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A two-fluid storage model with Lévy inputs and alternating outputs

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Abstract In this paper we consider a storage model with two types of inputs and outputs that are subject to seasonal switching. Inputs are assumed to occur in a fluid fashion whereas outputs occur at a unit rate so long as the corresponding storage is non-empty. The distribution properties of the storage levels $\{Z_1(t), Z_2(t)\}$ are derived at finite time as well as in stationary regime. We first investigate this process embedded at the successive switching points. This process is Markovian with independent components. In continuous time the components $\{Z_1(t), Z_2(t)\}$ are also independent for each finite t, but are dependent in stationary regime.

Keywords Seasonal Switching · Dependence model

Mathematics Subject Classification (2000) Primary 90B05 · Secondary 91B70

1 Introduction

We consider a storage with input of two types of items and output that occur alternatingly for these types over constant intervals of time. Specifically, we assume the following:

(i) Let $X_i(t)$ be the input of type i that occurs during the time-interval $(0, t]$ for $i = 1, 2$. We assume that $X_1 = \{X_1(t), t \geq 0\}$ and $X_2 = \{X_2(t), t \geq 0\}$ are two independent Lévy processes on the state space $[0, \infty)$ with zero drift and $X_1(0) = X_2(0) = 0$ (subordinators).
(ii) Outputs occur during the interval $(nc, nc+r]$ for type 1 items and during the interval $(nc + r, nc + r + g]$ for type 2 items ($n \geq 0$). Here $r > 0$, $g > 0$ and $c = g + r$. For convenience we refer to $(nc, nc + r]$ as the red period and to $(nc + r, nc + c]$ as the green period.
(iii) The output in each case occurs continuously at unit rate except when the item of that type is not available in storage.

We denote by $Z_i(t)$ the storage level of items of type i at time $t \geq 0$. Of main interest is the study of the process $\{Z_1(t), Z_2(t), t \geq 0\}$.

The assumption regarding the accumulated inputs implies that small as well as large inputs appear in time in a steady fashion. As a particular case we have the compound Poisson process in which inputs appear in a simple Poisson process and the amounts of successive inputs are independent and identically distributed random variables. This classical case is still important, but in recent research in this general area more general Lévy processes have been considered, leading to the fluid model.

A few historical remarks may be in order. In Moran's model for a dam (see [4]), inputs of water occur during the *wet season* and are stored for use until the *dry season*, when it is released. Dam models with ordered inputs were considered by Gani and Pyke in [2]. See also [5] for additional references.

The red and green periods of our model may be generalized to intervals of random lengths, that are either independent of the state of the input process or dependent. The

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motivation for this paper is the traffic flow at an intersection controlled by traffic lights. This explains the choice of the terminology: red and green periods.

In Sect. 2 we derive the equations that describe the dynamics of the storage process $\{Z_1(t), Z_2(t)\}$. We prove a result concerning the effect of the interchange of red and green periods and the two types of inputs. This result provides a certain amount of simplicity and elegance in the proofs of the main results. In Sect. 3 we present a preliminary result leading to the independence of the component processes $Z_1(t), Z_2(t)$ for finite t. In Sect. 4 we study the process embedded at the sequence of switching points. In Sect. 5 we extend this study to the continuous time process $\{Z_1(t), Z_2(t)\}$ to derive the stationary behavior of the storage process. In the first case the independence of the components remains intact, but in the general case we find that the stationary random variables Z_1 and Z_2 are negatively dependent.

2 The storage process

For the subordinator $X_i(t)$ we denote the Laplace transform

$$\mathbf{E}[e^{-\theta_i X_i(t)}] = e^{-t\phi_i(\theta_i)} \quad (\theta_i > 0), \tag{2.1}$$

where

$$\phi_i(\theta_i) = \int_0^\infty (1 - e^{-\theta_i x})\nu_i(dx), \tag{2.2}$$

ν_i being a Lévy measure ($i = 1, 2$). We assume that $X_i(t)$ has a finite mean $\rho_i t$, where $\rho_i = \phi_i'(0+)$ ($i = 1, 2$). Our assumptions imply that the process of $\{Z_1(t), Z_2(t)\}$ satisfies the integral equations

$$Z_i(t) = Z_i(0) + X_i(t) - \int_0^t \delta_i(s)\mathbf{1}_{\{Z_i(s)>0\}}ds, \tag{2.3}$$

for $i = 1, 2$, where for any $n \geq 0$

$$\delta_1(t) = \begin{cases} 0 & \text{if } nc < t \leq nc + r, \\ 1 & \text{if } nc + r < t \leq nc + c, \end{cases}$$

and $\delta_2(t) = 1 - \delta_1(t)$. We can rewrite (2.3) as

$$Z_i(t) = Z_i(0) + Y_i(t) + \int_0^t \delta_i(s)\zeta_i(s)ds, \tag{2.4}$$

where $\zeta_i(s) = \mathbf{1}_{\{Z_i(s)=0\}}$ and

$$Y_i(t) = X_i(t) - D_i(t), \quad D_i(t) = \int_0^t \delta_i(s)ds,$$

for $i = 1, 2$. We may refer to $Y_i(t)$ as the net input of type i ($i = 1, 2$).

The integral equation (2.4) has the unique non-negative measurable solution given by

$$Z_i(t) = \max\{Z_i(0) + Y_i(t), \sup_{0\leq s\leq t} [Y_i(t) - Y_i(s)]\}, \tag{2.5}$$

for $i = 1, 2$ (see [3, Theorem 1]), which hold almost surely (a.s.). It is more convenient to write (2.5) as

$$Z_i(t) = Z_i(0) + Y_i(t) + I_i(t),$$

where

$$I_i(t) = [Z_i(0) + \inf_{0\leq s\leq t} Y_i(s)]^-,$$

for $i = 1, 2$.

Considerable simplification is achieved in the analysis of the storage process by noting the following.

Lemma 2.1 *Choose $Z_1(0) \overset{d}{=} Z_2(r)$. Then the permutation*

$$(r, g, X_1, X_2) \to (g, r, X_2, X_1), \tag{2.6}$$

results in the permutation

$$\begin{aligned} &[Z_1(t), I_1(t), Z_2(t + r), I_2(t + r) - I_2(r)] \\ &\quad \overset{d}{\to} [Z_2(t + r), I_2(t + r) - I_2(r), Z_1(t), I_1(t)]. \end{aligned} \tag{2.7}$$

Proof It can be easily verified that the permutation (2.6) results in the permutation

$$[\delta_1(s), \delta_2(s + r)] \to [\delta_2(s + r), \delta_1(s)].$$

Consequently

$$\begin{aligned} Y_1(t) = X_1(t) - \int_0^t \delta_1(s)ds &\to X_2(t) - \int_0^t \delta_2(s + r)ds \\ &\overset{d}{=} X_2(t + r) - X_2(r) - \int_r^{t+r} \delta_2(s)ds \\ &= Y_2(t + r) - Y_2(r), \end{aligned}$$

and similarly $Y_2(t + r) - Y_2(r) \to Y_1(t)$. Thus

$$[Y_1(t), Y_2(t + r) - Y_2(r)] \to [Y_2(t + r) - Y_2(r), Y_1(t)]. \tag{2.8}$$

Now

$$\begin{aligned} I_1(t) &= [Z_1(0) + \inf_{0\leq s\leq t} Y_1(s)]^- \\ &\to [Z_1(0) + \inf_{0\leq s\leq t} \{Y_2(s + r) - Y_2(r)\}]^- \\ &\overset{d}{=} I_2(t + r) - I_2(r), \end{aligned}$$

and similarly $I_2(t+r)-I_2(r)\to I_1(t)$. Finally, using (2.8)

$$\begin{aligned} Z_1(t) &= Z_1(0)+Y_1(t)+I_1(t)\\ &\to Z_2(r)+[Y_2(t+r)-Y_2(r)]+[I_2(t+r)-I_2(r)]\\ &= Z_2(t+r), \end{aligned}$$

and similarly $Z_2(t+r)\to Z_1(t)$. Thus (2.7) is completely proved. □

3 Main results

Lemma 3.1 *For $nc<t\le nc+c$, $(n=0,1,\ldots)$, $\theta_1>0$, $\theta_2>0$ the identity*

$$\begin{aligned} &e^{-\theta_1[I_1(t)-I_1(nc)]-\theta_2[I_2(t)-I_2(nc)]}\\ &\quad=\left\{1-\theta_1\int_{nc+r}^{t\vee nc+r} e^{-\theta_1[I_1(s)-I_1(nc)]}dI_1(s)\right\}\\ &\qquad\times\left\{1-\theta_2\int_{nc}^{t\wedge nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s)\right\}, \end{aligned}\tag{3.1}$$

holds a.s.

Proof By integration by parts we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} e^{-\theta_1[I_1(s)-I_1(t_1)]-\theta_2[I_2(s)-I_2(t_1)]}[\theta_1 dI_1(s)+\theta_2 dI_2(s)]\\ &\quad=1-e^{-\theta_1[I_1(t_2)-I_1(t_1)]-\theta_2[I_2(t_2)-I_2(t_1)]}, \end{aligned}\tag{3.2}$$

for $0\le t_1<t_2$. For $t_1=nc$ and $t_2=t$ the left side of (3.1) equals

$$\begin{aligned} &1-\int_{nc}^{t} e^{-\theta_1[I_1(s)-I_1(nc)]-\theta_2[I_2(s)-I_2(nc)]}\\ &\quad\times[\theta_1 dI_1(s)+\theta_2 dI_2(s)]. \end{aligned}\tag{3.3}$$

For $nc<t\le nc+r$ this last expression is

$$1-\theta_2\int_{nc}^{t} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s),$$

since $I_1(s)=I_1(nc)$ and $dI_1(s)=0$ for $nc<s\le t$. We thus obtain the desired factorization (3.1) in this case since the second factor on the right side reduces to unity.

For $nc+r<t\le nc+c$ the expression (3.3) is

$$\begin{aligned} &1-\theta_2\int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s)\\ &\quad-\theta_1 e^{-\theta_2[I_2(nc+r)-I_2(nc)]}\int_{nc+r}^{t} e^{-\theta_1[I_1(s)-I_1(nc)]}dI_1(s), \end{aligned}$$

which leads to the desired factorization in this case, since

$$e^{-\theta_2[I_2(nc+r)-I_2(nc)]}=1-\theta_2\int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s),$$

on account of (3.2). The proof is thus completed. □

Theorem 3.2 *For each $t>0$ the random variables $Z_1(t)$ and $Z_2(t)$ are independent. Furthermore,*

$$\mathbf{E}[e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)}]=\mathbf{E}[e^{-\theta_1 Z_1(t)}]\mathbf{E}[e^{-\theta_2 Z_2(t)}],$$

where

$$\begin{aligned} &\mathbf{E}[e^{-\theta_1 Z_1(t)}]\\ &\quad=\begin{cases} \mathbf{E}[e^{-\theta_1 Z_1(nc)}]e^{-(t-nc)\phi_1(\theta_1)}\\ \quad(nc<t\le nc+r),\\ \mathbf{E}[e^{-\theta_1 Z_1(nc+r)}]e^{-(t-nc-r)[\phi_1(\theta_1)-\theta_1]}\\ \quad-\theta_1\int_0^{t-nc-r} e^{-s[\phi_1(\theta_1)-\theta_1]}F_1(0,t-s)ds\\ \quad(nc+r<t\le nc+c), \end{cases} \end{aligned}\tag{3.4}$$

and

$$\begin{aligned} &\mathbf{E}[e^{-\theta_2 Z_2(t)}]\\ &\quad=\begin{cases} \mathbf{E}[e^{-\theta_2 Z_2(nc)}]e^{-(t-nc)[\phi_2(\theta_2)-\theta_2]}\\ \quad-\theta_2\int_0^{t-nc} e^{-s[\phi_2(\theta_2)-\theta_2]}F_2(0,t-s)ds\\ \quad(nc<t\le nc+r),\\ \mathbf{E}[e^{-\theta_2 Z_2(nc+r)}]e^{-(t-nc-r)\phi_2(\theta_2)}\\ \quad(nc+r<t\le nc+c), \end{cases} \end{aligned}\tag{3.5}$$

and $F_i(0,s)=P[Z_i(s)=0]$, $i=1,2$.

Proof For $nc<t\le nc+c$, $(n\ge 0)$ we have a.s.

$$\begin{aligned} &e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)}\\ &\quad=e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)-\theta_1[Z_1(t)-Z_1(nc)]-\theta_2[Z_2(t)-Z_2(nc)]}\\ &\quad=e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)}\\ &\qquad\times e^{-\theta_1[Y_1(t)-Y_1(nc)+I_1(t)-I_1(nc)]}\\ &\qquad\times e^{-\theta_2[Y_2(t)-Y_2(nc)+I_2(t)-I_2(nc)]}\\ &\quad=e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)}e^{-\theta_1[Y_1(t)-Y_1(nc)]-\theta_2[Y_2(t)-Y_2(nc)]}\\ &\qquad\times\left\{1-\theta_2\int_{nc}^{t\wedge nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s)\right\}\\ &\qquad\times\left\{1-\theta_1\int_{nc+r}^{t\vee nc+r} e^{-\theta_1[I_1(s)-I_1(nc)]}dI_1(s)\right\}, \end{aligned}\tag{3.6}$$

where we have used the Lemma 3.1. For $t=nc+c$ it follows from (3.6) that

$$\begin{aligned} &e^{-\theta_1[Z_1(nc+c)-Z_1(nc)]-\theta_2[Z_2(nc+c)-Z_2(nc)]}\\ &\quad=e^{-\theta_1[Y_1(nc+c)-Y_1(nc)]} \end{aligned}$$

$$\times \left\{1-\theta_1\int_{nc+r}^{nc+c} e^{-\theta_1[I_1(s)-I_1(nc+r)]}dI_1(s)\right\}$$
$$\times e^{-\theta_2[Y_2(nc+c)-Y_2(nc)]}$$
$$\times \left\{1-\theta_2\int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}dI_2(s)\right\}. \tag{3.7}$$

Taking expectations in (3.7) we find that the increments

$$Z_1(nc+c)-Z_1(nc), \qquad Z_2(nc+c)-Z_2(nc),$$

are independent. Therefore $Z_1(nc)$ and $Z_2(nc)$ are independent and so are $Z_1(t)$ and $Z_2(t)$ on account of (3.6).

From (3.6) we obtain

$$e^{-\theta_1 Z_1(t)} = e^{-\theta_1 Z_1(nc)}e^{-\theta_1[Y_1(t)-Y_1(nc)]},$$

for $nc<t\le nc+r$ and

$$\begin{aligned} e^{-\theta_1 Z_1(t)} &= e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)]} \\ &\quad -\theta_1\int_{nc+r}^{t} e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)+Z_1(s)-Z_1(nc)-Y_1(s)-Y_1(nc)]} \\ &\quad \times \zeta_1(s)ds \\ &= e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)]} \\ &\quad -\theta_1\int_{nc+r}^{t} e^{-\theta_1[Y_1(t)-Y_1(s)]}\zeta_1(s)ds, \end{aligned}$$

for $nc+r<t\le nc+c$, since $Z_1(s)>0$ if and only if $\zeta_1(s)=1$. Taking expectations in these we arrive at the results of (3.4).

Again from (3.6) we obtain similarly for $nc<t\le nc+r$

$$e^{-\theta_2 Z_2(t)} = e^{-\theta_2[Z_2(nc)+Y_2(t)-Y_2(nc)]} - \theta_2\int_{nc}^{t} e^{-\theta_2[Y_2(t)-Y_2(s)]}\zeta_2(s)ds.$$

Taking expectations we obtain the first result in (3.5). For $nc+r<t\le nc+c$, we have

$$e^{-\theta_2 Z_2(t)} = e^{-\theta_2[Z_2(nc)+Y_2(t)-Y_2(nc)]} - \theta_2\int_{nc}^{nc+r} e^{-\theta_2[Y_2(t)-Y_2(s)]}\zeta_2(s)ds.$$

Taking expectations in this we obtain

$$\begin{aligned} \mathbf{E}[e^{-\theta_2 Z_2(t)}] &= \mathbf{E}[e^{-\theta_2 Z_2(nc)}]e^{-(t-nc-r)\phi_2(\theta_2)-r[\phi_2(\theta_2)-\theta_2]} \\ &\quad -\theta_2 e^{-(t-nc-r)\phi_2(\theta_2)} \\ &\quad \times\int_0^r e^{-s[\phi_2(\theta_2)-\theta_2]}F_2(0,nc+r-s)ds, \end{aligned}$$

since $Y_2(t)-Y_2(s) \stackrel{d}{=} X_2(t-nc-r)+X_2(nc+r-s)-(nc+r-s)$ when $nc<s\le nc+r$. This agrees with (3.5). □

4 Limit distributions

We are now in a position to derive the stationary distribution of $\{Z_1(t),Z_2(t)\}$. Our plan is to first consider the embedded sequence $\{Z_1(nc),Z_2(nc)\}$ and then extend the results to $\{Z_1(t),Z_2(t)\}$ using Theorem 3.2. We first note that according to the model description

$$\begin{aligned} Z_1(nc+r) &= Z_1(nc)+X_1(nc+r)-X_1(nc) \\ &\stackrel{d}{=} Z_1(nc)+X_1(r), \\ Z_2(nc+c) &= Z_2(nc+r)+X_2(nc+c)-X_2(nc+r) \\ &\stackrel{d}{=} Z_2(nc+r)+X_2(g). \end{aligned}$$

Thus it is more convenient to express the results for $\{Z_1(nc),Z_2(nc)\}$ equivalently in terms of $\{Z_1(nc), Z_2(nc+r)\}$.

We have the following results.

Theorem 4.1 *As $n\to\infty$, $\{Z_1(nc),Z_2(nc+r)\}\stackrel{d}{\to}\{Z_1(\infty), Z_{2r}(\infty)\}$, where $Z_1(\infty)$, $Z_{2r}(\infty)$ are finite if and only if $\rho_1<\frac{g}{c}<1-\rho_2$, in which case*

$$\mathbf{E}[e^{-\theta_1 Z_1(\infty)-\theta_2 Z_{2r}(\infty)}] = \mathbf{E}[e^{-\theta_1 Z_1(\infty)}]\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}],$$

where

$$\mathbf{E}[e^{-\theta_1 Z_1(\infty)}] = \frac{F_1(0)\theta_1}{\theta_1-\phi_1(\theta_1)}\frac{1-e^{-g[\phi_1(\theta_1)-\theta_1]}}{1-e^{-c\phi_1(\theta_1)+g\theta_1}}, \tag{4.1}$$

$$\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}] = \frac{F_2(0)\theta_2}{\theta_2-\phi_2(\theta_2)}\frac{1-e^{-r[\phi_2(\theta_2)-\theta_2]}}{1-e^{-c\phi_2(\theta_2)+r\theta_2}}, \tag{4.2}$$

and

$$F_1(0)=1-\frac{c}{g}\rho_1, \qquad F_2(0)=1-\frac{c}{r}\rho_2.$$

Proof The independence of $Z_1(\infty)$ and $Z_{2r}(\infty)$ follows from Theorem 3.2 and the remarks at the beginning of this section.

From Theorem 3.2 we obtain

$$\begin{aligned} &\mathbf{E}[e^{-\theta_1 Z_1(nc+c)}] \\ &\quad = \mathbf{E}[e^{-\theta_1 Z_1(nc)}]e^{-c\phi_1(\theta_1)+g\theta_1} \\ &\qquad -\theta_1\int_0^g e^{-s[\phi_1(\theta_1)-\theta_1]}F_1(0,nc+c-s)ds, \end{aligned}$$

and for $0<\alpha<1$

$$\sum_{n=0}^{\infty}\alpha^n\mathbf{E}[e^{-\theta_1 Z_1(nc)}] = \frac{e^{-\theta_1 Z_1(0)}-\theta_1\int_0^g e^{-s[\phi_1(\theta_1)-\theta_1]}J_1(s,\alpha)ds}{1-\alpha e^{-c\phi_1(\theta_1)+g\theta_1}}, \tag{4.3}$$

where $J_1(s,\alpha)=\sum_{n=0}^{\infty}\alpha^{n+1}F_1(0,nc+c-s)$.

Denote $\alpha^{-1} = e^{c\beta}$, $\beta > 0$. Then the denominator in (4.3) vanishes at η_1 where $\eta_1 \equiv \eta_1(\beta)$ is the unique positive root of the equation

$$\frac{g}{c}\eta_1 = \beta + \phi_1(\eta_1),$$

with $\eta_1(\infty) = \infty$. Since the left side of (4.3) is a bounded analytic function of θ_1 we must have

$$\int_0^g e^{-s[\phi_1(\eta_1)-\eta_1]} J_1(s,\alpha)ds = \frac{e^{-\eta_1 Z_1(0)}}{\eta_1}. \tag{4.4}$$

Here (4.4) is an integral equation of J_1. We shall not solve for J_1 explicitly since we are interested only in the limit behavior of $Z_1(nc)$. Proceeding as in the proof of Lemma 2 of [6, p. 114], we find that

(i) as $\beta \to 0+$, $\eta_1(\beta) \to \eta_1(0)$, where $\eta_1(0)$ is the least positive root of the equation

$$\frac{g}{c}\eta_1(0) = \phi_1(\eta_1(0)),$$

and $\eta_1(0) > 0$ if and only if $\rho_1 > g/c$,

(ii)

$$\eta_1'(0+) = \begin{cases} \left(\frac{g}{c} - \rho_1\right)^{-1} & \text{if } \rho_1 < g/c, \\ \infty & \text{if } \rho_1 = g/c. \end{cases}$$

Using these results in (4.4) we obtain

$$\lim_{\alpha\to 1-} (1-\alpha) \int_0^g e^{-s[\phi_1(\eta_1)-\eta_1]} J_1(s,\alpha)ds$$
$$= \lim_{\beta\to 0+} (1-e^{-c\beta}) \frac{e^{-\eta_1 Z_1(0)}}{\eta_1},$$

which reduces to

$$\int_0^g e^{-s[\phi_1(\eta_1(0))-\eta_1(0)]} F_1(0)ds$$
$$= \begin{cases} c\left(\frac{g}{c} - \rho_1\right) & \text{if } \rho_1 < g/c, \\ 0 & \text{if } \rho_1 \ge g/c\ . \end{cases}$$

We conclude that

$$F_1(0) = \begin{cases} 1 - \frac{c}{g}\rho_1 & \text{if } \rho_1 < g/c, \\ 0 & \text{if } \rho_1 \ge g/c. \end{cases}$$

Collecting all these results and using them in (4.3) we arrive at the desired result (4.1).

To prove (4.2) we note from Lemma 2.1 that the permutation

$$(r, g, X_1, X_2) \to (g, r, X_2, X_1),$$

results in the permutation

$$\{Z_1(nc), Z_2(nc+r)\} \to \{Z_2(nc+r), Z_1(nc)\},$$

and in the limit as $n \to \infty$

$$\{Z_1(\infty), Z_{2r}(\infty)\} \to \{Z_{2r}(\infty), Z_1(\infty)\}.$$

Thus (4.1) leads to (4.2). □

Remark 4.2 From (2.5) we see that $\{Z_1(nc), Z_2(nc+r), n = 0, 1, \ldots\}$ is a Markov process. Therefore the limit distribution given by Theorem 4.1 provides a stationary distribution for this process. That is, if we choose

$$\{Z_1(0), Z_2(r)\} \stackrel{d}{=} \{Z_1(\infty), Z_{2r}(\infty)\},$$

then for every $n \ge 0$

$$\{Z_1(nc), Z_2(nc+r)\} \stackrel{d}{=} \{Z_1(\infty), Z_{2r}(\infty)\}.$$

5 Dependence under stationarity

Having in mind the limit behaviour of the process $\{Z_1(t), Z_2(t)\}$, we recall some facts from the theory of the alternating renewal process.

Let $\{X_k, k \ge 1\}$ and $\{Y_k, k \ge 1\}$ be independent sequences of i.i.d. non-negative random variables with

$$P[X_k \le x] = F(x), \qquad P[Y_k \le x] = G(x),$$

and

$$\mathbf{E}[X_k] = \mu_1, \qquad \mathbf{E}[Y_k] = \mu_2,$$

for $0 < \mu_1 \le \infty$, $0 < \mu_2 \le \infty$. Denote $S_0 = 0$,

$$S_n = \sum_{k=1}^{n} (X_k + Y_k),$$

for $n \ge 1$. Then $\{S_n, n \ge 0\}$ is a renewal process with lifespan distribution H, where

$$H(z) = P[X_k + Y_k \le z],$$

for $k \ge 1$. Let $N(t) = \max\{n : S_n \le t\}$ and $\mathbf{E}N(t) = \sum_{n=0}^{\infty} P[S_n \le t] < \infty$. We are concerned with the two probabilities

$$Z_x(t) = P[S_{N(t)} < t \le S_{N(t)} + X_{N(t)+1}, t - S_{N(t)} \le x],$$
$$Z_y(t) = P[S_{N(t)} + X_{N(t)+1} < t \le S_{N(t)+1},$$
$$t - S_{N(t)} - X_{N(t)+1} \le y].$$

The following results describe the behavior of these at $t \to \infty$.

Theorem 5.1 (Feller [1]) *If F or G is non-arithmetic*:

(a) *Then we have*

$$\lim_{t\to\infty} Z_x(t) = \frac{1}{\mu_1+\mu_2}\int_0^x [1-F(s)]ds,$$

$$\lim_{t\to\infty} Z_y(t) = \frac{1}{\mu_1+\mu_2}\int_0^y [1-G(s)]ds,$$

the limit in each case being zero if μ_1 *or* μ_2 *is infinite.*

(b) *And if at least one of* μ_1 *or* μ_2 *is finite, then*

$$\lim_{t\to\infty} P[S_{N(t)} < t \le S_{N(t)} + X_{N(t)+1}] = \frac{\mu_1}{\mu_1+\mu_2},$$

$$\lim_{t\to\infty} P[S_{N(t)} + X_{N(t)+1} < t \le S_{N(t)+1}] = \frac{\mu_2}{\mu_1+\mu_2},$$

the limits being interpreted as 1 *and* 0 *if* $\mu_1 = \infty$ *and* 0 *and* 1 *if* $\mu_2 = \infty$.

The sequence of red and green periods represents an alternating renewal process with lifespans $X_k = r$ and $Y_k = g$ with probability one. For this we have $S_n = nc$, $n \ge 0$ and let $N \equiv N(t) = [t/c]$ be the integer part of the fraction t/c. We see that

$$Z_x(t) = P[Nc < t \le Nc + r, t - Nc \le x],$$

$$Z_y(t) = P[Nc + r < t \le Nc + c, t - Nc - r \le y].$$

Since

$$F(x) = \begin{cases} 0 & \text{for } 0 < x < r, \\ 1 & \text{for } x \ge r, \end{cases}$$

we have $r^{-1}[1 - F(x)] = r^{-1}$ for $0 < x < r$, which is the uniform density in $(0, r)$. Therefore

$$\lim_{t\to\infty} Z_x(t) = \begin{cases} \dfrac{x}{c} & \text{for } 0 < x < r, \\ \dfrac{r}{c} & \text{for } x \ge r. \end{cases}$$

Similar results hold for $Z_y(t)$.

Remark 5.2 When r and g are constant, the two processes $\{Z_1(t), t \ge 0\}$ and $\{Z_2(t), t \ge 0\}$ are periodic and consequently their limit distributions do not exist. Therefore we consider their stationary distributions. Let T be a random variable representing the first switching point, with

$$T \overset{d}{=} \begin{cases} U_r & \text{with probability } \dfrac{r}{c}, \\ U_g & \text{with probability } \dfrac{g}{c}, \end{cases} \tag{5.1}$$

where for any $a > 0$, U_a is a random variable having uniform density in $(0, a)$. We note that $t \to \infty$ through red and green periods. To be specific, let $T + (Nc, Nc + r]$ be the red period and $T + (Nc + r, Nc + c]$ be the green period that correspond to the given t. Clearly, $N \sim t/c$, as $t \to \infty$. It turns out that $t - Nc$ (under 't red') and $t - Nc - r$ (under 't green') both have stationary distributions. In fact

$$t - T - Nc \overset{d}{=} U_r, \qquad t - T - Nc - r \overset{d}{=} U_g. \tag{5.2}$$

In particular, in stationary regime the probability of 't red' is r/c and the probability of 't green' is g/c.

Further, for the net inputs over $T + (Nc, Nc + r]$ and $T + (Nc + r, Nc + c]$ we see $\{Y_1(t), Y_2(t)\} \overset{d}{=} (W_1, W_2)$, where

$$(W_1, W_2) = \begin{cases} \{X_1(U_r), X_2(U_r) - U_r + X_2(g)\} \\ \quad \text{with probability } \dfrac{r}{c}, \\ \{X_1(U_g) - U_g + X_1(r), X_2(U_g)\} \\ \quad \text{with probability } \dfrac{g}{c}. \end{cases} \tag{5.3}$$

Remark 5.3 If we choose $\{Z_1(0), Z_2(0)\} \overset{d}{=} \{Z_1(\infty), Z_{2r}(\infty)\}$ it is easily seen from Remarks 4.2 and 5.2 that for $t \in (0, T)$ the distribution of $\{Z_1(t), Z_2(t), t \ge 0\}$ is independent of t, with net inputs given by (W_1, W_2).

We now consider a new setup in which the initial state of the process $\{Z_1(t), Z_2(t), t \ge 0\}$ has the stationary distribution described above. It follows that for every finite t, the process has the same distribution.

Theorem 5.4 *If* $\{Z_1(0), Z_2(0)\} \overset{d}{=} \{Z_1(\infty), Z_{2r}(\infty)\}$, *then the stationary distribution of the process* $\{Z_1(t), Z_2(t)\} \overset{d}{=} \{Z_1, Z_2\}$, *satisfies*

$$\begin{aligned} &\mathbf{E}(e^{-\theta_1 Z_1 - \theta_2 Z_2}) \\ &\quad = \mathbf{E}[e^{-\theta_1 Z_1(\infty) - \theta_2 Z_{2r}(\infty)}]\mathbf{E}[e^{-\theta_1 W_1 - \theta_2 W_2}] \\ &\qquad - \frac{g}{c}\theta_1 F_1(0)\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}] \\ &\qquad \times \mathbf{E}\left[e^{-U_g\phi_2(\theta_2)}\int_0^{U_g} e^{-s(\phi_1(\theta_1)-\theta_1)}ds\right] \\ &\qquad - \frac{r}{c}\theta_2 F_2(0)\mathbf{E}[e^{-\theta_1 Z_1(\infty)}] \\ &\qquad \times \mathbf{E}\left[e^{-U_r\phi_1(\theta_1)}\int_0^{U_r} e^{-s(\phi_2(\theta_2)-\theta_2)}ds\right], \end{aligned} \tag{5.4}$$

where the random vector (W_1, W_2) *is defined by relation* (5.3).

Proof To be specific, let $T + (Nc, Nc + r]$ be the red period and $T + (Nc + r, Nc + c]$ is the green period that correspond to the given t. The other case comes analogously. From Theorem 3.2 we find that

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)}) &= \mathbf{E}(e^{-\theta_1 Z_1(T+Nc)-\theta_2 Z_2(T+Nc)}) \\ &\quad\times e^{-(t-T-Nc)[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]} \\ &\quad-\theta_2 \mathbf{E}(e^{-\theta_1 Z_1(T+Nc)})e^{-(t-T-Nc)\phi_1(\theta_1)} \\ &\quad\times\int_0^{t-T-Nc} e^{-s[\phi_2(\theta_2)-\theta_2]}F_2(0,t-s)ds,\end{aligned}$$

for $T+Nc<t\le T+Nc+r$ and

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)}) &= \mathbf{E}(e^{-\theta_1 Z_1(T+Nc+r)-\theta_2 Z_2(T+Nc+r)}) \\ &\quad\times e^{-(t-T-Nc-r)[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]} \\ &\quad-\theta_1 \mathbf{E}(e^{-\theta_2 Z_2(T+Nc+r)})e^{-(t-T-Nc-r)\phi_2(\theta_2)} \\ &\quad\times\int_0^{t-T-Nc-r} e^{-s[\phi_1(\theta_1)-\theta_1]}F_1(0,t-s)ds,\end{aligned}$$

for $T+Nc+r<t\le T+Nc+c$. Letting $t\to\infty$ and using (5.2)

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 Z_1-\theta_2 Z_2}) &= \mathbf{E}(e^{-\theta_1 Z_1(\infty)-\theta_2 Z_2(\infty)})\mathbf{E}e^{-U_r[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]} \\ &\quad-\theta_2 F_2(0)\mathbf{E}(e^{-\theta_1 Z_1(\infty)})\mathbf{E}e^{-U_r\phi_1(\theta_1)} \\ &\quad\times\int_0^{U_r} e^{-s[\phi_2(\theta_2)-\theta_2]}ds,\end{aligned}\tag{5.5}$$

with probability r/c and

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 Z_1-\theta_2 Z_2}) &= \mathbf{E}(e^{-\theta_1 Z_{1r}(\infty)-\theta_2 Z_{2r}(\infty)})\mathbf{E}e^{-U_g[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]} \\ &\quad-\theta_1 F_1(0)\mathbf{E}(e^{-\theta_2 Z_{2r}(\infty)})\mathbf{E}e^{-U_g\phi_2(\theta_2)} \\ &\quad\times\int_0^{U_g} e^{-s[\phi_1(\theta_1)-\theta_1]}ds,\end{aligned}\tag{5.6}$$

with probability g/c. Adding the limit results (5.5) and (5.6) with appropriate weights we arrive at the result (5.4) since

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 W_1-\theta_2 W_2}) &= \frac{r}{c}\mathbf{E}[e^{-\theta_1 X_1(U_r)-\theta_2 X_2(U_r)+\theta_2 U_r-\theta_2 X_2(g)}] \\ &\quad+\frac{g}{c}\mathbf{E}[e^{-\theta_1 X_1(r)-\theta_1 X_1(U_g)+\theta_1 U_g-\theta_2 X_2(U_g)}] \\ &= \frac{r}{c}\mathbf{E}[e^{-U_r[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]}]e^{-g\phi_2(\theta_2)} \\ &\quad+\frac{g}{c}\mathbf{E}[e^{-U_g[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]}]e^{-r\phi_1(\theta_1)}.\end{aligned}$$ □

Letting $\theta_2\to 0+$ and $\theta_1\to 0+$ respectively in (5.4) we obtain the following for the marginal distributions of Z_1 and Z_2.

Corollary 5.5 (i) *We have*

$$\begin{aligned}\mathbf{E}(e^{-\theta_1 Z_1}) &= \mathbf{E}(e^{-\theta_1 Z_1(\infty)})\mathbf{E}[e^{-\theta_1 W_1}] \\ &\quad-\frac{g}{c}\theta_1 F_1(0)\mathbf{E}\int_0^{U_g} e^{-s[\phi_1(\theta_1)-\theta_1]}ds,\end{aligned}\tag{5.7}$$

and

$$\begin{aligned}\mathbf{E}(e^{-\theta_2 Z_2}) &= \mathbf{E}(e^{-\theta_1 Z_{2r}(\infty)})\mathbf{E}[e^{-\theta_2 W_2}] \\ &\quad-\frac{r}{c}\theta_2 F_2(0)\mathbf{E}\int_0^{U_r} e^{-s[\phi_2(\theta_2)-\theta_2]}ds.\end{aligned}\tag{5.8}$$

(ii) *Assume that* $\operatorname{var}[X_i(1)]=\sigma_i^2<\infty$, $i=1,2$. *Then*

$$\mathbf{E}[Z_1]=\frac{\frac{c}{g}\sigma_1^2}{2(1-\frac{c}{g}\rho_1)},\qquad \mathbf{E}[Z_2]=\frac{\frac{c}{r}\sigma_2^2}{2(1-\frac{c}{r}\rho_2)},\tag{5.9}$$

and

$$\operatorname{cov}[Z_1,Z_2]=-\frac{1}{12}(r\rho_1)(g\rho_2)<0.\tag{5.10}$$

Proof The results (5.7) and (5.8) follow easily from (5.4). The result (5.9) follow from (4.1), (4.2), (5.7) and (5.8) by differentiation. Also, from (5.4) we obtain by differentiation

$$\mathbf{E}[Z_1 Z_2]=\mathbf{E}[Z_1]\mathbf{E}[Z_2]-\frac{1}{12}(r\rho_1)(g\rho_2).$$ □

Remark 5.6 From Lemma 2.1 we find that the permutation (2.7) results in the permutation

$$\{W_1,W_2\}\to\{W_2,W_1\},$$

as can be seen from (5.3). Theorem 5.4 shows that

$$\{Z_1,Z_2\}\to\{Z_2,Z_1\}.$$

Remark 5.7 The result (5.10) indicates the dependence of the random variables Z_1 and Z_2. This dependence is not surprising. In the equations (2.3) formulating the processes $Z_i(t)$, $(i=1,2)$ the inputs $X_1(t)$ and $X_2(t)$ are independent, so the components $Z_1(t)$ and $Z_2(t)$ are independent, as was claimed in Theorem 3.2. However, in stationary regime $X_1(t)$ and $X_2(t)$ have to be considered over the intervals of random lengths U_r and U_g. This is evident from the proof of Theorem 5.4, where the random vector (W_1,W_2) is defined in terms of $\{X_1(U_r),X_2(U_r)\}$ and $\{X_1(U_g),X_2(U_g)\}$ (see (5.3)).

However, if we consider the lengths r and g as non-degenerate random variables, then this contradiction of independence at finite times t but dependence in the stationary regime does not occur any more.

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