solvency of position X. This capital conservatism should be justified by the presence of a, b where $a < b$ with $(a,\, b) \in (0,\, 1] \times [0,\, \infty)$. Moreover, the reduction of the solvency capital based on the knowledge of the class of the heavy-tailed distribution of X, merits a special attention.

It is essential to define $AES_{a,\,b}$ on some $L^{1+\varepsilon}$-space, where $\varepsilon \geq 0$, since these are Banach spaces in which heavy-tailed random variables belong. The importance of the moment index is related to the local convexity of the topologies which are compatilbe with the dual pair

$$\left\langle L^{1+\varepsilon},\ L^{1+\frac{1}{\varepsilon}} \right\rangle\,,$$

for $\varepsilon \geq 0$. The compactness properties of these linear topologies affect the optimization problems related to the risk minimization of $AES_{a,\,b}$. For example, in the corresponding problems, if $\varepsilon = 0$, the uniqueness of solution via saddle-points cannot be assured by Theorem [20, Th. 4.2], since L^1 has not a separable dual.

Proposition 3. *$AES_{a,\,b,\,\varepsilon}$ is a $(\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cup \{0\},\ L_+^{1+\varepsilon})$-cone-coherent risk measure, for any $\varepsilon \geq 0$.*

Remark 3. Proposition 2 implies the Subadditivity of $AES_{a,\,b}$.

Let us assume that the distribution F_Y lies in the class $\mathcal{J}$. If we suppose that the interest force is zero, the surplus of an insurance company is equal to

$$Z_t = u + ct - S(t)\,,$$

for some time horizon $t \in [0,\, T]$ of a fixed length T, where c is the premium rate and u is the initial capital of the company. In order to remain inside the acceptance set of $AES_{a,\,b,\,\varepsilon}$ continuously, we need that $AES_{a,\,b,\,\varepsilon}(Z_t) = AES_{a,\,b,\,\varepsilon}(u + ct\,\mathbf{1} - S(t)) \leq 0$ and hence

$$AES_{a,\,b,\,\varepsilon}(Z_t) = \sup_{Q\in\mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-u-ct\,\mathbf{1}+S(t))$$
$$= -u-ct-\inf_{Q\in\mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q[-S(t)] \le 0\,. \qquad (2)$$

The process $S(t)$ represents a random sum in $L^{1+\varepsilon}(\Omega,\,\mathcal{F},\,\mu)\cap\mathfrak{W}^{\leftarrow}(\mathcal{J})$ for any $\varepsilon\ge 0$, hence $S(t)\in\mathfrak{W}^{\leftarrow}(\mathcal{J})$, given that $\mathbb{E}(Y^{1+\varepsilon})<\infty$. Let us introduce the set

$$\mathcal{D}_{a,\,b,\,\varepsilon} := \left\{ \frac{dQ}{d\mu}\in L^{1+\frac{1}{\varepsilon}}(\Omega,\,\mathcal{F},\,\mu)\;\Big|\; \frac{1}{b}\le\frac{dQ}{d\mu}\le\frac{1}{a},\mu-a.s.\right\}.$$

We restrict ourselves in the class of distributions with long and dominatedly varying tails $\mathcal{L}\cap\mathcal{D}$.

Lemma 6. *The set $\mathcal{D}_{a,\,b,\,\varepsilon}$ is a $\sigma(L^{1+1/\varepsilon},\;L^{1+\varepsilon})$-compact (weak-star) set in the space $L^{1+1/\varepsilon}(\Omega,\,\mathcal{F},\,\mu)$, for $\varepsilon\ge 0$.*

A quantitative advantage of $AES_{a,\,b}$ related to ES_a, is that $AES_{a,\,b}$ may be calculated via its dual representation. However, what is deduced is that $AES_{a,\,b}(X)$ as a supremum on the set of Radon-Nikodym derivatives $\mathcal{D}_{a,\,b,\,\varepsilon}$ represents the maximum over the subset of the probability measures $Q\ll\mu$ with

$$\frac{dQ}{d\mu}=\frac{1}{b}\mathbf{1}\,.$$

This property of $AES_{a,\,b}$ is not extensible in the case of ES_a.

Since $\rho := AES_{a,\,b,\,\varepsilon}$ is wedge-coherent for $X\in L^{1+\varepsilon}$, we find $\rho_{(\mathcal{A}_\rho,\,\mathbf{1})}=\rho$, where

$$\mathcal{A}_\rho = \{X\in L^{1+\varepsilon}|\rho(X)\le 0\}\,.$$

Capital conservatism is useful, given that the company avoids ruin during the time-horizon $[0,\,T]$. It is well-known by [17, Pr. 2.3] or [18, Def. 3.10], that

$$\rho_{(L_+^{1+\varepsilon},\,e)}(X)=\inf\{m\in\mathbb{R}\mid m\,e+X\in L_+^{1+\varepsilon}\}\,,$$

is a $(L_+^{1+\varepsilon},\,e)$-coherent risk measure, where $L_+^{1+\varepsilon}$ is wedge and e represents a radial interior point on the ordered linear space L. The precise value of the solvency capital is equal to $\rho_{(L_+^{1+\varepsilon},\,\mathbf{1})}(Z_T)$. Since

$$L_+^{1+\varepsilon}\subseteq\mathcal{A}_{AES_{a,\,b,\,\varepsilon}}=\mathcal{A}_\rho\,,$$

we have that $\rho_{(L_+^{1+\varepsilon},\,\mathbf{1})}(Z_T)\ge AES_{a,\,b,\,\varepsilon}(Z_T)$.

Hence, conditionally on the fact that the insurance company avoids ruin, the part of the solvency capital is equal to $AES_{a,\,b,\,\varepsilon}(Z_T)\overline{\psi}(u,\,T)$, where

$\psi(u, T) = 1 - \overline{\psi}(u, T)$ denotes the ruin probability of the company over the time-horizon $[0, T]$ with initial capital u. The actuarial solvency capital, conditionally on the ruin of the company through the time-horizon $[0, T]$, is equal to $\rho_{(L^{1+\varepsilon}_+, \mathbf{1})}(Z_T)$. Therefore, the total solvency capital permits a representation in the form

$$SRM_{a,\,b,\,\varepsilon,\,u}(Z_T) := AES_{a,\,b,\,\varepsilon}(Z_T)\overline{\psi}(u,\,T) + \rho_{(L^{1+\varepsilon}_+, \mathbf{1})}(Z_T)\psi(u,\,T),$$

named Solvency Risk Measure over the time-horizon $[0, T]$. The first term is negative and represents a reserve which has to be formed by the insurance company as a result of its survival, while the second positive term corresponds to the deficit which should be covered by extra finance backing (as for example through some loan).

The point-wise ordering, that is the partial ordering induced by the cone $L^p_+ = \{x \in L^p \mid x \geq 0,\ \mu - a.e.\}$ on L^p, makes the space L^p a **Banach lattice**, for $1 \leq p \leq \infty$. We find from [14, Th.4.4.4], that for an infinite-dimensional Banach lattice L, whose cone L_+ is well-based, the norm-interior of the positive cone L_+ is empty. A cone C is a well-based, if there exists a base of the form $\{x \in C \mid f(x) = 1\}$ with $f \in L^*$. The cone L^1_+ is well-based, since with $f = \mathbf{1} \in L^\infty$ such a base is defined. However, as μ is a probability measure on $(\Omega, \mathcal{F})$, then $\mathbf{1} \in L^1_+(\Omega, \mathcal{F}, \mu)$ is also probability measure on $(\Omega, \mathcal{F})$ but by the previous result it does not represent an interior point of L^1_+.

In [2, Th. 7.52], is indicated that in a Riesz pair $\langle L, L^* \rangle$ a vector $x \in L_+$ is **strictly positive** if and only if the ideal $L_x = \{y \in L \mid |y| \leq r\,|x|,\ r > 0\}$ is weakly dense in L. If L_x is weakly dense in L, it is also norm-dense in L, because L_x is a subspace of L, hence x is a **quasi-interior point** of L_+.

An element $x \in L_+$ is called **strictly positive** in L_+ if $x^*(x) > 0$ for any $x^* \in L^*_+ \setminus \{0\}$. We notice that $\mathbf{1}$ is a strictly positive element of L^1_+, hence it is a quasi-interior point of L^1_+. This holds because $\langle L^1, L^\infty \rangle$ is a **Riesz pair**, or else L^∞ is an ideal in the order dual space of L^1, since L^1 and L^∞ among others, are defined on a probability space.

Since the risk measure ρ is defined with respect to L^1_+ and $\mathbf{1}$, we use the fact that $\mathbf{1}$ is a quasi-interior point of L^1_+ to deduce properties of ρ.

Theorem 1. *The risk measure $SRM_{a,\,b,\,\varepsilon,\,u}$ over the time-horizon $[0, T]$, represents a cone-coherent risk measure, defined on $\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cap L^{1+\varepsilon}$, for $\varepsilon \geq 0$.*

The advantage of $AES_{a,\,b}$, where $0 < a < b \leq \infty$, with regard to ES_a is that due to its definition by a specific dual representation, the supremum indicated by this dual representation is attained on the set of probability measures on $(\Omega, \mathcal{F})$, which are absolutely continuous with respect to μ and their Radon-Nikodym derivative

$$\frac{dQ}{d\mu}\,,$$

is equal to $1/b$, $\mu - a.e.$. This makes the calculation of $AES_{a,\,b}$ easier either combined with Wald identity or with Blackwell Renewal Theorem, if we would like to find $AES_{a,\,b}(Z_T)$. The same attempt of calculation of $ES_a(Z_T)$ through its own dual representation (see [15, Th. 4.1]), does not permit such a simplification. Namely, despite the fact that the set of the Radon-Nikodym derivatives $\mathcal{D}_a$ in its dual representation is $\sigma(L^\infty,\ L^1)$-compact set, the extreme set on which the maximum is attained (implied by Krein-Milman Theorem), cannot be directly specified.

Finally, we have the following convergence:

Theorem 2. *For any $X \in L^{1+\varepsilon}$, $AES_{a,\,b}(X) \to ES_a(X)$, if $b \to \infty$, where $\varepsilon \geq 0$.*

3 Optimization in $L^{1+\varepsilon}$

Let us consider the following risk minimization problem

$$Minimize\ AES_{a,\,b,\,\varepsilon}(X)\ subject\, to\ X \in \mathcal{X}\,, \tag{3}$$

where we suppose the set $\mathcal{X}$ to be a convex, (closed and) bounded. Namely, it is a subset of an appropriate multiple of the closed unit ball of the space $L^{1+\varepsilon}$. By Eberlein-Šmulian and Alaoglou theorems of weak compactness, the closed unit ball in this case is weakly compact.

The meaning of this problem is that if $\mathcal{X}$ is a set of investments' payoffs at time-period T, the investor would like to know the minimal capital to put in advance, which is to be spent for the solvency of his investment.

According to previous setup, $\mathcal{X}$ represents a subset of the wedge

$$C = \overline{\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D})}^{d}\,,$$

under a possible change of the probability space, according to Lemma 1. Moreover, we consider this set $\mathcal{X}$ in the relevant $L^{1+\varepsilon}$-space, for some $\varepsilon > 0$. Since the convolution-closure holds for the classes of distributions $\mathcal{L} \cap \mathcal{D}$, and $\mathcal{D}$ (see [11], [7]), from Lemma 4 and Lemma 2 we see that these classes are also convex-closed. Hence the solution of the optimization problem (3) depends only on the existence of the moments

$$\mathbb{E}_\mu(Y^k)\,,$$

of the claim variable Y.

In order to solve the problem (3), we use the following min-max Theorem, suggested in [8, p. 10].

Let K be a compact, convex subset of a locally convex space $\mathcal{Y}$. Let L be a convex subset of an arbitrary vector space $\mathcal{E}$. Suppose that u is a bilinear *function* $u : \mathcal{E} \times \mathcal{Y} \to \mathbb{R}$. *For each* $l \in \mathcal{L}$, *we suppose that the partial (linear) function* $u(l, \cdot)$ *is continuous on* $\mathcal{Y}$. *Then we have that*

$$\inf_{l \in L} \sup_{k \in K} u(l,k) = \sup_{k \in K} \inf_{l \in L} u(l,k)\,. \tag{4}$$

A relevant dual system $\langle E,\ F \rangle$ is to be considered in the definition of the metric space on the functionals

$$M_2 = \left\{ \begin{array}{c} u\ :u(X,\,\cdot)\ \text{l.s.c. on}\ \varPi\,,\ u(\cdot,\,\pi)\ \text{u.s.c. on}\ \varSigma\,, \\ \sup_{\varSigma \times \varPi} |u(X,\,\pi)| < \infty\,, \\ \exists\,(X_0,\,\pi_0) \in \varSigma \times \varPi\ \text{saddle point of}\ u\,, \end{array} \right\}$$

for any functions $u\ :\ \varSigma \times \varPi \to \mathbb{R}$, endowed by the metric

$$\rho_2(u_1, u_2) = \sup_{\varSigma \times \varPi} |u_1(X,\pi) - u_2(X,\pi)|\,,$$

with $u_1, u_2 \in M_2$. The pair (M_2, ρ_2) is a metric space of the saddle-point problems mentioned in [20, Lem. 4.4]. Since under the weak topology, the weak compactness and the weak sequentially compactness coincide in a normed space according to the Eberlein-$\check{S}$mulian Theorem, $\varSigma$, $\varPi$ have to be weakly compact in the relevant weak topologies ($\sigma(E,F)$ and $\sigma(F,E)$ respectively). If we select our dual system to be $\left\langle L^1, L^\infty \right\rangle$, the order-intervals are $\sigma(L^\infty, L^1)$ (weak-star) compact in L^∞, see [2, Lem. 7.54]. By this way we may extend the Lemma 6 for the case of weak-star compactness for

$$\mathcal{D}_{a,\,b,0} \subset L^\infty(\varOmega,\,\mathcal{F},\,\mu)\,.$$

According to [20, Th. 4.2], the uniqueness of the saddle-point for the relevant optimization problem (3) depends on the compactness of the sets $\varSigma$, $\varPi$ (weak compactness and weak sequentially compactness respectively) of the sets of the arguments of the objective functional f.

Proposition 4. *The minimization problem* (3) *has a solution in* $L^{1+\varepsilon}$ *with* $\varepsilon \geq 0$.

Definition 5. A **residual subset** of a Hausdorff topological space M is a set $Q \subseteq M$ which contains the intersection of countable dense, open subsets of M.

Based on [20, Rem. 2.2], we find that the space $(L^{1+\varepsilon},\ \sigma(L^{1+\varepsilon},\ L^{1+1/\varepsilon}))$ belongs to the class $\varLambda$ of Hausdorff spaces with the property that every u.s.c. mapping $S_2\ :\ M_2 \to 2^{\varSigma \times \varPi}$ is almost l.s.c. on some dense residual subset of its domain (the almost l.s.c. at $u \in M_2$ property means that if there exists $x \in S_2(u)$ such that for each open neighborhood U of x, there exists and an open neighborhood O of u, such that $U \cap S_2(u') \neq \emptyset$ for each $u' \in O$). So, we can apply [20, Th. 4.2] to establish the uniqueness of the saddle-point of u.

Definition 6. If there exists a dense residual subset Q of the Baire space M such that for any $u \in Q$ a certain property P depending on u holds, then we say that the property P is **generic** in M.

In order to deduce the generic uniqueness of the saddle point for the problem (3), we have to examine whether this problem belongs to the space of the saddle point problems (M_2, ρ_2).

Lemma 7. *1. For $X \in L^{1+\varepsilon}$, the function $u_X : L^{1+1/\varepsilon} \to \mathbb{R}$, with $u_X(\pi) = u(X,\pi) = \pi(-X)$, for any $\pi \in L^{1+1/\varepsilon}$ is $\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})$ lower-semicontinuous on $L^{1+1/\varepsilon}$, for $\varepsilon > 0$.*

2. For $\pi \in L^{1+1/\varepsilon}$, the function $u_\pi : L^{1+\varepsilon} \to \mathbb{R}$, with $u_\pi = u(X,\pi) = \pi(-X)$, for any $X \in L^{1+\varepsilon}$ is $\sigma(L^{1+\varepsilon},\ L^{1+1/\varepsilon})$ upper semicontinuous on $L^{1+\varepsilon}$, for $\varepsilon > 0$.

3. The function $u : L^{1+\varepsilon} \times L^{1+1/\varepsilon} \to \mathbb{R}$, with $u(X,\pi) = \pi(-X)$, for $X \in L^{1+\varepsilon}$ and $\pi \in L^{1+1/\varepsilon}$ satisfies

$$\sup_{\mathcal{X} \times \mathcal{D}_{a,\, b,\, \varepsilon}} |\pi(-X)| < \infty \,.$$

Let us keep in mind [20, Th.4.2] and [20, Le.2.1], where this last Lemma together with Closed Graph Theorem (see [2, Th. 16.11]) and Theorem [2, Th. 16.12] imply that if Σ is Hausdorff then the conclusion of this Lemma can be replaced by the fact that S is an u.s.c. map, namely $S(u)$ is compact for any $u \in M$.

Theorem 3. *The saddle-point which solves the problem* (3) *is unique in the dual system $\left\langle L^{1+\varepsilon}(\Omega,\, \mathcal{F},\, \mu),\ L^{1+1/\varepsilon}(\Omega,\, \mathcal{F},\, \mu)\right\rangle$, for $\varepsilon > 0$.*

Proposition 5. *The problem* (3) *has generically a unique solution, if $F_X \in \mathcal{D}$ for any $X \in \mathcal{X}$ and $1 + \varepsilon < \delta_{\overline{F}_X}$, while $X \geq 0$, $\mu - a.s.$, where $\varepsilon \geq 0$, if $L^{1+\frac{1}{\epsilon}}$ is separable.*

The problem (3) *has generically a unique solution, if $F_X \in \mathcal{D} \cap \mathcal{L}$ for any $X \in \mathcal{X}$ and $1 + \varepsilon < \delta_{\overline{F}_X}$, while $X \geq 0$, $\mu - a.s.$, if $L^{1+\frac{1}{\epsilon}}$ is separable.*

Among the topological spaces of class Λ, we consider separately the metric spaces, since by Lemma [20, Lem. 2.2, Lem. 2.3], when S is a u.s.c. mapping then it is also l.s.c. at any $u \in Q$ for some residual subset Q.

If the Hausdorff spaces X, Y, endowed with the weak and the weak-star topology of $\Sigma = L^{1+\varepsilon}$ and $\Pi = L^{1+\frac{1}{\varepsilon}}$, respectively for any $\varepsilon > 0$, are weakly and weak-star compact sets, then holds the generic uniqueness of the saddle point, as shown in Theorem [20, Th. 4.2]. We remark that in this case, we use James Theorem (see [2, Th. 6.36]) and the reflexivity of the corresponding L^p spaces in order to apply [20, Lem. 2.1] for the map S_2 in [20, Th. 4.2], which requires the space $\Sigma \times \Pi$ to be compact under the product topology.

In case the financial positions belong to L^1, the generic uniqueness result in [20, Th. 4.2] is no more applicable. Therefore, the generic uniqueness in

Proposition 5 fails. The cause of this failure is that [20, Th. 4.2] together with [20, Lem. 2.1] require both the Σ, Π (of L^1 and L^∞ respectively) to be weakly compact and weakly-star compact respectively. Hence, for application of [20, Lem. 2.1], we need Σ to be a weakly compact subset of L^1 and moreover we must check the metrizability of X as a weakly compact subset of L^1, so that we verify its membership of the class Λ. This makes us to ask if a countable total subset of L^∞ exists, according to [2, Th. 6.36]. However, L^∞ spaces are not separable, since $L^\infty[0,1]$ is not separable. Indeed, there exists an uncountable set of $L^\infty[0,1]$ elements, such as the indicator functions of the intervals $[0,t]$ for any $t \in [0,1]$, for which $\|f-g\|_\infty = 1$ whenever $f \neq g$ (see [19, Pr. 1.21.1 and Exam. 1.24]). Furthermore, in case Ω is uncountable, we consider the set of the indicator functions G_A of the non-empty subsets of the uncountable $A \in \mathcal{F}$. Hence, if $C, B \in G_A$, then $\|\mathbf{1}_C - \mathbf{1}_B\|_\infty = 1$.

By the Extension Theorem [13, Th. 2.2], the canonical space of law-invariant convex risk measures is considered to be L^1. However, by the above argument this is not valid, due to the fact that the optimization results for uniqueness in Theorem 3 can not be extended to L^1.

3.1 Estimation of $AES_{a,b}(Z_T)$

Under the assumption that the second moment of the claim-size variable is finite, in the Lundberg risk model the random variable Z_T belongs to $L^{1+\varepsilon}$. Thus, the $AES_{a,\,b,\,\varepsilon}(-S(T))$ may be understood as solvency capital for the company at the end of the time interval $[0,T]$ and its minimization is deduced to the problem of minimization of a linear functional of $L^{1+\varepsilon}$ (namely the variable $-S(T)$), over a weakly compact and convex set of Radon-Nikodym derivatives $\mathcal{D}_{a,\,b,\,\varepsilon}$.

According to Krein-Milman Theorem (see [2, Th. 5.117]) and the Bauer Maximum Principle (see [2, Th. 5.118]) exists a Radon-Nikodym derivative in $\mathcal{D}_{a,\,b,\,\varepsilon}$, such that minimizes $AES_{a,\,b,\,\varepsilon}$ and it represents an extreme point of $\mathcal{D}_{a,\,b,\,\varepsilon}$, since the mean value $\mathbb{E}_Q(-S(T))$ represents the value of the evaluation map

$$\left\langle -S(T),\ \frac{dQ}{d\mu} \right\rangle,$$

in $L^{1+\varepsilon}(\Omega,\,\mathcal{F},\,\mu)$, which is weakly u.s.c. at the first variable.

Lemma 6 allows the computation of $AES_{a,\,b,\,\varepsilon}(Z_T)$ in the Lundberg risk model and it remains invariable for certain classes of heavy-tailed distributions.

Lemma 8. *In the Lundberg risk model, if $F_Y \in \mathcal{L} \cap \mathcal{D}$ and the inter-occurrence times follow the Exponential Distribution $Exp(\lambda)$, holds*

$$AES_{a,\,b,\,\varepsilon}(Z_T) = -u - c\,T + \frac{\lambda\,T}{b}\mathbb{E}_\mu(Y)\,, \tag{5}$$

if the claim-size $Y \in L^{1+\varepsilon}$ *for any* $\varepsilon \geq 0$.

Let us consider now the class of extended regularly varying tailed distributions $ERV(-\gamma,-\delta)$, where $0 \leq \gamma \leq \delta < \infty$, a subclass of the class $\mathcal{L} \cap \mathcal{D}$, for some more calculations, specifically relying on Matuszewska indexes (see [22]). This class includes all distributions F, such that

$$y^{-\delta} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\gamma}\,.$$

We remind that the lower Matuszewska index of a c.d.f. F is defined as follows:

$$\delta_{\overline{F}} = -\lim_{y\to\infty} \frac{\ln \overline{F}^*(y)}{\ln y}\,,$$

where

$$\overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\,.$$

Thereupon, we reach the next results for the solvency capital in the Lundberg risk model with respect to the Adjusted Expected Shortfall. We observe that the calculations of $AES_{a,\,b}(S_T)$ on certain subclasses of $\mathcal{D} \cap \mathcal{L}$ under the Lundberg or the renewal risk model are extended over $\mathcal{D}$, under some appropriate conditions, related to the values of the lower Matuszewska index of the claim-size distribution.

Theorem 4. *In the Lundberg risk model, if the inter-occurrence distribution is* $Exp(\lambda)$ *and the claim size distribution* F_Y *belongs to class* $\mathcal{D}$ *with* $1 < \delta_{\overline{F}_Y}$*, then relation* (5) *remains intact.*

Remark 4. In the Lundberg risk model, if the inter-occurrence time distribution is $Exp(\lambda)$*, the claim size distribution* F_Y *belongs to class* $\mathcal{L}\cap\mathcal{D}$ *or belongs to class* $ERV(-\gamma,-\delta)$ *with* $1 \leq \gamma \leq \delta < \infty$ *or belongs to class* $\mathcal{R}_{-(1+\varepsilon)}$ *with* $\varepsilon > 0$ *and* $1 < \delta_{\overline{F}_Y}$*, then relation* (5) *remains intact.*

Further, we calculate the solvency capital with respect to the Adjusted Expected Shortfall in the case of the Renewal Risk Model, by using the Blackwell Theorem (see [3, p. 118] or [12]).

Corollary 1. *If in the renewal risk model, where the distribution of the inter-occurrence of the claim-payments is* F_A *such that* $\mathbb{E}_\mu(A) < \infty$*, the claim size variable* Y *has the property that* F_Y *belongs to the class* $\mathcal{D}$ *or that* F_Y *belongs to the class* $\mathcal{L} \cap \mathcal{D}$ *or that* F_Y *belongs to the class* $ERV(-\gamma,-\delta)$*, where*

$1 \leq \gamma \leq \delta < \infty$ and $1 < \delta_{\overline{F}_Y}$ or that F_Y belongs to the class $\mathcal{R}_{-(1+\varepsilon)}$, where $\varepsilon > 0$ and $1 < \delta_{\overline{F}_Y}$, then

$$AES_{a,\,b,\,\varepsilon}(Z_T) = -u - c\,T + \frac{T}{b\mathbb{E}_\mu(A)}\mathbb{E}_\mu(Y)\,.$$

Remark 5. We observe that the calculation of $AES_{a,\,b}(Z_T)$ remains in tact for the Renewal Model under certain assumptions.

4 Proofs

Proof of Proposition 1

Let us consider a sequence $\{X_n, n \in \mathbb{N}\}$, where $X_n \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$ and $X_n \stackrel{d}{\to} X$ and another sequence $\{Y_n, n \in \mathbb{N}\}$, where $Y_n \in \mathfrak{W}^{\leftarrow}(\mathcal{J})$ and $Y_n \stackrel{d}{\to} Y$. Then we apply Proposition 1 to find the probability space $(A, \mathcal{G}, \nu)$ and the random variables

$$X'_n\,,\; X'\,,\; Y'_n\,,\; Y' \in L^0(A,\,\mathcal{G},\,\nu)\,,$$

such that hold $X'_n \stackrel{d}{=} X_n$ and $Y'_n \stackrel{d}{=} Y_n$ for any $n \in \mathbb{N}$. Hence we find $X' \stackrel{d}{=} X$, $Y' \stackrel{d}{=} Y$ and $X'_n \to X'$, $Y'_n \to Y'$ ν-a.s. The ν-a.s. convergence of both sequences implies $X'_n + Y'_n \to X' + Y'$, ν-a.s., which implies $X'_n + Y'_n \stackrel{d}{\to} X' + Y'$.

If

$$X_n \in \mathfrak{W}^{\leftarrow}(\mathcal{J})\,,$$

for any $n \in \mathbb{N}$ and further $X_n \stackrel{d}{\to} X$ as $n \to \infty$, again we obtain random variables

$$X'_n\,,\; X' \in L^0(A,\,\mathcal{G},\,\nu)\,,$$

with $X'_n \stackrel{d}{=} X_n$ and $X' \stackrel{d}{=} X$ such that $X'_n \to X'$, ν-a.s. In case $r = 0$ this means that 0 is the a.s. limit and even the weak limit of the sequence of $\{r\,X'_n\,,\; n \in \mathbb{N}\}$ and of the sequence $\{r\,X_n\,,\; n \in \mathbb{N}\}$ too. In case $r > 0$, since $X'_n \to X'$, ν-a.s., then we obtain $r\,X'_n \to r\,X'$, ν-a.s. and hence it implies $r\,X'_n \stackrel{d}{\to} r\,X'$. Therefore, $\overline{\mathfrak{W}^{\leftarrow}(\mathcal{J})}^d$ is a wedge of $L^0(A, \mathcal{G}, \nu)$. □

Proof of Lemma 2

Since $\mathcal{J}$ is wedge-closed, it is *convolution-closed*. Namely, for any distributions $F_{X_1}, F_{X_2} \in \mathcal{J}$, where X_1, X_2 are random variables, we have that $F_{X_1+X_2} \in \mathcal{J}$. We may also suppose that $X_1 = a_1\,Y_1$, for $a_1 > 0$ and $X_2 = a_2\,Y_2$, for $a_2 > 0$, due to the fact that $\mathcal{J}$ is wedge-closed. We define the random variable

$$D = \frac{a_1}{a_1 + a_2} Y_1 + \frac{a_2}{a_1 + a_2} Y_2 \,,$$

being the abstract convex combination of two variables which belong to $\mathcal{J}$. Due to the properties of the family $\mathcal{J}$, we have $F_D \in \mathcal{J}$. By induction we take the required result. □

Proof of Lemma 3

We have that $F_{X+a\mathbf{1}}(x) = \mu(X + a\mathbf{1} \leq x) = F_X(x - a)$, for any $x, a \in \mathbb{R}$. Hence

$$\limsup_{x\to\infty} \frac{\overline{F}_{X+a\mathbf{1}}(xy)}{\overline{F}_{X+a\mathbf{1}}(x)} = \limsup_{x\to\infty} \frac{\overline{F}_X(xy - a)}{\overline{F}_X(x - a)} \,.$$

We set $u := x - a$ and therefore we can notice that for a certain $a \in \mathbb{R}$ if $x \to \infty$, then $u \to \infty$. Then the last expression becomes

$$\limsup_{u\to\infty} \frac{\overline{F}_X(uy - [1 - y]\, a)}{\overline{F}_X(u)} \,,$$

which is equal to

$$\limsup_{u\to\infty} \frac{\overline{F}_X(uy)}{\overline{F}_X(u)} \,.$$

Hence the initial limit is finite for any $y \in (0, 1)$, since $F_X \in \mathcal{D}$.

As before, $F_{X+a\mathbf{1}}(x) = F_X(x - a)$ for any $x \in \mathbb{R}$ for a certain $a \in \mathbb{R}$. The assumption $F_X \in \mathcal{L}$ is equivalent to

$$\lim_{x\to\infty} \frac{\overline{F}_X(x - y)}{\overline{F}_X(x)} = 1 \,,$$

for all $y \in \mathbb{R}$. For $X + a\mathbf{1}$ we have to show that

$$\lim_{x\to\infty} \frac{\overline{F}_{X+a\mathbf{1}}(x - y)}{\overline{F}_{X+a\mathbf{1}}(x)} = 1 \,,$$

for all $y \in \mathbb{R}$. Indeed, this is equivalent to

$$\lim_{x\to\infty} \frac{\overline{F}_X(x - y - a)}{\overline{F}_X(x - a)} = 1 \,.$$

Put $u := x - a$ and then $u \to \infty$, as $x \to \infty$. Therefore

$$\lim_{u\to\infty} \frac{\overline{F}_X(u - y)}{\overline{F}_X(u)} = 1 \,,$$

for any $y \in \mathbb{R}$ by assumption. This completes the proof, since it shows that for any $a \in \mathbb{R}$, $X + a\mathbf{1} \in \mathfrak{W}^{\leftarrow}(\mathcal{L})$.

We finish with $\mathfrak{W}^{\leftarrow}(\mathcal{C})$. If $X \in \mathfrak{W}^{\leftarrow}(\mathcal{C})$, then

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy-a)}{\overline{F}(x)} = \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1\,,$$

since for any specific $a \in \mathbb{R}$, $F_{X+a\mathbf{1}}(x) = F_X(x-a)$ for any $x \in \mathbb{R}$. □

Proof of Lemma 4

We have to show that

$$\limsup_{x\to\infty}\frac{\overline{F}_{rX}(xy)}{\overline{F}_{rX}(x)} < \infty\,,$$

for any $y \in (0,1)$. This is equivalent with

$$\limsup_{x\to\infty}\frac{\overline{F}_{X}(xy/r)}{\overline{F}_{X}(x/r)} < \infty\,.$$

However, $u := x/r \to \infty$ as $x \to \infty$ and this last observation implies

$$\limsup_{u\to\infty}\frac{\overline{F}_{X}(uy)}{\overline{F}_{X}(u)} < \infty\,,$$

for any $y \in (0,1)$, which is the condition for $F_X \in \mathcal{D}$.

The assumption $F_X \in \mathcal{L}$ is equivalent to

$$\lim_{x\to\infty}\frac{\overline{F}_{X}(x-y)}{\overline{F}_{X}(x)} = 1\,,$$

for any $y \in \mathbb{R}$. So, for $r\,X$ we have to show that

$$\lim_{x\to\infty}\frac{\overline{F}_{rX}(x-y)}{\overline{F}_{rX}(x)} = 1\,,$$

for some $y \in \mathbb{R}$. Indeed, this is equivalent to

$$\lim_{x\to\infty}\frac{\overline{F}_{X}\left([x-y]/r\right)}{\overline{F}_{X}(x/r)} = 1\,.$$

From $x \to \infty$, we find that $u = x/r \to \infty$. Then for any $y' = y/r \in \mathbb{R}$ we need that

$$\lim_{u\to\infty}\frac{\overline{F}_{X}(u-y')}{\overline{F}_{X}(u)} = 1\,,$$

as we assumed. We finish with $\mathfrak{W}^{\leftarrow}(\mathcal{C})$. If $X \in \mathfrak{W}^{\leftarrow}(\mathcal{C})$, then

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy/r)}{\overline{F}(x/r)} = \lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1\,,$$

since for any specific $r > 0$,

Therefore

$$F_{rX}(x) = F_X\left(\frac{x}{r}\right)\,,$$

for any $x \in \mathbb{R}$ and also for $r = 0$ the subsequent asymptotic form $0/0$ may be posed by definition equal to 1. □

Proof of Lemma 5

Let us consider $0 < u < 1$. We show first that $a\,F_{Y_1} + (1-a)\,F_{Y_2} \in \mathcal{D}$ or equivalently

$$\limsup_{x\to\infty} \frac{1 - a\,F_{Y_1}(u\,x) - (1-a)\,F_{Y_2}(u\,x)}{1 - a\,F_{Y_1}(x) - (1-a)\,F_{Y_2}(x)} := \limsup_{x\to\infty} R(x) < \infty\,.$$

Indeed, we can write

$$\begin{aligned} R(x) &= \frac{a\,\overline{F}_{Y_1}(u\,x) + (1-a)\,\overline{F}_{Y_2}(u\,x)}{a\,\overline{F}_{Y_1}(x) + (1-a)\,\overline{F}_{Y_2}(x)} \\ &= \frac{\overline{F}_{Y_1}(u\,x)}{\overline{F}_{Y_1}(x) + \dfrac{1-a}{a}\,\overline{F}_{Y_2}(x)} + \frac{\overline{F}_{Y_2}(u\,x)}{\dfrac{a}{1-a}\,\overline{F}_{Y_1}(x) + \overline{F}_{Y_2}(x)} \\ &\leq \frac{\overline{F}_{Y_1}(u\,x)}{\overline{F}_{Y_1}(x)} + \frac{\overline{F}_{Y_2}(u\,x)}{\overline{F}_{Y_2}(x)}\,, \end{aligned}$$

which has finite superior limit by the assumption $F_{Y_1},\, F_{Y_2} \in \mathcal{D}$.

Now we employ [11, Pr. 2] to establish that if $F_{Y_1},\, F_{Y_2} \in \mathcal{D}\cap\mathcal{S}$ then

$$a\,F_{Y_1} + (1-a)\,F_{Y_2} \in \mathcal{S}\cap\mathcal{D}\,.$$

So, it remains to take into account that $\mathcal{S}\cap\mathcal{D} = \mathcal{L}\cap\mathcal{D}$. □

Proof of Proposition 2

The sum $S_1(t)+S_2(t)$ represents a compound Poisson process with Poisson point process $N(t)$ with intensity $\lambda = \lambda_1 + \lambda_2 > 0$ and at each point of this process appears a step with random height

$$Y = p\,Y_1 + (1-p)\,Y_2\,,$$

as mixture of two random variables Y_1 and Y_2 with weights $p = \lambda_1/\lambda$ and $1-p = \lambda_2/\lambda$ respectively.

By assumption we know that $Y_1,\, Y_2 \in \mathcal{L}\cap\mathcal{D}$. Thus by Lemma 5 we find that $Y \in \mathcal{L}\cap\mathcal{D}$ and this through [10, Th. 3] gives the result. □

Proof of Proposition 3

It suffices to prove that the four properties in Definition 3 are valid.

1. For any $t \in \mathbb{R}$, such that $X + t\mathbf{1} \in C$ we find that

$$AES_{a,\,b,\,\varepsilon}(X + t\mathbf{1}) = \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-X) - t = AES_{a,\,b,\,\varepsilon}(X) - t\,.$$

2. For any $X_1, X_2 \in C$, we see that $X_1 + X_2 \in C$ and

$$\begin{aligned} AES_{a,\,b,\,\varepsilon}(X_1 + X_2) &= \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} [\mathbb{E}_Q(-X_1) + \mathbb{E}_Q(-X_2)] \\ &\le \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-X_1) + \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} (\mathbb{E}_Q(-X_2) \\ &= AES_{a,\,b,\,\varepsilon}(X_1) + AES_{a,\,b,\,\varepsilon}(X_2)\,. \end{aligned}$$

3. For any $X \in C$ we see that for any $r \ge 0$ we have $r\,X \in C$ and

$$\begin{aligned} AES_{a,\,b,\,\varepsilon}(r\,X) &= \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} r\,\mathbb{E}_Q(-X) \\ &= r \sup_{Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-X) = r\,AES_{a,\,b,\,\varepsilon}(X)\,. \end{aligned}$$

4. For any $X, Y \in C$ such that $Y \ge X$ under the usual partial ordering in $L^{1+\varepsilon}$, where $\varepsilon \ge 0$, we see that $-X \ge -Y$ and for any $Q \in \mathcal{Z}_{a,\,b,\,\varepsilon}$ we have

$$\mathbb{E}_Q(-X) \ge \mathbb{E}_Q(-Y)\,.$$

By taking supremums over $\mathcal{Z}_{a,\,b,\,\varepsilon}$ we obtain

$$AES_{a,\,b,\,\varepsilon}(X) \ge AES_{a,\,b,\,\varepsilon}(Y)\,. \ \square$$

Proof of Lemma 6
Since

$$\left\langle L^{1+1/\varepsilon},\ L^{1+\varepsilon} \right\rangle\,,$$

is a Riesz pair (see [2, Lem. 8.56]) we obtain that the order-interval $[\mathbf{1}/b\,,\ \mathbf{1}/a]$ is $\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})$-compact, for $\varepsilon \ge 0$. Then, it suffices to prove that

$$\mathcal{D}_{a,\,b,\,\varepsilon} \subseteq \left[\frac{\mathbf{1}}{b}\,,\ \frac{\mathbf{1}}{a}\right]\,,$$

is $\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})$-closed. Let us consider a net $(Q_i)_{i \in I} \subseteq \mathcal{Z}_{a,\,b,\,\varepsilon}$ such that

$$\frac{dQ_i}{d\mu} \overset{\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})}{\longrightarrow} f\,. \tag{6}$$

From the fact that

$$\frac{dQ_i}{d\mu} \in \mathcal{D}_{a,\,b,\,\varepsilon}\,,$$

for any $i \in I$, we obtain that

$$\frac{dQ_i}{d\mu} \in L^{1+1/\varepsilon}(\Omega,\,\mathcal{F},\,\mu)\,.$$

We have to prove that f is a Radon-Nikodym derivative of some measure $Q_1 \in \mathcal{Z}_{a,\,b,\,\varepsilon}$ with respect to μ. Let us consider the map $Q_0 : \mathcal{F} \to [0,1]$ where

$$Q_0(A) = \int_\Omega f\,\chi_A d\mu\,,$$

and χ_A is the indicator function of the set A. In order to show that Q_0 is a probability measure, we should establish

$$Q_0(\Omega) = \int_\Omega f d\mu = 1\,.$$

Indeed, it is equal to

$$\lim_{i\in I} \int_\Omega dQ_i\,,$$

and every of the terms of the net of real numbers

$$\left(\int_\Omega dQ_i\right)_{i\in I}\,,$$

is equal to 1. By the same argument, we may deduce that $Q_0(\emptyset) = 0$. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$Q_i\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} Q_i(A_k)\,,\; i \in I,\; n \in \mathbb{N}\,.$$

Hence,

$$Q_0\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} Q_0(A_k)\,,\; n \in \mathbb{N}\,,$$

due to the weak star convergence (6). With limiting $n \to \infty$, we get that

$$Q_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Q_0(A_n)\,,$$

from the $\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})$-convergence in (6), the definition of Q_0 and the fact that any indicator function $\chi_A, A \in \mathcal{F}$ belongs to $L^{1+\varepsilon}(\Omega,\ \mathcal{F},\ \mu)$ and also to the space $L^1(\Omega,\ \mathcal{F},\ \mu)$. We make use of the Monotone Convergence Theorem [2, Th. 11.18], where the indicator function of the set

$$\bigcup_{n=1}^{\infty} A_k \,,$$

is the μ-integrable function deduced by the Theorem. This holds, because the indicator functions of the sets

$$\bigcup_{k=1}^{n} A_k \,,$$

for $n \in \mathbb{N}$ belong to the space $L^1(\Omega,\ \mathcal{F},\ \mu)$, since they belong to $L^{1+\varepsilon}(\Omega, \mathcal{F}, \mu)$. For the μ-continuity of Q_0, we have that if holds $\mu(A) = 0$ for a set $A \in \mathcal{F}$, then

$$Q_i(A) = 0 = \int_A \frac{dQ_i}{d\mu} d\mu \,,$$

for any $i \in I$, since $Q_i, i \in I$ is μ-continuous. But from (6) we obtain

$$Q_0(A) = \int_A f d\mu = \lim_{i \in I} \int_A \frac{dQ_i}{d\mu} d\mu = 0 \,.$$

Therefore, Q_0 is μ-continuous. Since $Q_i, i \in I$ are probability measures, holds

$$\frac{dQ_i}{d\mu}(\omega) \geq 0 \,,$$

μ-a.s. From the fact that Q_0 is a μ-continuous probability measure, by Radon-Nikodym Theorem we have

$$\frac{dQ_0}{d\mu} = f \,,$$

μ-a.s. and $f(\omega) \geq 0$, μ-a.s. In order to show that

$$\frac{\mathbf{1}}{b} \leq f \leq \frac{\mathbf{1}}{a} \,, \tag{7}$$

with respect to the usual (point-wise) partial ordering on $L^{1+1/\varepsilon}(\Omega,\ \mathcal{F},\ \mu)$, we use the convergence argument

$$\int_A \frac{dQ_i}{d\mu} d\mu \to \int_A f d\mu \,,$$

for any $A \in \mathcal{F}$. This implies that

$$\int_A f d\mu \in \left[\frac{1}{b},\ \frac{1}{a}\right] , \tag{8}$$

for any $A \in \mathcal{F}$ and therefore holds (7). Indeed, if we suppose that it does not hold, then we could find some $B \in \mathcal{F}$ such that either $f(\omega) > 1/a$, or $f(\omega) < 1/b$ for any $\omega \in B$. Then, we would have either

$$\int_B f d\mu > \frac{1}{a} ,$$

or

$$\int_B f d\mu < \frac{1}{b} ,$$

which is the absurd, because of (8). Therefore, the $\mathcal{D}_{a,\, b,\, \varepsilon}$ is $\sigma(L^{1+1/\varepsilon},\ L^{1+\varepsilon})$-closed, for $\varepsilon \geq 0$. □

Proof of Theorem 1

The measure $SRM_{a,\, b,\, 0,\, u}$ is a cone-coherent risk measure on the cone $[\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cup \{0\}] \cap L^1$, since it is a convex combination of two cone-coherent risk measures. The first one, the $AES_{a,\, b,\, \varepsilon}$ is cone-coherent on $[\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cup \{0\}] \cap L^1$, due to Propositions 1, 2 and 3.

The second one $\rho_{(L^1_+, \mathbf{1})}$ is cone-coherent as a restriction of the $(L^1_+, \mathbf{1})$-coherent risk measure $\rho_{(L^1_+, \mathbf{1})}$ on the cone $[\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cup \{0\}] \cap L^1$. Indeed, we observe that $\mathbf{1}$ is a **quasi-interior point** in L^1_+. This means that for any $x \in L^1$, the convergence $\|x - u_n\| \to 0$ is established, where $u_n \in I_{\mathbf{1}}$, and

$$I_{\mathbf{1}} = \bigcup_{n \in \mathbb{N}} [-n\mathbf{1}, n\mathbf{1}]_{L^1_+} .$$

Hence $x = u_n + t_n$, where $t_n \to 0$, and there are $k_n \in \mathbb{N}$ with $|u_n| \leq k_n \mathbf{1}$. This implies the existence of a subsequence of $\{t_n\}$, called again $\{t_n\}$, which converges μ-a.s. to 0, see [1, Th. 12.6]. But, according to [1, Lem. 7.16], this convergence is in fact an order convergence. This implies that the elements of the subsequence $\{t_n\}$ may be initially replaced by the elements of the sequence of y_n, since $|t_n| \geq y_n$, where $y_n \downarrow 0$. Hence, there exists some $y_n \geq 0$ and the inequalities

$$-k_n\mathbf{1} + y_n \leq u_n + y_n \leq k_n\mathbf{1} + y_n ,$$

imply $y_n \leq x \leq 2k_n\mathbf{1} + y_n$, hence there is $m = k_n > 0$ such that $x + m\mathbf{1} \in L^1_+$ and $\rho_{(L^1_+, \mathbf{1})}(x) \neq +\infty$.

In order to prove that $\rho_{(L^1_+, \mathbf{1})}(x) \neq -\infty$ for any $x \in L^1$, let us suppose at moment the existence of some $x_0 \in L^1$, for which $\rho_{(L^1_+, \mathbf{1})}(x_0) = -\infty$. Thus, we have that for any $k \notin \{m \in \mathbb{R} \mid m\mathbf{1} + x_0 \in L^1_+\}$, there is $h \in \{m \in \mathbb{R} \mid m\mathbf{1} + x_0 \in L^1_+\}$ such that $h < k$. This necessarily holds, because in the opposite case, we would have $k \leq \rho_{(L^1_+,\mathbf{1})}(x_0) = -\infty$ which leads to absurd. Then, for some such k, $k - h > 0$ and by the properties of L^1_+ as a wedge, we take that $x_0 + k\,\mathbf{1} = x_0 + h\,\mathbf{1} + (k-h)\,\mathbf{1} \in L^1_+$, which is in contradiction. Hence, there is not any $x_0 \in L^1$, such that $\rho_{(L^1_+,\mathbf{1})}(x_0) = -\infty$. This implies that this risk measure takes only finite values and the properties of coherence as they are indicated in [17, Th. 2.3].

Since $\mathbf{1}$ is a **quasi-interior point** in $L^{1+\varepsilon}_+$, with $\varepsilon > 0$, the same proof holds for $SRM_{a,\,b,\,\varepsilon,\,u}$ defined on $(\mathfrak{W}^{\leftarrow}(\mathcal{L} \cap \mathcal{D}) \cup \{0\}) \cap L^{1+\varepsilon}$. □

Proof of Theorem 2

For the real parameters from $AES_{a,b}$, we have $a < b$ and their domain is $(a,\,b) \in (0,\,1] \times (0,\,\infty]$. We will show the Theorem by using Berge Maximum Theorem (see [2, Th. 16.31]). For any $X \in L^{1+\varepsilon}$, we define

$$\phi_X : (0,1] \times (0,\infty] \to 2^{L^{1+\frac{1}{\varepsilon}}}\,,$$

where $\phi_X(a,\,b) = \mathcal{D}_{a,\,b,\,\varepsilon}$. We have to prove both the upper and lower hemicontinuity of ϕ_X. The upper hemicontinuity of ϕ_X is verified as follows. Consider some sequence of points $(a_n,\,b_n) \subset (0,1] \times (0,\infty]$. For this sequence we assume that $(a_n,\,b_n) \to (a_0,\,b_0)$ with respect to the product topology. If we pick a sequence in $\phi_X(a_n,\,b_n)$, it has limit points in $\phi_X(a_0,\,b_0)$. For the lower hemicontinuity of ϕ_X we notice that for any $\pi \in \mathcal{D}_{a_0,\,b_0}$, there is a subnet $(\pi_\lambda)_{\lambda\in\Lambda}$ in it, such that

$$\pi_\lambda \overset{\sigma(L^\infty,\,L^1)}{\longrightarrow} \pi\,.$$

Without loss of generality, we may suppose that $\Lambda = (0,\,\infty)$. Consider $(a_\lambda,\,b_\lambda) \to (a_0,\,b_0)$ with respect to the product topology. There exist a subnet $(a_{\lambda_\kappa},\,b_{\lambda_\kappa})$, such that $|a_{\lambda_\kappa} - a_0| < 1/\kappa$ and $|b_{\lambda_\kappa} - b_0| < 1/\kappa$ with $\kappa \in (0,\,\infty)$. These convergence inequalities imply the existence of a sequence $(\pi_{\lambda_\kappa})_{\kappa\in\mathbb{N}} \subset \mathcal{D}_{a_0,\,b_0}$, satisfying the following inequalities with respect to usual partial ordering of L^∞

$$\frac{1}{b_0 + 1/\kappa}\,\mathbf{1} < \frac{1}{b_{\lambda_\kappa} + 1/\kappa}\,\mathbf{1} \leq \pi_{\lambda_\kappa} \leq \frac{1}{a_{\lambda_\kappa} - 1/\kappa}\,\mathbf{1} < \frac{1}{a_0 - 1/\kappa}\,\mathbf{1}\,.$$

Due to the weak-star compactness of $\mathcal{D}_{a_0,\,b_0}$, a $\sigma(L^\infty,\,L^1)$-converging to the subnet $(\pi_{\lambda_\kappa})_{\kappa\in(0,\,\infty)}$. Finally, we notice that $\pi_{\lambda_\kappa} \in \mathcal{D}_{a_{\lambda_\kappa},\,b_{\lambda_\kappa}}$.

For any $X \in L^1$ the function $f_X : Gr\phi_X \to \mathbb{R}$ is defined as follows

$$f_X((a,\,b),\,\pi_Q) = \mathbb{E}_Q[-X]\,,$$

for $Q \in \phi_X(a,\,b) = \mathcal{D}_{a,\,b}$. The corresponding value function is

$$m(X) = \max_{\pi_Q \in \phi_X(a,\,b)} \mathbb{E}_Q[-X]\,,$$

while the values of the *argmax* correspondence are defined as follows

$$\mu_X(a,\, b) = \{\pi_Q \in \mathcal{D}_{a,\, b} \mid m(x) = \mathbb{E}_Q[-X]\}\,.$$

The conclusion of the Berge Maximum Theorem implies that the correspondence μ_X has weak-star compact values and it is upper hemicontinous. Namely, if we consider a sequence $a_n,\, b_n \to (a_0,\, \infty)$ and $\pi_{Q_0} \in \mu(a_0,\, \infty)$, then there exists a sequence $\pi_{Q_n} \in \mu_X(a_n,\, b_n)$, such that

$$\pi_{Q_n} \overset{\sigma(L^\infty, L^1)}{\longrightarrow} \pi_{Q_0}\,.$$

The last weak-star convergence implies that $\pi_{Q_n}(-X) = AES_{a_n,\, b_n}(X) \to \pi_{Q_0}(-X) = ES_{a_0}(X)$, namely adjusted expected shortfall converges to expected shortfall. $\square$

Proof of Proposition 4

If we apply the min-max theorem (see [8, p. 10]), we find $\mathcal{Y} = L^{1+1/\varepsilon}$ endowed with the $\sigma(L^{1+1/\varepsilon}, L^{1+\varepsilon})$-topology and $\mathcal{E} = L^{1+\varepsilon}$, where $\varepsilon > 0$. We also suppose that $K = \mathcal{D}_{a,\, b,\, \varepsilon}$, $L = \mathcal{X}$. The space $\mathcal{Y}$ is locally convex, the space $\mathcal{E} = L^{1+\varepsilon}$ is linear and $u : L^{1+\varepsilon} \times L^{1+1/\varepsilon} \to \mathbb{R}$, where $u(X, \pi) = \pi(-X)$, $X \in L^{1+\varepsilon}, \pi \in L^{1+1/\varepsilon}$. By Lemma 6 we obtain that the set $\mathcal{D}_{a,\, b,\, \varepsilon}$ is $\sigma(L^{1+1/\varepsilon}, L^{1+\varepsilon})$-compact and the set $L = \mathcal{X}$ is convex.

For any $X \in \mathcal{X}$ and in general for any $X \in L^{1+\varepsilon}$, the partial function $u(X, \cdot)$ represent the function $u_X : L^{1+1/\varepsilon} \to \mathbb{R}$, where $u_X(\pi) = \pi(-X)$. This function is weak-star continuous for any $X \in L^{1+\varepsilon}$, and hence for any $X \in \mathcal{X}$ too. If we consider a weak-star convergent net $(\pi_i)_{i\in I}$ of K, the convergence implies that for any $X \in L^{1+\varepsilon}$ we have $\pi_i(X) \to \pi(X)$, which also implies

$$u(X, \pi_i) = \pi_i(-X) \to u(X, \pi) = \pi(-X)\,,$$

that is equivalent to $u_X(\pi_i) \to u_X(\pi)$. The function u is bilinear by definition. Then, the following result (see [4, Pr. 3.1]) hold.

A function u satisfies the min-max equality (4) *if and only if it has a saddle-point.*

So, the min-max equation for u implies the existence of some saddle-point $(X_*, \pi_{Q^*}) \in \mathcal{X} \times \mathcal{D}_{a,\, b,\, \varepsilon}$, such that

$$\inf_{X\in\mathcal{X}} AES_{a,\, b,\, \varepsilon}(X) = \inf_{X\in\mathcal{X}} \sup_{\pi_Q\in\mathcal{D}_{a,\, b,\, \varepsilon}} u(X, \pi_Q) = \sup_{\pi_Q\in\mathcal{D}_{a,\, b,\, \varepsilon}} \inf_{X\in\mathcal{X}} u(X, \pi_Q)$$

$$= u(X_*, \pi_{Q^*}) = AES_{a,\, b,\, \varepsilon}(X_*)\,. \square$$

Proof of Lemma 7

1. Consider a net

$$(\pi_i)_{i\in I} \subseteq L^{1+1/\varepsilon}\,,$$

such that for the specific $a \in \mathbb{R}$ the inequality $u(X, \pi_i) \leq a$ holds. Then, if

$$\pi_i \overset{\sigma(L^{1+1/\varepsilon}, L^{1+\varepsilon})}{\rightarrow} \pi\,,$$

we have that $\pi_i(X) \to \pi(X)$, hence $\pi_i(-X) \to \pi(-X)$, which implies

$$u(X, \pi_i) \to u(X, \pi)\,.$$

But from the inequalities $u(X, \pi_i) \leq a$ for any $i \in I$ and the weak convergence, the inequality $u(X, \pi) \leq a$ holds, which concludes the first point.

2. Consider a net

$$(X_i)_{i \in I} \subseteq L^{1+\varepsilon}\,,$$

such that for the specific $a \in \mathbb{R}$ the inequality $u(X, \pi_i) \geq a$ holds. Then, if

$$X_i \overset{\sigma(L^{1+\varepsilon}, L^{1+1/\varepsilon})}{\rightarrow} X\,,$$

we have that $\pi(X_i) \to \pi(X)$, hence $\pi(-X_i) \to \pi(-X)$, which implies

$$u(X_i, \pi) \to u(X, \pi)\,.$$

But from the inequalities $u(X_i, \pi) \geq a$ for any $i \in I$ and the weak convergence holds the inequality $u(X, \pi) \geq a$, which concludes the second point.

3. The inequality $\|X\|_{1+\varepsilon} \leq k$ holds for some $k > 0$, since we assume that $\mathcal{X}$ is bounded. The inequality $\|\pi\|_{1+1/\varepsilon} \leq k_1$ holds for any $k_1 > 0$, since $\mathcal{D}_{a,\,b,\,\varepsilon}$ is norm-bounded, because of the order intervals in $L^{1+1/\varepsilon}$ are norm-bounded. This could be interpreted as a consequence of the fact that $L^{1+1/\varepsilon}$ is a Banach lattice. Hence, from Hölder inequality we obtain the third point. □

Proof of Theorem 3

We just apply Theorem [20, Th. 4.2]. Indeed, there is a saddle point $(X_0, \pi_0) \in \mathcal{X} \times \mathcal{D}_{a,\,b,\,\varepsilon}$, such that the min-max equality holds. For the assumptions concerning the uniqueness of this saddle-point, we only mention that the sets $\mathcal{X}, \mathcal{D}_{a,\,b,\,\varepsilon}$ are $\sigma(L^{1+\varepsilon}, L^{1+1/\varepsilon})$-compact and $\sigma(L^{1+1/\varepsilon}, L^{1+\varepsilon})$-compact respectively and both of these weak topologies are Hausdorff topologies on the corresponding spaces. □

Proof of Proposition 5

The family of distribution with dominatedly varying tails $\mathcal{D}$ is convex-closed and the same holds for the family $\mathcal{L} \cap \mathcal{D}$ (see Lemma 5). □

Proof of Lemma 8

By the Wald identity we obtain from (2)

$$AES_{a,\,b,\,\varepsilon}(Z_T) = -\inf_{Q\in\mathcal{Z}_{a,\,b,\,\varepsilon}} \mathbb{E}_Q(-S(T)) - c\,T - u$$

$$= -\frac{1}{b}\mathbb{E}_\mu(-S(T)) - c\,T - u = \frac{\lambda\,T}{b}\mathbb{E}_\mu(Y) - c\,T - u\,. \ \Box$$

Proof of Theorem 4

By the Proof of [22, Lem. 3.5] we find that if $\overline{F}_Y(x) = o(x^{-v})$ for any $v < \delta_{\overline{F}_X}$ then

$$\mathbb{E}_\mu(Y^{p_1}) < \infty\,,$$

if $1 \le p_1 < \delta_{\overline{F}_Y}$ and X takes positive values. Together with the fact, that the distribution F_Y belongs to the class $\mathcal{D}$, we have the result. $\Box$

Proof of Corollary 1

By the Wald identity, we find $\mathbb{E}_\mu(S(T)) = \mathbb{E}_\mu(Y)\mathbb{E}_\mu(N(T))$, where the $N(T)$ is the random number of inter-occurrence times θ_k in $[0,\,T]$, with common distribution is F_θ. By the Blackwell Renewal Theorem

$$\frac{\mathbb{E}_\mu(N(T))}{T} \to \frac{1}{\mathbb{E}_\mu(\theta)}\,.$$

We replace $\mathbb{E}_\mu(N(T))$ by its limit and the proof is the same for the renewal risk model. $\Box$

References

1. Aliprantis, C.D., Burkinshaw, O. Locally Solid Riesz Spaces. Academic Press (1978)
2. Aliprantis, C.D., Border, K.C. Infinite Dimensional Analysis, A Hitchhiker's Guide, 2nd edition. Springer (1999)
3. Asmussen, S. Applied Probability and Queues. Wiley, Chichester (1987)
4. Barbu, V., Precupanu, T. Convexity and Optimization in Banach Spaces. D. Riedel Publishing Company, Kluwer Academic Publishers Group (1986)
5. Billingsley, P. Probability and Measure, 3rd edition. Wiley, New York (1995)
6. Billingsley, P. Convergence of Probability Measures, 2nd edition. Wiley, New York (1999)
7. Cai, J., Tang, Q. On max-sum equivalence and convolution closure of heavy-tailed distributions and their applications. J. Appl. Probab., **41** 117-130 (2004)
8. Delbaen, F. Coherent risk measures on general probability spaces. In Advances in finance and stochastics: essays in honor of Dieter Sondermann. Springer-Verlag: Berlin, New York, pp. 1-38 (2002)
9. Dhaene, G., Goovaerts, M.J., Kaas, R., Tang, Q., Vanduffel, S., Vyncke, D. Solvency Capital, Risk Measures and Comonotonicity: A Review. Research Report OR0416, Department of Applied Economics, Catholic University of Leuven (2003)
10. Embrechts, P., Goldie, C.M., Veraverbeke, N. Subexponentiality and infinite divisibility. Z. Warhscheinlickkeitstheorie verw. Gebiete, **49** 335-347 (1979)
11. Embrechts, P., Goldie, C.M. On closure and factorization properties of subexponential and related distributions. J. Austral. Math. Soc., **29** 243-256 (1980)

12. Feller, W. An Introduction to Probability Theory and Its Applications, Vol. II. Wiley, New York (1971)
13. Filipović, D., Svindland, G. The Canonical Space for Law-Invariant Convex Risk Measures is L^1. Math. Financ., **22** 585-589 (2012)
14. Jameson, G. Ordered Linear Spaces, Lecture Notes in Mathematics 141. Springer-Verlag (1970)
15. Kaina, M., Rüschendorf, L. On convex risk measures on L^p-spaces. Math. Methods Oper. Res., **69** 475-495 (2009)
16. Konstantinides, D.G. Extreme Subexponentiality in Ruin Probabilities. Commun. Statist. - Theory and Methods, **40** 2907-2918 (2011)
17. Konstantinides, D.G., Kountzakis, C. Risk measures in ordered normed linear spaces with non-empty cone-interior. Insur. Math. Econ., **48** 111-122 (2011)
18. Konstantinides, D.G., Kountzakis, C. The restricted convex risk measures in actuarial solvency. Decisions Econ. Finan., DOI 10.1007/s10203-012-0134-6 (2014)
19. Megginson, R. An Introduction to Banach Spaces. Springer, New York (1998)
20. Peng, D., Yu, J., Xiu, N. Generic uniqueness theorems with some applications. J. Glob. Optimiz., DOI 10.1007/s10898-012-9903-6 (2012)
21. Skorohod, A. V. Limit Theorems for Stochastic Processes. Theory Probab. Appl., **1** 261-290 (1956)
22. Tang, Q., Tsitsiashvili, G. Precise estimations for the ruin probability in finite-horizon in a discrete-time model with heavy-tail insurance and financial risks. Stoch. Process. Appl., **108** 299-325 (2003)