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PRECISE LARGE DEVIATIONS FOR CONSISTENTLY VARYING-TAILED DISTRIBUTIONS IN THE COMPOUND RENEWAL RISK MODEL

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Abstract. We investigate precise large deviations for heavy-tailed random sums. We prove a general asymptotic relation in the compound renewal risk model for consistently varying-tailed distributions. This model was introduced in [Q. Tang, C. Su, T. Jiang, and J.S. Zang, Large deviation for heavy-tailed random sums in compound renewal model, *Stat. Probab. Lett.*, 52:91–100, 2001] as a more practical risk model. The proof is based on the inequality found in [D. Fuk and S.V. Nagaev, Probability for sums of independent random variables, *Theory Probab. Appl.*, 16:600–675, 1971].

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1 INTRODUCTION

1.1 Heavy-tailed distributions

In this paper, we are interesting in probabilities of precise large deviations in the compound renewal risk model. Let us consider a sequence $\{X_n,\, n \geqslant 1\}$ of i.i.d. r.v.'s with common distribution function F and finite mean μ. We write $F(x) = \mathbf{P}(X \leqslant x)$ and denote the tail by $\overline{F} = 1 - F$. We suppose that the sequence is independent of a nonnegative renewal counting process $\{N(t),\, t \geqslant 0\}$ representing the claim arrival process. We denote $\lambda(t) = \mathbf{E}N(t) < \infty$ for all $0 \leqslant t < \infty$ and assume that $\lambda(t) \to \infty$ as $t \to \infty$. We say that X (or its distribution F) is heavy tailed if it has no exponential moments. We recall some interesting subclasses of subexponential distributions.

- *The class* $\mathcal{R}_{-\alpha}$: A distribution function F with support on $[0, \infty)$ belongs to $\mathcal{R}_{-\alpha}$ if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad y > 1,$$

  for some $\alpha > 1$. Such a distribution function F is said to have a regularly varying tail.

- *The class* ERV: A distribution function F with support on $[0, \infty)$ belongs to $\mathrm{ERV}(-\alpha, -\beta)$ if

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}, \quad y > 1,$$

for some $0 < \alpha \leqslant \beta < \infty$. In this case, the distribution function F is said to have an extended regularly varying tail.

- *The class* $\mathcal{C}$: A distribution function F with support on $[0, \infty)$ belongs to $\mathcal{C}$ if

$$\lim_{y \searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1$$

or, equivalently,

$$\lim_{y \nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Now the distribution function F is said to have a consistently varying tail.
Some results on the class $\mathcal{C}$ can be found in [2, 4].

- *The class* $\mathcal{D}$: A distribution function F with support on $[0, \infty)$ belongs to $\mathcal{D}$ if

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for any $0 \leqslant y \leqslant 1$ (or, equivalently, for $y = 1/2$). Such a distribution function F is said to have a dominatedly varying tail.

The following inclusions are well known: for any $0 < \alpha \leqslant \gamma \leqslant \beta < \infty$,

$$\mathcal{R}_{-\gamma} \subset \mathrm{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D}.$$

For heavy-tailed distributions, we refer to [5].

For two positive functions $a(\cdot)$ and $b(\cdot)$, we write

$$\begin{aligned}
a(x) &\asymp b(x) \quad \text{if } 0 < \liminf_{x\to\infty} \frac{a(x)}{b(x)} \leqslant \limsup_{x\to\infty} \frac{a(x)}{b(x)} < \infty,\\
a(x) &\lesssim b(x) \quad \text{if } \limsup_{x\to\infty} \frac{a(x)}{b(x)} \leqslant 1, \quad \text{and}\\
a(x) &\gtrsim b(x) \quad \text{if } \liminf_{x\to\infty} \frac{a(x)}{b(x)} \geqslant 1.
\end{aligned}$$

For a distribution function F, the upper Matuszewska index γ_F is defined as follows:

$$\gamma(y) := \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad \gamma_F := \inf\left\{-\frac{\log\gamma(y)}{\log y} \colon y > 1\right\}.$$

For more information about the Matuszewska index, see in [1, 4].

1.2 The two risk models

1.2.1 The renewal risk model

This model has the following structure:

1. The individual claim sizes $\{X_n,\ n \geqslant 1\}$ are i.i.d. nonnegative r.v.'s with common distribution function F and finite mean $\mu = \mathbf{E}X_1$.

2. The inter-arrival times $\{Y_n,\, n \geqslant 1\}$ are i.i.d. nonnegative r.v.'s with finite mean $\mathbf{E}Y_1 = 1/\lambda$, independent of $\{X_n,\, n \geqslant 1\}$.
3. The number of claims in the interval $[0, t]$ is denoted by

$$N(t) = \sup\{n \geqslant 1 \colon T_n \leqslant t\}, \quad t \geqslant 0,$$

where $T_n = \sum_{i=1}^{n} Y_i$.

1.2.2 The compound renewal risk model

In [20], in relation to an application in insurance and finance, there was introduced a more realistic risk model, called *compound renewal model*. It was noticed that, for every accident moment, it is natural to expect more than one individual claim.

That risk model has the following structure:

1. The individual claim sizes $\{X_n,\, n \geqslant 1\}$ are i.i.d. nonnegative r.v.'s with common distribution function F and finite mean $\mu = \mathbf{E}X_1$.
2. The inter-arrival times for the accident $\{Y_n,\, n \geqslant 1\}$ are i.i.d. nonnegative r.v.'s with finite mean $\mathbf{E}Y_1 = 1/\lambda$, independent of $\{X_n,\, n \geqslant 1\}$.
3. The number of accidents in the interval $[0, t]$ is denoted by

$$\tau(t) = \sup\{n \geqslant 1 \colon T_n \leqslant t\}$$

for every $t \geqslant 0$ with $\mathbf{E}(\tau(t)) = \nu(t)$.

The number of individual claims caused by the nth accident is a nonnegative, integer-valued r.v. Z_n, and $\{Z_n,\, n \geqslant 1\}$ constitutes a sequence of i.i.d. r.v.'s with common distribution function G and finite mean θ, independent of $\{X_n,\, n \geqslant 1\}$ and $\{Y_n,\, n \geqslant 1\}$. The total of claims up to time t is given by

$$N(t) = \sum_{i=1}^{\tau(t)} Z_i, \quad t \geqslant 0, \tag{1.1}$$

with

$$\lambda(t) = \mathbf{E}\big(N(t)\big) = \mathbf{E}\left(\sum_{i=1}^{\tau(t)} Z_i\right) = \theta\nu(t).$$

In catastrophe insurance, for instance, $\tau(t)$ is the number of events that occur in the interval $[0, t]$. Such events are natural catastrophe events like windstorms, earthquakes, etc. The sequence $\{Z_n,\, n \geqslant 1\}$ represents the number of individual claims caused by the nth catastrophic event.

1.3 Precise large deviations

We consider the asymptotic relation

$$\mathbf{P}(S_n - n\mu > x) \sim n\overline{F}(x) \tag{1.2}$$

as $n \to \infty$, where

$$S_n = \sum_{i=1}^{n} X_i \tag{1.3}$$

denotes the nth partial sum of the sequence $\{X_i,\, i \geqslant 1\}$.

We recall the following propositions established in the literature on precise large deviations for nonrandom sums.

Proposition 1. (*See* [14, 15].) *If* $F \in \mathcal{R}_{-\alpha}$ *with* $a > 1$*, then for any fixed* $\gamma > 0$*, relation* (1.2) *holds uniformly for* $x \geqslant \gamma n$*.*

Proposition 2. (*See* [3].) *If* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *with* $1 < \alpha \leqslant \beta < \infty$*, then for any fixed* $\gamma > 0$*, relation* (1.2) *holds uniformly for* $x \geqslant \gamma n$*.*

Proposition 3. (*See* [16].) *If* $F \in \mathcal{C}$*, then for any fixed* $\gamma > 0$*, relation* (1.2) *holds uniformly for* $x \geqslant \gamma n$*.*

More results in the field of precise large deviations for nonrandom sums can be found in [8, 9, 7, 12, 13]. A very general treatment of large deviations for subexponentials can be found in [18].

Further, precise large deviations for random sums of the type

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad t > 0, \tag{1.4}$$

were investigated by many researchers, where the asymptotic relation (1.2) was transformed into

$$\mathbf{P}\big(S_{N(t)} - \mu\lambda(t) > x\big) \sim \lambda(t)\overline{F}(x) \tag{1.5}$$

as $t \to \infty$, uniformly for $x \geqslant \gamma\lambda(t)$.

In the sequel, we need the following well-known propositions.

Proposition 4. (*See* [10, *Thm.* 3.1].) *Let* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $1 < \alpha \leqslant \beta < \infty$*. Furthermore, suppose that* $\{N(t),\, t \geqslant 0\}$ *is an integer counting process satisfying*

$$\textit{Assumption } N_1\text{:} \quad \frac{N(t)}{\lambda(t)} \xrightarrow{\mathbf{P}} 1$$

as $t \to \infty$ *and*

$$\textit{Assumption } N_2\text{:} \quad \sum_{k > (1+\delta)\lambda(t)} (1+\epsilon)^k \mathbf{P}\big(N(t) = k\big) = o(1)$$

as $t \to \infty$ *for any* $\delta > 0$ *and some sufficiently small* $\epsilon > 0$*. Then, for any fixed* $\gamma > 0$*, relation* (1.5) *holds uniformly for* $x \geqslant \gamma\lambda(t)$*.*

Proposition 5. (*See* [20, *Thm.* 2.1].) *Let* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $1 < \alpha \leqslant \beta < \infty$*. Furthermore, suppose that* $\{N(t),\, t \geqslant 0\}$ *is an integer counting process satisfying*

$$\textit{Assumption } A\text{:} \quad \mathbf{E}\big(\big(N(t)\big)^p I_{(N(t) > (1+\delta)\lambda(t))}\big) = O\big(\lambda(t)\big)$$

as $t \to \infty$ *for some* $p > \beta$ *and all* $\delta > 0$*. Then, for any fixed* $\gamma > 0$*, relation* (1.5) *holds uniformly for* $x \geqslant \gamma\lambda(t)$*.*

Proposition 6. (*See* [16, *Thm.* 4.1].) *Let* $F \in \mathcal{C}$ *and suppose that* $\{N(t),\, t \geqslant 0\}$ *is an integer counting process satisfying Assumption A for some* $p > \gamma_F$ *and all* $\delta > 0$*. Then, for any fixed* $\gamma > 0$*, relation* (1.5) *holds uniformly for* $x \geqslant \gamma\lambda(t)$*.*

Proposition 7. (*See* [20, *Thm.* 2.3].) *Suppose that* $Z_1 \geqslant 1$*,* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $1 < \alpha \leqslant \beta < \infty$*, and* $\mathbf{E}Z_1^p < \infty$ *for some* $p > \beta + 2$*. Then the compound renewal process* $\{N(t),\, t > 0\}$ *is an integer counting process satisfying Assumption A, and, consequently, for any fixed* $\gamma > 0$*,* (1.5) *holds uniformly for* $x \geqslant \gamma\lambda(t)$*.*

A nice review on recent developments on large deviations for random sums is given in [11]. In the case of independent r.v.'s, the precise large deviation has been established in [17] and [19].

Remark 1. Assumption N_1 has the following equivalent formulation given in [10]: There exists a positive function $\epsilon(t)$ such that $\epsilon(t) \to 0$ and

$$\mathbf{P}\big[\big|N(t) - \lambda(t)\big| > \epsilon(t)\lambda(t)\big] = o(1).$$

2 MAIN RESULTS

Theorem 1. *Let $\{X_k,\, k \geqslant 1\}$ be a sequence of i.i.d. nonnegative r.v.'s with common distribution function $F \in \mathcal{C}$ and finite mean μ, and let $\{Z_i,\, i \geqslant 1\}$ be a sequence of i.i.d. integer-valued r.v.'s with $Z_1 \geqslant 1$, $\mathbf{E}Z_1^p < \infty$ for some $p > \gamma_F + 2$ and common distribution function G with finite mean θ. Also, let $\{N(t),\, t \geqslant 0\}$ be a compound renewal process driven by a sequence of i.i.d. nonnegative random variables $\{Y_k,\, k \geqslant 1\}$ with finite mean. Suppose that the sequences $\{X_k,\, k \geqslant 1\}$, $\{Z_i,\, i \geqslant 1\}$, and $\{Y_k,\, k \geqslant 1\}$ are mutually independent. Then for any fixed $\gamma > 0$, the asymptotic relation* (1.5) *holds uniformly for $x \geqslant \gamma\lambda(t)$.*

For the proof, we need the following lemmas.

Lemma 1. (*See* [16, *Lemma* 2.1].) *For a distribution function $F \in \mathcal{D}$ with finite expectation, we have $1 \leqslant \gamma_F < \infty$, and, for any $\gamma > \gamma_F$,*

$$x^{-\gamma} = o\big(\overline{F}(x)\big) \quad \textit{as } x \to \infty.$$

The next lemma presents a reformulation of the standard Fuk and Nagaev inequality.

Lemma 2. (*See* [16, *Lemma* 2.3].) *Let $\{X_k,\, k \geqslant 1\}$ be a sequence of i.i.d. nonnegative random variables with common distribution function F and finite expectation μ. Then, for all $u > 0$, $x > 0$ and $n \geqslant 1$, we have*

$$\mathbf{P}(S_n > x) \leqslant n\overline{F}\left(\frac{x}{u}\right) + \left(\frac{\mathrm{e}\mu n}{x}\right)^u.$$

Lemma 3. (*See* [20, *Lemma* 3.5].) *Suppose that $\{N(t),\, t \geqslant 0\}$ is a renewal counting process. Then, for any positive δ and m, we have*

$$\sum_{k > (1+\delta)\lambda(t)} k^m \mathbf{P}\big(N(t) \geqslant k\big) = o(1), \quad t \to \infty.$$

Remark 2. From [20, Thm. 2.3] we obtain that the compound renewal counting process satisfies Assumption A, and from Theorem 2.1 in the same paper we get that Assumption A implies Assumption N_1. Hence, the compound renewal counting process satisfies Assumption N_1.

3 THE PROOF

Let us observe that

$$\mathbf{P}\big(S_{N(t)} - \mu\lambda(t) > x\big) = \sum_{k=1}^{\infty} \mathbf{P}\big(S_k - \mu\lambda(t) > x\big)\mathbf{P}\big(N(t) = k\big),$$

which we can split into three parts

$$\sum_{k < (1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t) \leqslant k \leqslant (1+\delta)\lambda(t)} + \sum_{k > (1+\delta)\lambda(t)} := I_1 + I_2 + I_3, \tag{3.1}$$

where δ is arbitrarily chosen from the interval $(0, \gamma/\mu)$.

We have

$$\begin{aligned} I_1 &= \sum_{k<(1-\delta)\lambda(t)} \mathbf{P}\big(S_k - \mu\lambda(t) > x\big)\mathbf{P}\big(N(t)=k\big) \\ &\leqslant \sum_{k<(1-\delta)\lambda(t)} \mathbf{P}\big[S_{(1-\delta)\lambda(t)} > x + \mu\lambda(t)\big]\mathbf{P}\big(N(t)=k\big) \\ &= \sum_{k<(1-\delta)\lambda(t)} \mathbf{P}\big[S_{(1-\delta)\lambda(t)} - (1-\delta)\lambda(t)\mu > x + \mu\lambda(t) - (1-\delta)\lambda(t)\mu\big]\mathbf{P}\big(N(t)=k\big). \end{aligned}$$

Combining this expression with the result in Proposition 3, we get

$$\begin{aligned} I_1 &\lesssim \sum_{k<(1-\delta)\lambda(t)} (1-\delta)\lambda(t)\overline{F}\big(x + \mu\lambda(t) - (1-\delta)\lambda(t)\mu\big)\mathbf{P}\big(N(t)=k\big) \\ &= \sum_{k<(1-\delta)\lambda(t)} (1-\delta)\lambda(t)\overline{F}\big(x + \delta\mu\lambda(t)\big)\mathbf{P}\big(N(t)=k\big) \\ &\leqslant (1-\delta)\lambda(t)\overline{F}(x) \sum_{k<(1-\delta)\lambda(t)} \mathbf{P}\big(N(t)=k\big) \\ &= (1-\delta)\lambda(t)\overline{F}(x)\mathbf{P}\big(N(t) < (1-\delta)\lambda(t)\big) \\ &= (1-\delta)\lambda(t)\overline{F}(x)\mathbf{P}\big(N(t) - \lambda(t) < -\delta\lambda(t)\big). \end{aligned}$$

Now we use Assumption N_1 to get

$$I_1 = o\big(\lambda(t)\overline{F}(x)\big) \tag{3.2}$$

for every fixed $0 < \delta < \gamma/\mu$.

Next, we deal with

$$\begin{aligned} I_2 &= \sum_{(1-\delta)\lambda(t)\leqslant k\leqslant(1+\delta)\lambda(t)} \mathbf{P}\big(S_k - \mu\lambda(t) > x\big)\mathbf{P}\big(N(t)=k\big) \\ &\leqslant \mathbf{P}\big(S_{(1+\delta)\lambda(t)} - \mu\lambda(t) > x\big)\mathbf{P}\left[(1-\delta) \leqslant \frac{N(t)}{\lambda(t)} \leqslant (1+\delta)\right]. \end{aligned}$$

Under Assumption N_1, we have

$$I_2 \lesssim \mathbf{P}\big(S_{(1+\delta)\lambda(t)} - (1+\delta)\lambda(t)\mu > x + \mu\lambda(t) - (1+\delta)\lambda(t)\mu\big)$$

as $t \to \infty$. Now we use the result of Proposition 3 to get

$$I_2 \lesssim (1+\delta)\lambda(t)\overline{F}\big[x + \mu\lambda(t) - (1+\delta)\lambda(t)\mu\big].$$

Since

$$x + \mu\lambda(t) - (1+\delta)\lambda(t)\mu \geqslant \left(1 - \frac{\delta\mu}{\gamma}\right)x \quad \text{or, equivalently,} \quad x \geqslant \gamma\lambda(t),$$

we get

$$I_2 \lesssim (1+\delta)\lambda(t)\overline{F}\left[\left(1 - \frac{\delta\mu}{\gamma}\right)x\right]$$

as $t \to \infty$. Also,

$$I_2 = \sum_{(1-\delta)\lambda(t)\leqslant k\leqslant(1+\delta)\lambda(t)} \mathbf{P}\big(S_k - \mu\lambda(t) > x\big)\mathbf{P}\big(N(t) = k\big)$$
$$\geqslant \mathbf{P}\big(S_{(1-\delta)\lambda(t)} - \mu\lambda(t) > x\big)\mathbf{P}\bigg((1-\delta) \leqslant \frac{N(t)}{\lambda(t)} \leqslant (1+\delta)\bigg).$$

By Assumption N_1 we obtain

$$I_2 \gtrsim \mathbf{P}\big(S_{(1-\delta)\lambda(t)} - (1-\delta)\lambda(t)\mu > x + \mu\lambda(t) - (1-\delta)\lambda(t)\mu\big)$$

as $t \to \infty$. Further, from the result of Proposition 3 we get

$$I_2 \gtrsim (1-\delta)\lambda(t)\overline{F}\big(x + \mu\lambda(t) - (1-\delta)\lambda(t)\mu\big).$$

By the inequality

$$x + \mu\lambda(t) - (1-\delta)\lambda(t)\mu \leqslant \left(1 + \frac{\delta\mu}{\gamma}\right)x \quad \text{or, equivalently,} \quad x \geqslant \gamma\lambda(t),$$

we obtain

$$I_2 \gtrsim (1-\delta)\lambda(t)\overline{F}\left[\left(1 + \frac{\delta\mu}{\gamma}\right)x\right]$$

as $t \to \infty$. From $F \in \mathcal{C}$ we get

$$\lim_{\delta\searrow 0}\limsup_{x\to\infty} \frac{1}{\overline{F}(x)}\,\overline{F}\left[\left(1 + \frac{\delta\mu}{\gamma}\right)x\right] = \lim_{\delta\searrow 0}\limsup_{x\to\infty} \frac{1}{\overline{F}(x)}\,\overline{F}\left[\left(1 - \frac{\delta\mu}{\gamma}\right)x\right] = 1,$$

and so

$$\lim_{\delta\searrow 0}\limsup_{t\to\infty}\sup_{x\geqslant\gamma\lambda(t)}\left|\frac{I_2}{\lambda(t)\overline{F}(x)} - 1\right| = 0. \tag{3.3}$$

Now we calculate the

$$I_3 = \sum_{k>(1+\delta)\lambda(t)} \mathbf{P}\big(S_k - \mu\lambda(t) > x\big)\mathbf{P}\big(N(t) = k\big) \leqslant \sum_{k>(1+\delta)\lambda(t)} \mathbf{P}(S_k > x)\mathbf{P}\big(N(t) = k\big).$$

From Lemma 2, choosing $\gamma_F \leqslant u < p - 1$, we get the inequality

$$\leqslant \sum_{k>(1+\delta)\lambda(t)} \left[k\overline{F}\left(\frac{x}{u}\right) + \left(\frac{\mathrm{e}\mu k}{x}\right)^u\right]\mathbf{P}\big(N(t) = k\big)$$
$$\leqslant \sum_{k>(1+\delta)\lambda(t)} k\overline{F}\left(\frac{x}{u}\right)\mathbf{P}\big(N(t) = k\big) + \left(\frac{\mathrm{e}\mu}{x}\right)^u \sum_{k>(1+\delta)\lambda(t)} k^u\mathbf{P}\big(N(t) = k\big)$$
$$:= K_1 + K_2.$$

For the first term, we obtain

$$K_1 \asymp \sum_{k>(1+\delta)\lambda(t)} k\overline{F}(x)\mathbf{P}\big(N(t) = k\big)$$

$$
\begin{aligned}
&\leqslant \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{n\geqslant k/(\tau+\theta)} \mathbf{P}\left(\sum_{i=1}^{n} Z_i = k\right) \mathbf{P}\big(\tau(t)=n\big) \\
&\quad + \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{1\leqslant n\leqslant k/(\tau+\theta)} \mathbf{P}\left(\sum_{i=1}^{n} Z_i = k\right) \mathbf{P}\big(\tau(t)=n\big) \\
&= J_1 + J_2,
\end{aligned}
$$

where $\theta = \mathbf{E}Z_1 \geqslant 1$, and $\tau > 0$ is such that

$$\frac{(1+\delta)\theta}{\tau+\theta} > 1.$$

Further, we deal with

$$
\begin{aligned}
J_1 &= \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{n\geqslant k/(\tau+\theta)} \mathbf{P}\left(\sum_{i=1}^{n} Z_i = k\right) \mathbf{P}\big(\tau(t)=n\big) \\
&\leqslant \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{n\geqslant k/(\tau+\theta)} \mathbf{P}\big(\tau(t)=n\big) = \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \mathbf{P}\left(\tau(t) \geqslant \frac{k}{\tau+\theta}\right).
\end{aligned}
$$

From Lemma 3 we get the expression

$$J_1 = o(1)\overline{F}(x) = o\big(\overline{F}(x)\big).$$

Further, we write

$$
\begin{aligned}
J_2 &= \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{1\leqslant n<k/(\tau+\theta)} \mathbf{P}\left(\sum_{i=1}^{n} Z_i = k\right) \mathbf{P}\big(\tau(t)=n\big) \\
&\leqslant \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \sum_{1\leqslant n<k/(\tau+\theta)} \mathbf{P}\left(\sum_{i=1}^{n} Z_i = k\right) \\
&\leqslant \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \mathbf{P}\left(\sum_{1\leqslant i<k/(\tau+\theta)} Z_i \geqslant k\right) \\
&\leqslant \overline{F}(x) \sum_{k>(1+\delta)\lambda(t)} k \mathbf{P}\left(\sum_{1\leqslant i<k/(\tau+\theta)} (Z_i-\theta) \geqslant \frac{\tau k}{\tau+\theta}\right).
\end{aligned}
$$

By [6, Corol. 3], which says that, for i.i.d. r.v.'s $\{U_n,\, n \geqslant 1\}$,

$$\mathbf{P}\left(\sum_{k=1}^{n} U_k \geqslant x\right) \leqslant \left(1+\frac{2}{p}\right)^{p} n\mathbf{E}|U_1|^p x^{-p} + \exp\left\{-2(p+2)^{-2}\mathrm{e}^{-p}x^2\big(n\mathbf{E}U_1^2\big)^{-1}\right\}$$

if $\mathbf{E}U_1 = 0$ and $\mathbf{E}|U_1|^p < \infty$ for some $p \geqslant 3$, we get

$$\mathbf{P}\left(\sum_{1\leqslant i\leqslant k/(\tau+\theta)} (Z_i-\theta) \geqslant \frac{\tau k}{\tau+\theta}\right)$$

$$\leqslant \left(1+\frac{2}{p}\right)^p \frac{k}{\tau+\theta}\mathbf{E}|Z_i-\theta|^p\left(\frac{\tau k}{\tau+\theta}\right)^{-p}$$
$$+\exp\left\{-2(-p+2)^{-2}\mathrm{e}^{-p}\left(\frac{k}{\tau+\theta}\mathbf{E}(Z_i-\theta)^2\right)^{-1}\left(\frac{\tau k}{\tau+\theta}\right)^2\right\}$$
$$\leqslant C_p k^{-p+1}+\exp\{-D_p k\},$$

where C_p and D_p are positive constants. Therefore, we get the asymptotic expressions

$$J_2 = \overline{F}(x)o(1) = o\big(\overline{F}(x)\big)$$

and

$$K_1 = J_1 + J_2 = o\big(\overline{F}(x)\big).$$

Now we examine

$$K_2 = \left(\frac{\mathrm{e}\mu}{x}\right)^u \sum_{k>(1+\delta)\lambda(t)} k^u\mathbf{P}\big(N(t)=k\big) = (\mathrm{e}\mu)^u x^{-u} \sum_{k>(1+\delta)\lambda(t)} k^u\mathbf{P}\big(N(t)=k\big).$$

In combination with Lemma 1, we get

$$K_2 = (\mathrm{e}\mu)^u o\big(\overline{F}(x)\big) \sum_{k>(1+\delta)\lambda(t)} k^u\mathbf{P}\big(N(t)=k\big).$$

In the same way, we obtain

$$\sum_{k>(1+\delta)\lambda(t)} k^u\mathbf{P}\big(N(t)=k\big) = o(1),$$

and so we get the relation

$$K_2 = o\big(\overline{F}(x)\big);$$

hence,

$$I_3 = K_1 + K_2 = o\big(\overline{F}(x)\big). \tag{3.4}$$

Finally, substituting (3.2)–(3.4) into (3.1), we obtain the result.

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