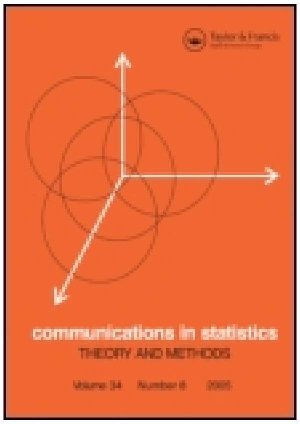

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Precise Large Deviations for Sums of Negatively Dependent Random Variables with Common Long-Tailed Distributions

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We investigate the precise large deviations for negatively dependent random variables. We prove general asymptotic relations for both the partial sums S_n for the long tailed distributions and the random sums S_t for the subexponential distributions, where the N_t is an integer counting process. It is found out that the precise large deviations for negatively dependent random variables are insensitive to this kind of dependence. Finally, we present applications on the classical counting processes, Poisson, and renewal.

Keywords Long tails; Negatively associated; Precise large deviations; Subexponetial tails.

Mathematics Subject Classification Primary 60F10; 60F05; 60K05.

1. Introduction

Our study is about the precise large deviations with dependent random variables. Especially in this article, we examine random variables with heavy-tailed distributions, which appear for modeling the large claims in actuarial and financial problems. One of the main concepts we use is the negative dependence, that was introduced in Block et al. (1982), and Ebrahimi and Ghosh (1981).

We call a sequence of random variables $\{X_k, k \geq 1\}$:

(1) Lower Negatively Dependent (LND) if for each $n = 1, 2, \ldots$ and all $x_1, x_2, \ldots, x_n$,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{k=1}^{n} P(X_k \leq x_k). \tag{1.1}$$

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(2) Upper Negatively Dependent (UND) if for each $n = 1, 2, \ldots$ and all $x_1, x_2, \ldots, x_n$,

$$P(X_1 > x_1, \ldots, X_n > x_n) \le \prod_{k=1}^{n} P(X_k > x_k). \tag{1.2}$$

(3) Negatively Dependent (ND) if both (1.1), (1.2) holds for each $n = 1, 2, \ldots$ and all $x_1, x_2, \ldots, x_n$.

For the ND random variables, the follow properties hold, that are found in Block et al. (1982, p. 769).

(1) If $\{X_k, k \ge 1\}$ are non negative UND then for each $n \ge 1$

$$E\left(\prod_{i=1}^{n} X_k\right) \le \prod_{i=1}^{n} E(X_k).$$

(2) Let $\{X_k, k \ge 1\}$ are LND (UND) and $\{f_k, k \ge 1\}$ be increasing functions, then $\{f(X_k), k \ge 1\}$ are still LND (UND).
(3) Let $\{X_k, k \ge 1\}$ are LND (UND) and $\{f_k, k \ge 1\}$ be decreasing functions, then $\{f(X_k), k \ge 1\}$ are still LND (UND).
(4) Let $\{X_k, k \ge 1\}$ are ND and $\{f_k, k \ge 1\}$ be increasing or decreasing functions, then $\{f(X_k), k \ge 1\}$ are still ND.

Let $\{X_n, n \ge 1\}$ be a sequence of non negative ND random variables with common df F and finite mean μ. We write $F(x) = P(X \le x)$ and we denote the tail by $\overline{F} = 1 - F$. We say X (or its df F) is heavy-tailed ($F \in \mathscr{K}$) if

$$\int_0^{\infty} \exp\{\varepsilon x\} dF(x) = \infty,$$

for any $\varepsilon > 0$, which means that there are no exponential moments.

We recall some important subclasses of heavy-tailed distributions. A distribution F with positive support, belongs to the class $\mathscr{L}$, if

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$$

holds for all $y \in (-\infty, \infty)$.

A distribution F with positive support, belongs to $\mathscr{S}$, if

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$$

holds for any $n \ge 2$ (or equivalently for $n = 2$), where F^{*n} denotes the nth convolution of F.

A distribution F with positive support, belongs to the class $\mathscr{D}$, if

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

holds for any $0 \le y \le 1$ (or equivalently for $y = 1/2$).

A distribution function F with positive support, belongs to the class $\mathscr{C}$, if the following hold:

$$\lim_{y\searrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1 \quad \text{or equivalently} \quad \lim_{y\nearrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1.$$

The following inclusions are well known:

$$\mathscr{C}\subset\mathscr{D}\cap\mathscr{L}\subset\mathscr{S}\subset\mathscr{L}\subset\mathscr{K}.$$

For more information about heavy-tailed distributions, see Embrechts et al. (1997).

In this article, we use the following notation. For two positive functions, $a(\cdot)$ and $b(\cdot)$,

$$a(x)\sim b(x) \quad if \quad \lim_{x\to\infty}\tfrac{a(x)}{b(x)}=1, \quad a(x)\lesssim b(x) \quad if \quad \limsup_{x\to\infty}\tfrac{a(x)}{b(x)}\leq 1$$

$$a(x)\gtrsim b(x) \quad if \quad \liminf_{x\to\infty}\tfrac{a(x)}{b(x)}\geq 1, \quad a(x)=o(b(x)) \quad if \quad \lim_{x\to\infty}\tfrac{a(x)}{b(x)}=0.$$

In this article, we are interested in probabilities of precise large deviations for non random sums and random sums.

We denote

$$S_n=\sum_{k=i}^{n}X_k$$

as the partial sum and we consider the asymptotic relations

$$P(S_n-n\mu>x)\gtrsim n\overline{F}(x+n\mu) \tag{1.3}$$

as $n\to\infty$, which holds uniformly when $x\geq\gamma n$ for any fixed $\gamma>0$ and

$$P(S_n>x)\gtrsim n\overline{F}(x) \tag{1.4}$$

as $n\to\infty$, which holds uniformly when $x\geq\gamma n$ for any fixed $\gamma>0$.

The above relations are understood in the following sense:

$$\liminf_{n\to\infty}\sup_{x\geq\gamma n}\frac{P(S_n-n\mu>x)}{n\overline{F}(x+n\mu)}\geq 1,$$

and

$$\liminf_{n\to\infty}\sup_{x\geq\gamma n}\frac{P(S_n>x)}{n\overline{F}(x)}\geq 1.$$

Let $\{N_t,t>0\}$ be a non negative integer valued process representing the claim arrival process, independent of the sequence $\{X_k,k\geq 1\}$. We denote $\lambda_t=EN_t<\infty$ for all $0\leq t<\infty$ and $\lambda_t\to\infty$ as $t\to\infty$. Further, the asymptotic relations of precise large deviations for random sum of the type

$$S_t=\sum_{k=1}^{N_t}X_k, \quad t>0,$$

become

$$P(S_t - \mu\lambda_t > x) \gtrsim \lambda_t \overline{F}(x + \mu\lambda_t), \tag{1.5}$$

as $t \to \infty$ uniformly for $x \geq \gamma\lambda_t$ and

$$P(S_t > x) \gtrsim \lambda_t \overline{F}(x), \tag{1.6}$$

as $t \to \infty$ uniformly for $x \geq \gamma\lambda_t$.

Equivalently,

$$\liminf_{t\to\infty} \sup_{x \geq \gamma\lambda_t} \frac{P(S_t - \mu\lambda_t > x)}{\lambda_t \overline{F}(x + \mu\lambda_t)} \geq 1,$$

and

$$\liminf_{t\to\infty} \sup_{x \geq \gamma\lambda_t} \frac{P(S_t > x)}{\lambda_t \overline{F}(x)} \geq 1.$$

The first works in this field with i.i.d. random variables are Heyde (1967a,b, 1968) and Nagaev (1969a,b). Later the large deviations were studied in the context of regularly varying-tailed class of distributions in Nagaev (1973, 1979). Further, we find in Cline and Hsing (1991) the precise large deviations over a larger class of extended regularly varying-tailed distributions. A general approach on subexponential distributions was examined in Pinelis (1985). The first article on precise large deviations with random sums is Klüppelberg and Mikosch (1997). In Mikosch and Nagaev (1998, 2001) there is a review of large deviations results for heavy-tailed distributions. Recent results in this field are found in Tang et al. (2001) and on the class of consistently varying-tailed distribution are found in Ng et al. (2004). For independent but not identically distributed random variables some results we met in Skučaitė (2004).

For a distribution F, the upper Matuszewska index γ_F was introduced as follows:

$$\gamma(y) := \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad \gamma_F := \inf\left\{-\frac{\log \gamma(y)}{\log y} : y > 1\right\}.$$

We remember Bingham et al. (1987) a classical book on Matuszewska indexes. More properties of Matuszewska indexes there exist in Cline and Samorodnitsky (1994).

Remark 1.1. We know from (Cline and Hsing, 1991, Th. 1.1) that the upper bound in asymptotics at (1.3) and (1.4) is possible only in the frame of the subclass $\mathscr{S}$. Therefore, we study only the lower asymptotic bound for the class $\mathscr{L}$.

The main results in the recent literature, that established for the precise large deviations of ND random variables, are contained in the following propositions.

Proposition 1.1 (Tang, 2006, Th. 1.1). *Let* $\{X_k, k \geq 1\}$ *be ND with common d.f.* $F \in \mathscr{C}$ *and mean* 0 *satisfying*

$$xF(-x) = o(\overline{F}(x))$$

as $x \to \infty$. *Then, for each fixed* $\gamma > 0$, *holds*

$$P(S_n > x) \sim n\overline{F}(x), \tag{1.7}$$

uniformly for $x \geq \gamma n$.

Proposition 1.2 (Chen and Zhang, 2007, Th. 1.2). *Suppose* $\{X_k, k \geq 1\}$ *is a sequence ND random variables with common d.f.* $F \in \mathscr{C}$ *and finite expectation* $\mu < 0$ *satisfying* $xF(-x) = o(\overline{F}(x))$ *independent of non-negative and integer-valued process* $\{N_t, t \geq 0\}$. *Assume that* N_t *satisfies*

$$Assumption\ N_1: \quad \frac{N_t}{\lambda_t} \xrightarrow{P} 1,$$

as $t \to \infty$,*and* $E|X_1|^r < \infty$ *for some* $r > 1$. *Then for any fixed* $\gamma > |\mu|$,

$$P(S_{N_t} - \mu\lambda_t > x) \sim \lambda_t \overline{F}(x), \tag{1.8}$$

holds uniformly for $x \geq \gamma\lambda_t$.

2. Large Deviations for Non-Random Sums

In the next Theorem, we extend the results on non-random sums, about precise large deviations in a wider class of heavy-tailed distributions namely the long-tailed ones.

Theorem 2.1. *Let* $\{X_k, k \geq 1\}$ *be a sequence of ND non negative random variables with common distribution* $F \in \mathscr{L}$ *and finite mean* μ. *Then for any* $\gamma > 0$

$$P(S_n - n\mu > x) \gtrsim n\overline{F}(x + n\mu), \tag{2.9}$$

holds uniformly for $x \geq \gamma n$ *and*

$$P(S_n > x) \gtrsim n\overline{F}(x), \tag{2.10}$$

holds uniformly for $x \geq \gamma n$.

Remark 2.1. It is worth mentioning that in (2.9) we obtain asymptotic relation for centered partial sum with argument $(x + n\mu)$ in the tail of the distribution F. Equation (2.10) gives us the classical form of precise large deviations.

$$\begin{aligned}
P(S_n - n\mu > x) &\geq P\Big(S_n - n\mu > x,\ \max_{1\leq i\leq n} X_i > x + n\mu\Big)\\
&\geq \sum_{i=1}^{n} P(S_n > x + n\mu,\ X_i > x + n\mu)\\
&\quad - \sum_{1\leq k<l\leq n} P(S_n - n\mu > x, X_k > x + n\mu, X_l > x + n\mu)\\
&\geq \sum_{i=1}^{n} P(X_i > x + n\mu) - (n\overline{F}(x + n\mu))^2
\end{aligned}$$

$$\geq n\overline{F}(x+n\mu)(1-n\overline{F}(x)).$$

So, we obtain

$$\frac{P(S_n - n\mu > x)}{n\overline{F}(x+n\mu)} \geq (1-n\overline{F}(x))$$

we know that for a sequence $\{X_k, k \geq 1\}$ of non negative random variables with distribution F and finite mean, it

$$n\overline{F}(x) \to 0$$

holds uniformly for $x \geq \gamma n$ as $n \to \infty$. So, we obtain

$$n\overline{F}(x+n\mu) \lesssim P(S_n - n\mu > x) \tag{2.11}$$

as $n \to \infty$. Furthermore, if we use the relation (2.9) we obtain

$$P(S_n > x) = P(S_n - n\mu > x - n\mu) \gtrsim n\overline{F}(x). \tag{2.12}$$

3. Large Deviations for Random Sums

Now we prove an asymptotic relation, when we have subexponential-tailed distributions and a general integer valued counting process $\{N_t,\ t \geq 0\}$.

Theorem 3.1. *Let $\{X_k, k \geq 1\}$ be a sequence of ND non negative random variables with common distribution $F \in \mathscr{S}$ and finite mean μ independent of non negative and integer valued process $\{N_t, t \geq 0\}$. Assume N_t satisfies*

$$\textit{Assumption } N_1: \quad \frac{N_t}{\lambda_t} \xrightarrow{P} 1,$$

as $t \to \infty$, and that

$$\textit{Assumption } N_2: \quad \sum_{k>(1+\delta)\lambda_t} (1+\epsilon)^k P(N_t = k) = o(1)$$

as $t \to \infty$, for any $\delta > 0$ and some small enough $\epsilon > 0$. Then for any $\gamma > 0$

$$P(S_{N_t} - \mu\lambda_t > x) \gtrsim \lambda_t \overline{F}(x+\mu\lambda_t) \tag{3.13}$$

holds, uniformly for $x \geq \gamma\lambda_t$ and furthermore

$$P(S_{N_t} > x) \gtrsim \lambda_t \overline{F}(x), \tag{3.14}$$

holds uniformly for $x \geq \gamma\lambda_t$.

Proof. We observe that

$$P(S_{N_t} - \mu\lambda_t > x) = \sum_{k=1}^{\infty} P(S_k - \mu\lambda_t > x) P(N_t = k)$$

which we can split into three parts:

$$= \sum_{k<(1-\delta)\lambda_t} + \sum_{(1-\delta)\lambda_t \le k \le (1+\delta)\lambda_t} + \sum_{k>(1+\delta)\lambda_t}$$
$$:= I_1 + I_2 + I_3 \ge I_2 + I_3, \tag{3.15}$$

where $0 < \delta < 1$ is arbitrary. We deal with I_2

$$I_2 = \sum_{((1-\delta)\lambda_t) \le k \le (1+\delta)\lambda_t} P(S_k - \mu\lambda_t > x) P(N_t = k)$$
$$\ge P(S_{((1-\delta)\lambda_t)} - \mu\lambda_t > x) P\left(1 - \delta \le \frac{N_t}{\lambda_t} \le 1 + \delta\right),$$

the Assumption N_1 gives the asymptotic relation

$$I_2 \gtrsim P(S_{(1-\delta)\lambda_t} - (1-\delta)\lambda_t\mu > x + \mu\lambda_t - (1-\delta)\lambda_t\mu),$$

and from Theorem 2.1

$$I_2 \gtrsim ((1-\delta)\lambda_t)\overline{F}(x + \mu\lambda_t). \tag{3.16}$$

Then from relations (3.16) we induce

$$\lim_{\delta\to 0} \liminf_{t\to\infty} \sup_{x \ge \gamma\lambda_t} \frac{I_2}{\lambda_t \overline{F}(x + \mu\lambda_t)} \ge 1. \tag{3.17}$$

Finally, we consider the last sum

$$I_3 = \sum_{k>(1+\delta)\lambda_t} P(S_k - \mu\lambda_t > x) P(N_t = k)$$
$$\le \sum_{k>(1+\delta)\lambda_t} P(S_k > x + \mu\lambda_t) P(N_t = k).$$

Since $F \in \mathscr{S}$ then holds, the following (see Chistyakov, 1964). For any $\epsilon > 0$, there exist a positive fixed $K = K(\epsilon)$ such that,

$$P(S_n > x) \le K(1+\epsilon)^n \overline{F}(x).$$

Therefore

$$I_3 \le \sum_{k>(1+\delta)\lambda_t} (K(1+\epsilon)^n \overline{F}(x + \mu\lambda_t)) P(N_t = k)$$
$$\le K\overline{F}(x + \mu\lambda_t) \sum_{k>(1+\delta)\lambda_t} (1+\epsilon)^k P(N_t = k),$$

and the Assumption N_2 leads to

$$I_3 \sim o(\lambda_t \overline{F}(x + \mu\lambda_t)). \tag{3.18}$$

as $t \to \infty$.

Finally, substituting (3.17), (3.18) into (3.15), it yields to the relation (3.13). Furthermore, if we use the relation (3.13) we obtain

$$P(S_{N_t} > x) = P(S_{N_t} - \mu\lambda_t > x - \mu\lambda_t) \gtrsim \lambda_t \overline{F}(x),$$

as $t \to \infty$ uniformly for $x \geq \gamma\lambda_t$. □

For the practical applications of the previous theorem we consider two special cases, the Poisson counting process and the renewal counting process.

Firstly, from (Klüppelberg and Mikosch, 1997, Lem. 2.1) we find that the Poisson counting process satisfies the Assumptions N_1 and N_2.

Secondly, from (Klüppelberg and Mikosch, 1997, Lem. 2.3) we find that, if $\{Y_n, n \geq 1\}$ represents a renewal counting process $\{N_t, t > 0\}$ and satisfies the stochastic ordering relation

$$P(Y_1 \leq x) \leq P(E_1 \leq x), \quad x \in \mathbb{R} \tag{3.19}$$

for an exponential random variable E_1, then the renewal counting process N_t satisfies both Assumptions N_1 and N_2. Hence, we conclude the following.

Proposition 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of ND non-negative random variables with finite mean and common distribution $F \in \mathcal{S}$. Let $\{N_t, t > 0\}$ be a Poisson counting process. Suppose that the sequences $\{X_n, n \geq 1\}$ and $\{N_t, t > 0\}$ are independent. Then for any $\gamma > 0$*

$$P(S_{N_t} - \mu\lambda_t > x) \gtrsim \lambda_t \overline{F}(x + \mu\lambda_t),$$

holds uniformly for $x \geq \gamma\lambda_t$ and

$$P(S_{N_t} > x) \gtrsim \lambda_t \overline{F}(x),$$

holds uniformly for $x \geq \gamma\lambda_t$.

Proposition 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of ND non-negative random variables with finite mean and common distribution $F \in \mathcal{S}$. Let $\{Y_n, n \geq 1\}$, with $\mathbf{E}[Y_1] < \infty$ and* $\mathrm{var}[Y_1] < \infty$, *constitute a renewal counting process $\{N_t, t > 0\}$ that satisfies the relation* (3.19). *Suppose that the sequences $\{X_n, n \geq 1\}$ and $\{N_t, t > 0\}$ are independent. Then for any $\gamma > 0$*

$$P(S_{N_t} - \mu\lambda_t > x) \gtrsim \lambda_t \overline{F}(x + n\mu),$$

holds uniformly for $x \geq \gamma\lambda_t$. and

$$P(S_{N_t} > x) \gtrsim \lambda_t \overline{F}(x),$$

holds uniformly for $x \geq \gamma\lambda_t$.

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