NUMERICAL SOLUTION OF A SYSTEM OF TWO FIRST ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS ARISING IN ULTIMATE RUIN THEORY

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Abstract. This paper considers a model of ultimate ruin theory in the form of a system of two 1st order Volterra integro-differential equations by Albrecher and Boxma (2004) and presents numerical results obtained from the application of polynomial collocation methods.

1. Introduction

In a recent article (Makroglou (2004) ([14]), models of ruin theory in the form of second order integro-differential equations were presented together with the computational treatment of one of them by collocation methods.

Ruin theory models for the ultimate time case in the form of first order integro-differential equations can be found for example in the papers by Dickson and Gray (1984) ([4]), Peters and Mangel (1990) ([20]), Makroglou, Harper, Smith (2000) ([15]), Lin, Willmot and Drekic (2003) ([11]), Albrecher and Boxma (2004) ([1]), Dickson and Waters (1996) ([6]), Möller (1996) ([19]), Michaud (1996) ([17]).

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For a brief bibliographical overview of methods of solution of integral and integro-differential equation models used in ruin theory, we refer for example to Makroglou (2003) ([13]).

The notation used is that of Dickson and Waters (2002) ([7]) which was also followed in Makroglou (2004) ([14]). So we let:

$U(t)$	:	surplus process, $t \geq 0$
u	:	initial surplus
X_i	:	amount of the $i-th$ claim
$N(t)$	:	counting process for the number of claims up to time t
$S(t)$	:	the accumulated claims process
T_i	:	random variables for the claim inter-arrival times, $i = 1, \ldots, \infty$
T	:	time to ruin
c	:	insurer's premium income/unit time assumed to be received continuously and such that $cE(T_i) > E(X_i), \forall i$
F	:	distribution function of X_i with density function f

We also let:

$\psi(u)$	=	$Pr(T < \infty)$ the probability of ultimate ruin from initial reserve u
$\phi(u)$	=	$1 - \psi(u)$ the probability of ultimate non-ruin from initial reserve u
δ	:	non negative parameter (interest force, or a dummy variable)
m_k	=	$E(X_i^k)$.

For the classical risk model we have:

$$U(t) = u + ct - S(t) \tag{1.1}$$

where

$$S(t) = \sum_{i=1}^{N(t)} X_i. \tag{1.2}$$

The accumulated claims process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process with Poisson parameter λ. The claim sizes $X_i, i = 1, 2, \ldots, N(t)$ are assumed to be independent and identically distributed.

For continuous risk models where the risk process is a variation/extension of the classical compound Poisson process, see for example Dickson and dos Reis (1997)([5]), Klüppelberg and Stadtmüller (1998) ([10]), Stanford, Stroiński and Lee (2000) ([22]), Kalashnikov and Konstantinides (2000) ([9], Lin, Willmot and Drekic (2003) ([11]).

Other types of continuous risk models include ones where the risk process is perturbed by Brownian motion (cf. Schlegel (1998) ([21])) and ones which allow for the insurance company to invest part of the surplus in bonds and part in the stock market (cf. Gaier and Grandits (2004) ([8])), or allow dependence between claim sizes and claim intervals (cf. Albrecher and Boxma (2004) ([1])).

For an introduction to models of the claims number process and the claim size distributions, see for example Mikosch (2004) ([18], chapters 2, 3).

This paper presents numerical results obtained from the application of collocation methods to a model applying to ultimate time ruin theory which is in the form of a system of two first order integro-differential equations developed by Albrecher and Boxma (2004) ([1]). It is based on a talk presented at the HERCMA 2006 Conference, Athens University of Economics and Business, 24-26 September, 2006. See also Makroglou and Konstantinides (2005) ([16]).

The organisation of the paper is as follows. Section 2 contains a description of the application of polynomial collocation method to a system of two 1st order Volterra integro-differential equations.

In section 3 the model by Albrecher and Boxma (2004) ([1]) is presented.

Section 4 contains the computational results and section 5 some conclusions.

2. Collocation methods for a system of first order Volterra integro-differential equations

The general form of a system of r first order Volterra integro-differential equations (VIDEs) is

$$y_i'(x) = G_i(x, y_1(x), \ldots, y_r(x), \int_0^x K_i(x, s, y_1(s), \ldots, y_r(s))ds, \quad (2.1)$$
$$0 \leq x \leq X$$
$$y_i(0) = y_{i0}, i = 1, 2, \ldots, r, \quad (2.2)$$

where $y_1(x), y_2(x), \ldots, y_r(x)$ are the unknown functions and $G_i, K_i, i = 1, 2, \ldots, r$ are given functions. Methods of numerical solution include quadrature methods, linear multistep methods, collocation methods, defect correction methods, Galerkin methods. For an introduction to the subject we refer for example to the books by Linz (1985) ([12]) and by Brunner (2004) ([2]).

(Following for example Brunner, Makroglou, Miller (1997)([3])) and extending to systems of VIDEs).

Preliminaries, notation

Consider mesh points

$$\Pi_N : 0 = u_0 < u_1 < \ldots < u_N (N \geq 1)$$

and set

$$h_n = u_{n+1} - u_n, n = 0, 1, \ldots, N-1,$$
$$\sigma_0 = [u_0, u_1], \sigma_n = (u_n, u_{n+1}], n = 1, 2, \ldots, N-1,$$
$$Z_N = \{u_n : n = 1, 2, \ldots, N-1\}, \tilde{Z}_N = Z_N \cup \{u_N\}.$$

The mesh sequence is assumed to be quasi-uniform, i.e.,

$$\frac{\max_{1 \leq n \leq N} h_n}{\min_{1 \leq n \leq N} h_n} \leq \gamma < \infty$$

uniformly for $N \in \mathbf{N}$.

Let $h = \max_{1 \leq n \leq N} h_n$ and $m \geq 1$ be a given integer.

$$U_n = \{u_{nj} = u_n + h_n c_j, j = 1, 2, \ldots, m\}, \quad U(N) = \cup_{n=0}^{N-1} U_n,$$

and the collocation parameters c_j be such that $0 \leq c_1 < \ldots < c_m \leq 1$.

The collocation method to be described next, approximates the solution $\vec{y}(x) = (y_1(x), y_2(x), \ldots, y_r(x))^T$ in (2.1)-(2.2) by approximating functions $\vec{v}(x) = (v_1(x), v_2(x), \ldots, v_r(x))^T$ on each subinterval σ_n and will find values of $\vec{v}(x)$ for $x \in U(N)$. These values will be determined by requiring that the system of VIDEs (2.1)-(2.2) is satisfied on $U(N)$, i.e. that (in vector form)

$$\vec{v}'(x) = \vec{G}(x, \vec{v}(x), \int_0^x \vec{K}(x, s, \vec{v}(s))ds, \forall x \in U(N), \tag{2.3}$$

with

$$\vec{v}(0) = \vec{y}(0) = \vec{y}_0, \tag{2.4}$$

or

$$\vec{v_n}'(x) = \vec{G}(x, \vec{v_n}(x), \sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} \vec{K}(x, s, \vec{v_i}(s))ds + \int_{u_n}^{x} \vec{K}(x, s, \vec{v_n}(s))ds, \forall x \in U_n, n = 0, \ldots, N-1, \tag{2.5}$$

where $v_n(x)$ is the restriction of $v(x)$ in σ_n.

Application of the collocation method.

Discretizing equation (2.5) at points $x = u_{nj}$ we obtain

$$\vec{v_n}'(u_{nj}) = \vec{G}(u_{nj}, \vec{v_n}(u_{nj}), \sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} \vec{K}(u_{nj}, s, \vec{v_i}(s))ds + \int_{u_n}^{u_{nj}} \vec{K}(u_{nj}, s, \vec{v_n}(s))ds \tag{2.6}$$

$$= \vec{G}(u_{nj}, \vec{v_n}(u_{nj}), \sum_{i=0}^{n-1} h_i \int_0^1 \vec{K}(u_{nj}, u_i + h_i s, \vec{v_i}(u_i + h_i s))ds + c_j h_n \int_0^1 \vec{K}(u_{nj}, u_n + c_j h_n s, \vec{v_n}(u_n + c_j h_n s))ds. \tag{2.7}$$

We now write

$$\vec{v_n}'(u_n + \tau h_n) = \sum_{k=1}^{m} B_k(\tau) \vec{v_n}'(u_{nk}) \tag{2.8}$$

and thus

$$\vec{v_n}(u_n + \tau h_n) = \vec{v_n}(u_n) + h_n \sum_{k=1}^{m} w_k(\tau) \vec{_n v}'(u_{nk}), \tag{2.9}$$

where

$$w_k(\tau) = \int_0^\tau B_k(s) ds, \tag{2.10}$$

and $B_k(s)$ are the Lagrange fundamental polynomials.

Using (2.9), (2.10) equation (2.7) is written

$$\begin{aligned} \vec{v_n}'(u_{nj}) &= \vec{G}\big(u_{nj}, \vec{v_n}(u_n) + h_n \sum_{k=1}^{m} w_k(c_j) \vec{v_n}'(u_{nk}), \\ & \sum_{i=0}^{n-1} h_i \int_0^1 \vec{K}\big(u_{nj}, u_i + h_i s, \vec{v_i}(u_i + h_i s)\big) ds \\ & + c_j h_n \int_0^1 \vec{K}\big(u_{nj}, u_n + c_j h_n s, \\ & \vec{v_n}(u_n) + h_n \sum_{k=1}^{m} w_k(c_j s) \vec{v_n}'(u_{nk})\big) ds\big). \end{aligned} \tag{2.11}$$

Approximating the integrals by quadrature rules of interpolatory type we obtain the approximate equations

$$\begin{aligned} \vec{V}_{nj} &= \vec{G}\big(u_{nj}, \vec{v_n}(u_n) + h_n \sum_{k=1}^{m} w_k(c_j) \vec{V}_{nk}, \\ & \sum_{i=0}^{n-1} h_i \sum_{\lambda=1}^{m} w_\lambda(1) \vec{K}\big(u_{nj}, u_{i\lambda}, \vec{v_i}(u_{i\lambda})\big) \\ & + c_j h_n \sum_{\lambda=1}^{m} w_\lambda(1) \vec{K}\big(u_{nj}, u_n + c_j h_n c_\lambda, \vec{v_n}(u_n) + h_n \sum_{k=1}^{m} w_k(c_j c_\lambda) \vec{V}_{nk}\big)\big), \\ & n = 0, 1, \ldots, N; j = 1, 2, \ldots, m, \end{aligned} \tag{2.12}$$

where we have set

$$\vec{V}_{nj} \simeq \vec{v_n}'(u_{nj}). \tag{2.13}$$

We are using $\vec{v_n}(u_n) = \vec{v_{n-1}}(u_n)$ in the case that $c_1 = 0, c_m = 1$.

For each $n = 0, \ldots, N-1$, equations (2.12) are in general a nonlinear system of rm equations in $(\vec{V}_{nj})_i, i = 1, 2, \ldots, r$.

3. The Albrecher and Boxma ([1]) example

Albrecher and Boxma (2004) ([1]) consider a generalization of the classical risk model, where the distribution of the time between two claim occurences depends on the size of the previous claim (claim sizes - claim inter-occurence times dependence risk model). They derive exact solutions for the probability of survival using Laplace-Stieltjes transforms.

In one of their models (Model 1), the claim occurence process is assumed to be such that if a claim X_i is larger than a certain threshold H_i, then the time until the next claim is exponentially distributed with rate λ_1, otherwise it is exponentially distributed with rate λ_2. It is also assumed that $\{H_i\}$ are independent and identically distributed random variables with distribution function $H(\cdot)$.

The system of integro-differential equations given for $\phi_i(u), i = 1, 2$, the probability of survival with initial reserve u, given that the first claim occurs according to the exponential distribution with rate λ_i, are ([1], p. 246)

$$c\phi_1'(u) - \lambda_1\phi_1(u) + \lambda_1 \int_0^u \mathbb{P}(H \leq y)\phi_1(u-y)dF(y) + \lambda_1 \int_0^u \mathbb{P}(H > y)\phi_2(u-y)dF(y) = 0, \tag{3.1}$$

$$c\phi_2'(u) - \lambda_2\phi_2(u) + \lambda_2 \int_0^u \mathbb{P}(H \leq y)\phi_1(u-y)dF(y) + \lambda_2 \int_0^u \mathbb{P}(H > y)\phi_2(u-y)dF(y) = 0. \tag{3.2}$$

A number of variations of this model are considered together with several examples in the Albrecher and Boxma (2004) ([1]) paper.

4. Numerical results with exponentially distributed claims

Numerical results are presented for one example, that of the system of VIDEs (3.1)-(3.2) from the paper of Albrecher and Boxma (2004) ([1], p. 246), and in particular

$$\begin{aligned} \phi_1'(u) &= \frac{\lambda_1}{c}\phi_1(u) - \frac{\lambda_1}{c}\int_0^u [\mathbb{P}(H \le y)\phi_1(u-y) \\ &+ \mathbb{P}(H > y)\phi_2(u-y)]dF(y), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \phi_2'(u) &= \frac{\lambda_2}{c}\phi_2(u) - \frac{\lambda_2}{c}\int_0^u [\mathbb{P}(H \le y)\phi_1(u-y) \\ &+ \mathbb{P}(H > y)\phi_2(u-y)]dF(y). \end{aligned} \tag{4.2}$$

Using $H(y) = 1 - e^{-\mu y}$ and $F(y) = 1 - e^{-vy}$ and setting $w = u - y$, we obtain

$$\begin{aligned} \phi_1'(u) &= \frac{\lambda_1}{c}\phi_1(u) - \frac{\lambda_1 v}{c}\int_0^u [e^{-v(u-w)}\phi_1(w) \\ &- e^{-(\mu+v)(u-w)}\phi_1(w) + e^{-(\mu+v)(u-w)}\phi_2(w)]dw, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \phi_2'(u) &= \frac{\lambda_2}{c}\phi_2(u) - \frac{\lambda_2 v}{c}\int_0^u [e^{-v(u-w)}\phi_1(w) \\ &- e^{-(\mu+v)(u-w)}\phi_1(w) + e^{-(\mu+v)(u-w)}\phi_2(w)]dw. \end{aligned} \tag{4.4}$$

True solution

(1)

$$c = 2, \lambda_1 = 3, \lambda_2 = 1, \mu = 2, v = 1.$$

$$\begin{aligned} &\phi_1(u) = 1 - 0.006822555643524832e^{-3.1612304061510854u} \\ &-0.9384348924092282e^{-0.0645182293014091u} \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\phi_2(u) = 1 - 0.002895338477817656e^{-3.1612304061510854u} \\ &-0.8669330294870135e^{-0.0645182293014091u} \end{aligned} \tag{4.6}$$

([1], Example 3, p. 252 and private communication of H. Albrecher).

(2)

$$c = 2, \lambda_1 = 1, \lambda_2 = 2, \mu = 1, v = 1.$$

$$\phi_1(u) = 1 - 0.6322060547624042e^{-0.35541572677584504u} + 0.017076223892762123e^{-1.8892285591291944u} \quad (4.7)$$

$$\phi_2(u) = 1 - 0.7979824803908411e^{-0.35541572677584504u} + 0.028242142130125e^{-1.8892285591291944u} \quad (4.8)$$

([1], Example 4, p. 252 and private communication of H. Albrecher).

The true solution of (3.1)-(3.2) was used to find $\phi_1(0), \phi_2(0)$. In the Albrecher and Boxma (2004) ([1]) paper, formulae for $\phi_1(0), \phi_2(0)$ are given in terms of c, λ_1, λ_2, $\chi_1(s) = \int_0^\infty e^{-sx} H(x) dF(x)$, $\chi_2(s) = \int_0^\infty e^{-sx}(1 - H(x))dF(x)$ and σ, the unique zero with $\Re(\sigma) > 0$ of

$$\left(cs + \frac{\lambda_1 \mu v}{(v+s)(v+\mu+s)} - \lambda_1\right) \left(cs + \frac{\lambda_2 v}{v+\mu+s} - \lambda_2\right) - \frac{\lambda_1 \lambda_2 \mu v^2}{(v+\mu+s)^2(v+s)} = 0.$$

The numerical results in the following tables show computed values of $\phi_1(u)$ and $\phi_2(u)$ in the first and the second row respectively, for 2 step sizes ($h = 0.1(X = 10, N = 100), h = 0.05(X = 10, N = 200)$) and absolute errors using the true solution.

Tables 4.1-4.4 contain results for Example 1 and Tables 4.5-4.8 for Example 2.

The collocation (COL) methods were used with $m = 3$, and collocation parameters $c_1 = 0, c_2 = 0.5, c_3 = 1$ which makes the method equivalent to the Simpson's method and with $m = 4$ and collocation parameters $c_1 = 0$, $c_2 = 0.5(1 - \sqrt{0.2})$, $c_3 = 0.5(1 + \sqrt{0.2})$, $c_4 = 1$, the Lobatto points in $[0, 1]$.

TABLE 1. Example (1),
$m = 3, c_1 = 0, c_2 = 0.5, c_3 = 1$

u	$h = 0.1$	True sol	abs error
0.1	$0.62626819E-01$	$0.62626797E-01$	$0.22479376E-07$
	$0.13653165E+00$	$0.13653165E+00$	$0.47238664E-08$
0.5	$0.89950758E-01$	$0.89950691E-01$	$0.67125716E-07$
	$0.15999119E+00$	$0.15999119E+00$	$0.67865370E-08$
1.0	$0.11991042E+00$	$0.11991034E+00$	$0.76612375E-07$
	$0.18711110E+00$	$0.18711111E+00$	$0.10153176E-07$
1.5	$0.14806868E+00$	$0.14806861E+00$	$0.64903008E-07$
	$0.21300923E+00$	$0.21300926E+00$	$0.38220057E-07$
2.0	$0.17515802E+00$	$0.17515798E+00$	$0.38632692E-07$
	$0.23801095E+00$	$0.23801102E+00$	$0.74177443E-07$
5.0	$0.32031891E+00$	$0.32031984E+00$	$0.93062680E-06$
	$0.37210614E+00$	$0.37210649E+00$	$0.35040290E-06$
8.0	$0.43989763E+00$	$0.43992671E+00$	$0.29077939E-04$
	$0.48260277E+00$	$0.48260019E+00$	$0.25730761E-05$
10.0	$0.50739768E+00$	$0.50772811E+00$	$0.33043082E-03$
	$0.54527564E+00$	$0.54523562E+00$	$0.40022689E-04$

TABLE 2. Example (1),
$m = 3, c_1 = 0, c_2 = 0.5, c_3 = 1$

u	$h = 0.05$	true sol	abs error
0.1	$0.62626798E-01$	$0.62626797E-01$	$0.13828571E-08$
	$0.13653165E+00$	$0.13653165E+00$	$0.28691352E-09$
0.5	$0.89950695E-01$	$0.89950691E-01$	$0.40303382E-08$
	$0.15999119E+00$	$0.15999119E+00$	$0.37751263E-09$
1.0	$0.11991034E+00$	$0.11991034E+00$	$0.42890287E-08$
	$0.18711111E+00$	$0.18711111E+00$	$0.73286877E-09$
1.5	$0.14806862E+00$	$0.14806861E+00$	$0.29467294E-08$
	$0.21300926E+00$	$0.21300926E+00$	$0.25270754E-08$
2.0	$0.17515798E+00$	$0.17515798E+00$	$0.20825816E-09$
	$0.23801102E+00$	$0.23801102E+00$	$0.47746166E-08$
5.0	$0.32031969E+00$	$0.32031984E+00$	$0.14645911E-06$
	$0.37210648E+00$	$0.37210649E+00$	$0.12403337E-07$
8.0	$0.43992144E+00$	$0.43992671E+00$	$0.52757322E-05$
	$0.48260079E+00$	$0.48260019E+00$	$0.59312340E-06$
10.0	$0.50766734E+00$	$0.50772811E+00$	$0.60765648E-04$
	$0.54524317E+00$	$0.54523562E+00$	$0.75496452E-05$

Table 3. Example (1),
$m = 4, c_1 = 0, c_2 = 0.5(1 - \sqrt{0.2}), c_3 = 0.5(1 + \sqrt{0.2}), c_4 = 1$

u	$h = 0.1$	true sol	abs error
0.1	$0.62626797E-01$	$0.62626797E-01$	$0.10418263E-10$
	$0.13653165E+00$	$0.13653165E+00$	$0.92914565E-12$
0.5	$0.89950691E-01$	$0.89950691E-01$	$0.37317635E-10$
	$0.15999119E+00$	$0.15999119E+00$	$0.67967854E-12$
1.0	$0.11991034E+00$	$0.11991034E+00$	$0.65560793E-10$
	$0.18711111E+00$	$0.18711111E+00$	$0.44000359E-11$
1.5	$0.14806861E+00$	$0.14806861E+00$	$0.10918097E-09$
	$0.21300926E+00$	$0.21300926E+00$	$0.13639700E-10$
2.0	$0.17515798E+00$	$0.17515798E+00$	$0.18816149E-09$
	$0.23801102E+00$	$0.23801102E+00$	$0.28674563E-10$
5.0	$0.32031985E+00$	$0.32031984E+00$	$0.69832501E-08$
	$0.37210649E+00$	$0.37210649E+00$	$0.93608099E-09$
8.0	$0.43992699E+00$	$0.43992671E+00$	$0.27698438E-06$
	$0.48260016E+00$	$0.48260019E+00$	$0.35023847E-07$
10.0	$0.50773133E+00$	$0.50772811E+00$	$0.32154748E-05$
	$0.54523522E+00$	$0.54523562E+00$	$0.40522472E-06$

TABLE 4. Example (1),
$m = 4, c_1 = 0, c_2 = 0.5(1 - \sqrt{0.2}), c_3 = 0.5(1 + \sqrt{0.2}), c_4 = 1$

u	$h = 0.05$	true sol	abs error
0.1	$0.62626797E-01$	$0.62626797E-01$	$0.16205093E-12$
	$0.13653165E+00$	$0.13653165E+00$	$0.13988810E-13$
0.5	$0.89950691E-01$	$0.89950691E-01$	$0.57676086E-12$
	$0.15999119E+00$	$0.15999119E+00$	$0.79658502E-14$
1.0	$0.11991034E+00$	$0.11991034E+00$	$0.10044049E-11$
	$0.18711111E+00$	$0.18711111E+00$	$0.74218409E-13$
1.5	$0.21300926E+00$	$0.21300926E+00$	$0.22107316E-12$
	$0.15082095E+00$	$0.15082095E+00$	$0.17501278E-11$
2.0	$0.17515798E+00$	$0.17515798E+00$	$0.28508307E-11$
	$0.23801102E+00$	$0.23801102E+00$	$0.45699555E-12$
5.0	$0.32031984E+00$	$0.32031984E+00$	$0.10559092E-09$
	$0.37210649E+00$	$0.37210649E+00$	$0.14258372E-10$
8.0	$0.43992672E+00$	$0.43992671E+00$	$0.41901815E-08$
	$0.48260019E+00$	$0.48260019E+00$	$0.53005494E-09$
10.0	$0.50772816E+00$	$0.50772811E+00$	$0.48644755E-07$
	$0.54523561E+00$	$0.54523562E+00$	$0.61306801E-08$

Table 5. Example (2),
$m = 3, c_1 = 0, c_2 = 0.5, c_3 = 1$

u	$h = 0.1$	True sol	abs error
0.1	$0.40400548E + 00$	$0.40400549E + 00$	$0.28641088E - 08$
	$0.25326124E + 00$	$0.25326125E + 00$	$0.77315149E - 08$
0.5	$0.47736507E + 00$	$0.47736507E + 00$	$0.87887970E - 08$
	$0.34292082E + 00$	$0.34292085E + 00$	$0.29220979E - 07$
1.0	$0.55947987E + 00$	$0.55947988E + 00$	$0.96874941E - 08$
	$0.44497827E + 00$	$0.44497832E + 00$	$0.47931404E - 07$
1.5	$0.63004475E + 00$	$0.63004476E + 00$	$0.74389767E - 08$
	$0.53342858E + 00$	$0.53342865E + 00$	$0.68841873E - 07$
2.0	$0.68982821E + 00$	$0.68982821E + 00$	$0.27715154E - 08$
	$0.60864807E + 00$	$0.60864816E + 00$	$0.98475122E - 07$
5.0	$0.89307563E + 00$	$0.89307538E + 00$	$0.25334961E - 06$
	$0.86503739E + 00$	$0.86503824E + 00$	$0.85432232E - 06$
8.0	$0.96318899E + 00$	$0.96318559E + 00$	$0.34057188E - 05$
	$0.95352482E + 00$	$0.95353215E + 00$	$0.73341840E - 05$
10.0	$0.98193145E + 00$	$0.98191545E + 00$	$0.16003596E - 04$
	$0.97714152E + 00$	$0.97717334E + 00$	$0.31816912E - 04$

TABLE 6. Example (2),
$m = 3, c_1 = 0, c_2 = 0.5, c_3 = 1$

u	$h = 0.05$	True sol	abs error
0.1	$0.40400549E + 00$	$0.40400549E + 00$	$0.17810758E - 09$
	$0.25326125E + 00$	$0.25326125E + 00$	$0.48182353E - 09$
0.5	$0.47736507E + 00$	$0.47736507E + 00$	$0.54571220E - 09$
	$0.34292085E + 00$	$0.34292085E + 00$	$0.18199870E - 08$
1.0	$0.55947988E + 00$	$0.55947988E + 00$	$0.60062677E - 09$
	$0.44497832E + 00$	$0.44497832E + 00$	$0.29853972E - 08$
1.5	$0.63004476E + 00$	$0.63004476E + 00$	$0.46117343E - 09$
	$0.53342865E + 00$	$0.53342865E + 00$	$0.42914111E - 08$
2.0	$0.68982821E + 00$	$0.68982821E + 00$	$0.17273982E - 09$
	$0.60864816E + 00$	$0.60864816E + 00$	$0.61462068E - 08$
5.0	$0.89307539E + 00$	$0.89307538E + 00$	$0.15794830E - 07$
	$0.86503819E + 00$	$0.86503824E + 00$	$0.53516331E - 07$
8.0	$0.96318580E + 00$	$0.96318559E + 00$	$0.21303671E - 06$
	$0.95353169E + 00$	$0.95353215E + 00$	$0.45932545E - 06$
10.0	$0.98191645E + 00$	$0.98191545E + 00$	$0.10016733E - 05$
	$0.97717134E + 00$	$0.97717334E + 00$	$0.19922586E - 05$

TABLE 7. Example (2),
$m = 4, c_1 = 0, c_2 = 0.5(1-\sqrt{0.2}), c_3 = 0.5(1+\sqrt{0.2}), c_4 = 1$

u	$h = 0.1$	true sol	abs error
0.1	$0.40400549E+00$	$0.40400549E+00$	$0.86930463E-13$
	$0.25326125E+00$	$0.25326125E+00$	$0.99786845E-12$
0.5	$0.47736507E+00$	$0.47736507E+00$	$0.15498713E-12$
	$0.34292085E+00$	$0.34292085E+00$	$0.41088244E-11$
1.0	$0.55947988E+00$	$0.55947988E+00$	$0.28976821E-12$
	$0.44497832E+00$	$0.44497832E+00$	$0.71163075E-11$
1.5	$0.63004476E+00$	$0.63004476E+00$	$0.12867485E-11$
	$0.53342865E+00$	$0.53342865E+00$	$0.10210055E-10$
2.0	$0.68982821E+00$	$0.68982821E+00$	$0.29293235E-11$
	$0.60864816E+00$	$0.60864816E+00$	$0.14139800E-10$
5.0	$0.89307538E+00$	$0.89307538E+00$	$0.51693094E-10$
	$0.86503824E+00$	$0.86503824E+00$	$0.11197010E-09$
8.0	$0.96318559E+00$	$0.96318559E+00$	$0.51928939E-09$
	$0.95353215E+00$	$0.95353215E+00$	$0.10150354E-08$
10.0	$0.98191545E+00$	$0.98191545E+00$	$0.23252146E-08$
	$0.97717333E+00$	$0.97717334E+00$	$0.44813431E-08$

TABLE 8. Example (2),
$m = 4, c_1 = 0, c_2 = 0.5(1-\sqrt{0.2}), c_3 = 0.5(1+\sqrt{0.2}), c_4 = 1$

u	$h = 0.05$	true sol	abs error
0.1	$0.40400549E+00$	$0.40400549E+00$	$0.13877788E-14$
	$0.25326125E+00$	$0.25326125E+00$	$0.15709656E-13$
0.5	$0.47736507E+00$	$0.47736507E+00$	$0.25535130E-14$
	$0.34292085E+00$	$0.34292085E+00$	$0.64448447E-13$
1.0	$0.55947988E+00$	$0.55947988E+00$	$0.39968029E-14$
	$0.44497832E+00$	$0.44497832E+00$	$0.11152190E-12$
1.5	$0.63004476E+00$	$0.63004476E+00$	$0.19206858E-13$
	$0.53342865E+00$	$0.53342865E+00$	$0.15976109E-12$
2.0	$0.68982821E+00$	$0.68982821E+00$	$0.44741988E-13$
	$0.60864816E+00$	$0.60864816E+00$	$0.22126745E-12$
5.0	$0.89307538E+00$	$0.89307538E+00$	$0.80491169E-12$
	$0.86503824E+00$	$0.86503824E+00$	$0.17515989E-11$
8.0	$0.96318559E+00$	$0.96318559E+00$	$0.81097351E-11$
	$0.95353215E+00$	$0.95353215E+00$	$0.15869306E-10$
10.0	$0.98191545E+00$	$0.98191545E+00$	$0.36332048E-10$
	$0.97717333E+00$	$0.97717334E+00$	$0.70048523E-10$

5. NUMERICAL RESULTS WITH PARETO CLAIMS

Let us see not the case with pareto distributed claims. Namely, for $\kappa > 1$, we consider the distribution of claim sizes,

$$F(y) = \left[1 - \left(\frac{\kappa}{y+\kappa}\right)^{\alpha}\right] \mathbf{1}_{[y>0]} \,,$$

with density

$$f(y) = \frac{\alpha\, \kappa^{\alpha}}{(y+\kappa)^{\alpha+1}} \,,$$

for $y > 0$ and mean value

$$\frac{\alpha\, \kappa}{\alpha - 1} \,.$$

Then we obtain for $\alpha \neq 1, 2, \ldots$

$$\begin{aligned}\chi_1(s) &= \int_0^\infty e^{-s\,x}\,\mathbb{P}[H \leq x]\,dF(x) \qquad (5.1)\\ &= \alpha\,\kappa^\alpha \int_0^\infty e^{-s\,x}\,(x+\kappa)^{-\alpha-1}\,dx - \alpha\,\kappa^\alpha \int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx\,,\\ \chi_2(s) &= \int_0^\infty e^{-s\,x}\,\mathbb{P}[H > x]\,dF(x) = \alpha\,\kappa^\alpha \int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx\,,\end{aligned}$$

and thus σ is the unique solution s with $Res > 0$ of the equation

$$\left[c\,s + \lambda_2\,\alpha\,\kappa^\alpha \int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx - \lambda_2\right]$$

$$\left[c\,s + \lambda_1\,\alpha\,\kappa^\alpha \left\{\int_0^\infty e^{-s\,x}\,(x+\kappa)^{-\alpha-1}\,dx - \int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx\right\} - \lambda_1\right]$$

$$= \lambda_1\,\lambda_2\,\alpha^2\,\kappa^{2\alpha}\left\{\int_0^\infty e^{-s\,x}\,(x+\kappa)^{-\alpha-1}\,dx - \int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx\right\}$$

$$\left(\int_0^\infty e^{-(s+\mu)\,x}\,(x+\kappa)^{-\alpha-1}\,dx\right)\,,$$

With thus chosen σ we obtain

$$\phi_2(0+) = \frac{c\,\sigma + \lambda_2\,\chi_2(\sigma) - \lambda_2}{\lambda_1\,\chi_2(\sigma)}\phi_1(0+) = \frac{\lambda_2\,\chi_1(\sigma)}{c\,\sigma + \lambda_1\,\chi_1(\sigma) - \lambda_1}\phi_1(0+)\,.$$

So together with the relation

$$[1-\phi_1(0+)]\frac{\mathbb{P}[X > H]}{\lambda_1} + [1-\phi_2(0+)]\frac{\mathbb{P}[X \leq H]}{\lambda_2} = \frac{\alpha\,\kappa}{c\,(\alpha-1)}\,,$$

we find the Laplace-Stieltjes transforms of the survival probabilities ϕ_1 and ϕ_2

$$\frac{c\,\phi_1(0+)\,[c\,s + \lambda_2\,\chi_2(s) - \lambda_2] - c\,\lambda_1\,\chi_2(s)\,\phi_2(0+)}{[c\,s + \lambda_1\,\chi_1(s) - \lambda_1]\,[c\,s + \lambda_2\,\chi_2(s) - \lambda_2] - \lambda_1\,\lambda_2\,\chi_1(s)\,\chi_2(s)}\,, \qquad (5.2)$$

$$\frac{c\,\phi_2(0+)\,[c\,s + \lambda_1\,\chi_1(s) - \lambda_1] - c\,\lambda_2\,\chi_1(s)\,\phi_1(0+)}{[c\,s + \lambda_1\,\chi_1(s) - \lambda_1]\,[c\,s + \lambda_2\,\chi_2(s) - \lambda_2] - \lambda_1\,\lambda_2\,\chi_1(s)\,\chi_2(s)}\,, \qquad (5.3)$$

respectively.

6. Conclusions

Numerical results for a model of ultimate ruin theory in the form of a system of two first order Volterra integro-differential equations (Model 1, in Albrecher and Boxma (2004) ([1])) were presented. They were obtained by the application of polynomial collocation methods extended to apply to systems of r 1st order VIDEs.

The results were in good agreement with these found by analytical methods in [1]. The observed order of convergence was $O(h^4)$ and $O(h^6)$ for the choices of the collocation parameters being $c_1 = 0, c_2 = 0.5, c_3 = 1$ and $c_1 = 0, c_2 = 0.5(1 - \sqrt{0.2})$, $c_3 = 0.5(1 + \sqrt{0.2})$, $c_4 = 1$ respectively. These are in agreement with the theoretical orders of convergence (cf. Brunner (2004) ([2]), p. 174, 175). For the method using $c_1 = 0, c_2 = 0.5, c_3 = 1$ applied to Example 1, the order of convergence for large u was verified by computations obtained with $h = 0.025$.

Keywords: Numerical solution, collocation methods, Volterra integro-differential equations, actuarial risk management, ultimate time ruin theory.

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