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RELIABILITY ESTIMATION OF A COMPLEX RENEWABLE SYSTEM WITH AN UNBOUNDED NUMBER OF REPAIR UNITS

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Abstract

In this study an asymptotical analysis of the reliability of a complex renewable system with an unbounded number of repair units is provided. The system state is given by a binary vector $e(t) = [e_1(t), \cdots, e_n(t)]$, $e_i(t) = 0(1)$, if at moment t the ith element is failure-free (failed). We assume that at the state e the ith element has failure intensity $\lambda_i(e)$. At the instant of failure of every element the renewal work begins and the renewal time has distribution function $G_i(t)$. Let E_- be the set of failed system states. The goal of this study is the asymptotic estimation of the distribution of the time until the first system failure, $\tau = \inf\{t : e(t) \in E_- \mid e(0) = \bar{0}\}$.

RELIABILITY ASYMPTOTIC ANALYSIS; REGENERATIVE PROCESS

1. System description and problem statement

We examine a system of n elements, which can be in failure-free or failure state. The system state is given through the element states as a binary vector

$$e(t) = [e_1(t), e_2(t), \cdots, e_n(t)],$$

where $e_i(t) = 0$, if the ith element at moment t is failure-free and $e_i(t) = 1$ in the opposite case. The set $E = \{e\}$ of the system states is divided into two subsets $E = E_+ \cup E_-$, where E_+ is the subset of the failure-free states and E_- is the subset of the failure states. We assume that this division introduces a monotonic (coherent) structure (see Barlow and Proschan (1975)). The failure intensity of the ith element depends only on the other element states and is denoted by $\lambda_i(e) > 0$, $i = 1, \cdots, n$. Every element after its failure enters the repair procedure immediately. The number of the repair units is unbounded. We denote by η the repair time of the ith element, and $G_i(t) = P\{\eta_i < t\}$, $i = 1, \cdots, n$. When the repair is completed the element comes back to its place at once.

The random process $e(t)$ so created describes the behaviour of the complex renewable system. Let $\lambda(e) = \Sigma_{i=1}^{n} \lambda_i(e)$. This process represents a regenerative process of special

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type (see Barzilovich et al. (1983), p. 52), whose free period, when all the elements are operational, has an exponential distribution with parameter $\lambda(\bar{0})$ and the busy period has a certain distribution $\phi(x)$.

The main reliability characteristic of such a system represents the time until the first system failure $\tau = \inf\{t : e(t) \in E_- \mid e(0) = \bar{0}\}$. The distribution of τ, in general, cannot be found in a closed form. The estimation of this distribution by simulation modelling is almost impossible to determine, because, in the case of highly-reliable systems, the process $e(t)$ changes its state many times before the system ultimately fails and the simulation demands a very long period of computer time. However, based on the fact that the repair time of the elements, in most real cases, is many times less than the lifetime of the elements, we can use asymptotic methods for the reliability estimation. Barzilovich et al. ((1983), p. 107) provide the following theorem. Let $\bar{\lambda} = \max_{e \in E_+} \lambda(e)$, $G(x) = \min_i G_i(x)$, $T = \int_0^\infty x dG(x)$ and q be the probability of the system failure during a regenerative period.

Theorem 1. If $\lambda_i(e)$, $G_i(t)$, $i = 1, \cdots, n$ and n change so that $\bar{\lambda} T \to 0$, then $P\{\lambda(\bar{0}) q \tau > x\} \to e^{-x}$.

(For a complete proof see Gnedenko and Solovyev (1975).)

In order to apply this theorem effectively for reliability estimations, it is necessary to estimate the probability q, which in the majority of real systems cannot be found in a closed form. An estimation of this kind was provided in Gnedenko and Solovyev (1975), under the assumptions that the $G_i(t)$ and n are fixed, there is the moment $\int_0^\infty x^{m+1} dG(x) < \infty$, where $m = \max_{e \in E_+} \Sigma_{i=1}^n e_i$ and the intensities $\lambda_i(e)$ have the form $\lambda_i(e) = \lambda_i^0(e)\varepsilon$, where $\lambda_i^0(e)$ are fixed and $\varepsilon \to 0$. In fact these conditions mean that all the element reliabilities are of the same order and all the lifetimes of the elements are also of the same order. In most real cases, this condition does not apply, because the failure intensities $\lambda_i(e)$ and the lifetimes η_i of different elements differ widely. In this case, the estimation of q provided in this paper is not applicable.

The goal of the present paper is the estimation of the probability q under certain assumptions when, by the limit procedure, all the intensities $\lambda_i(e)$ and distributions $G_i(t)$ change in an arbitrary way.

We introduce certain concepts and notations. We call the path π a sequence of system states that the process $e(t)$ visits from the beginning of the busy period until the moment of the system failure in this period. That is, $\pi = [\bar{0} = e^{(0)}, e^{(1)}, \cdots, e^{(m+1)}]$, where $e^{(i)} \in E_+ \setminus \{\bar{0}\}$, $i = 1, \cdots, m$, $e^{(m+1)} \in E_-$. The set of all possible paths is denoted by $\Pi = \{\pi\}$. A path π is called monotonic if $e^{(i)} < e^{(i+1)}$, $i = 0, \cdots, m$ (here the order is componentwise). The condition of monotonicity means that from the beginning of the busy period and until the moment of the system failure, none of the failed elements has already been repaired. The class of monotonic paths is denoted by Π_0, and $\Pi_1 = \Pi \setminus \Pi_0$ is the class of non-monotonic paths. A failure of an element in a path π is called essential if it has not already been repaired before the system fails. Obviously, in monotonic paths all the failures are essential.

Let a busy period begin at moment 0 and let the process $e(t)$ pass through the path $\pi_1 \in \Pi_1$. The moments of the essential failures are denoted by $x_1, x_2, \cdots, x_{m+1}$ $(0 \leqq x_1 < x_2 < \cdots < x_{m+1})$. At the moment x_{m+1} a system failure takes place. From the relation $\pi_1 \in \Pi_1$, it follows that in the interval $(0, x_{m+1})$ (at least) one non-essential failure appears.

For every $\pi_1 \in \Pi_1$ there is a corresponding monotonic path π_0, whose ordered sequence of numbers of the failed elements coincides with the sequence of the numbers of the essential failures. Conversely, for every $\pi_0 \in \Pi_0$ there is a corresponding class of non-monotonic paths $\Pi_1(\pi_0) \subseteq \Pi_1$ whose sequence of numbers of the essential failures forms the monotonic path π_0.

We take only the non-essential failures from a path $\pi_1 \in \Pi_1$. They form certain (at least one) busy periods, which terminate before the system failure at moment x_{m+1}.

2. Estimation of the probability q

Let $q(\pi)$ be the probability that a system failure appears in a busy period, specifically through the path π. Then

$$q = \sum_{\pi \in \Pi} q(\pi) = \sum_{\pi_0 \in \Pi_0} q(\pi_0) + \sum_{\pi_1 \in \Pi_1} q(\pi_1) \equiv q_0 + q_1.$$

The probability q_0 can be found as in Gnedenko and Solovyev (1975), p. 92:

$$q(\pi_0) = \frac{\Lambda(\pi_0)}{\lambda(\bar{0})} \int_{\Delta_m} \cdots \int \exp(-s(\pi_0)) \bar{G}_{i_1}(u_1) \cdots \bar{G}_{i_m}(u_m) du_1 \cdots du_m, \tag{1}$$

where

$$\Lambda(\pi_0) = \lambda_{i_1}(\bar{0}) \lambda_{i_2}(e^{(1)}) \cdots \lambda_{i_{m+1}}(e^{(m)}), \quad u_k = x_{m+1} - x_k, \qquad k = 1, \cdots, m,$$

$$s(\pi_0) = \lambda(e^{(1)})(u_1 - u_2) + \lambda(e^{(2)})(u_2 - u_3) + \cdots + \lambda(e^{(m)}) u_m,$$

$$\Delta_m = \{(u_1, \cdots, u_m) : u_1 > u_2 > \cdots > u_m\}, \qquad \bar{G}(u) = 1 - G(u).$$

Since the number of monotonic paths is finite, the probability q_0 is expressed as a finite sum of integrals like (1): $q_0 = \Sigma_{\pi_0 \in \Pi_0} q(\pi_0)$.

In its turn the probability $q(\pi_0)$ can be estimated by a simpler expression. We denote

$$\tilde{q}(\pi_0) = \frac{\Lambda(\pi_0)}{\lambda(\bar{0})} \int_{\Delta_m} \cdots \int \bar{G}_{i_1}(u_1) \cdots \bar{G}_{i_m}(u_m) du_1 \cdots du_m,$$

and correspondingly $\tilde{q}_0 = \Sigma_{\pi_0 \in \Pi_0} \tilde{q}(\pi_0)$.

First we show the equivalence between q_0 and $\tilde{q}_0$.

Lemma 1. If $\lambda_i(e)$ and $G_i(t)$ change so that

$$\frac{\bar{\lambda}_i}{\underline{\lambda}_i} \leqq c < \infty, \quad \frac{1}{T_i} \int_0^\infty (1 - \exp(-\bar{\lambda} x)) \bar{G}_i(x) dx \to 0, \qquad i = 1, \cdots, n, \tag{2}$$

where $\bar{\lambda}_i = \max_{e\in E_+} \lambda_i(e)$, $\lambda_i = \min_{e\in E_+, e_i=0} \lambda_i(e)$, $T_i = \int_0^\infty x dG_i(x)$, then $q_0 \approx \bar{q}_0$.

Proof. Let e be a boundary failed state, in which the process $e(t)$ falls in following the path π_0. If, through this, failed elements correspond to the numbers $i_1, i_2, \cdots, i_{m+1}$, then in the state e the units lie in the positions with numbers $i_1, i_2, \cdots, i_{m+1}$. We consider all the monotonic paths that end at the state e, the last failure having number i_{m+1}. They can be taken by permutation of the indexes $(i_1, \cdots, i_m)$. The class of these paths is denoted by $\Pi_0(e, i_{m+1})$. Correspondingly,

$$q_0(e, i_{m+1}) = \sum_{\pi_0 \in \Pi_0(e, m+1)} q(\pi_0), \quad \bar{q}_0(e, i_{m+1}) = \sum_{\pi_0 \in \Pi_0(e, m+1)} \bar{q}(\pi_0).$$

We estimate the quantity

$$0 \leqq \frac{\bar{q}_0(e, i_{m+1}) - q_0(e, i_{m+1})}{\bar{q}_0(e, i_{m+1})}$$

$$\leqq c^{m+1} \frac{\sum \int \cdots \int_{\Delta_m} (1 - \exp(-\bar{\lambda} u_1)) \prod_{k=1}^{m} \bar{G}_{s_k}(u_k) du_k}{\sum \int \cdots \int_{\Delta_m} \prod_{k=1}^{m} \bar{G}_{s_k}(u_k) du_k} \equiv B,$$

where the sums in the numerator and denominator are taken over all the permutations $(s_1, s_2, \cdots, s_m)$ of the indexes $(i_1, i_2, \cdots, i_m)$. Performing in every integral the reverse permutation of the integral variables, we take

$$B = c^{m+1} \frac{\int_0^\infty \cdots \int_0^\infty (1 - \exp(-\bar{\lambda} u_{\max})) \prod_{k=1}^{m} \bar{G}_{i_k}(u_k) du_k}{\int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{m} \bar{G}_{i_k}(u_k) du_k},$$

where $u_{\max} = \max_k u_k$. But $1 - \exp(-\bar{\lambda} u_{\max}) \leqq \sum_{k=1}^{m} (1 - \exp(-\bar{\lambda} u_k))$, so

$$B \leqq c^{m+1} \sum_{k=1}^{m} \frac{1}{T_{i_k}} \int_0^\infty (1 - \exp(-\bar{\lambda} u_k)) \bar{G}_{i_k}(u_k) du_k \to 0.$$

Hence, under the conditions (2), $q_0(e, i_{m+1}) \approx \bar{q}_0(e, i_{m+1})$ and so

$$q_0 = \sum_{e, i_{m+1}} q_0(e, i_{m+1}) \approx \sum_{e, i_{m+1}} \bar{q}_0(e, i_{m+1}) = \bar{q}_0,$$

where the sums are taken at first over all the numbers i_{m+1} of the failed elements in the state e and then over all the boundary failed states e. The lemma is proved.

Further, we show that $q_1 = o(q_0) = o(\bar{q}_0)$ applies under certain conditions. For this, we introduce some concepts in order to estimate the probability q_1. Let π_0 be a monotonic path. We have already introdued above the class of the non-monotonic paths $\Pi_1(\pi_0)$. We divide this class into two subclasses: $\Pi_1(\pi_0) = \Pi_1'(\pi_0) \cup \Pi_1''(\pi_0)$, where $\Pi_1'(\pi_0)$ is the class of the non-monotonic paths, whose first essential failure occurs at the instant of the beginning of the busy period, and $\Pi''_1(\pi_0) = \Pi_1(\pi_0) \setminus \Pi'_1(\pi_0)$. Correspondingly the class of the non-monotonic paths Π_1 is divided into two subclasses $\Pi_1 = \Pi'_1 \cup \Pi''_1$, $\Pi'_1 = \bigcup_{\pi_0 \in \Pi_0} \Pi'_1(\pi_0)$, $\Pi''_1 = \bigcup_{\pi_0 \in \Pi_0} \Pi''_1(\pi_0)$. We introduce the corresponding notation for the probabilities

$$q'_1(\pi_0) = \sum_{\pi_1 \in \Pi'_1(\pi_0)} q(\pi_1), \qquad q''_1(\pi_0) = \sum_{\pi_1 \in \Pi''_1(\pi_0)} q(\pi_1),$$

$$q'_1 = \sum_{\pi_0 \in \Pi_0} q'_1(\pi_0), \qquad q''_1 = \sum_{\pi_0 \in \Pi_0} q''_1(\pi_0), \qquad q_1 = q'_1 + q''_1.$$

Lemma 2. If the conditions (2) are fulfilled, then

$$q'_1 = o(q_0). \tag{3}$$

Proof. At moments $x_1 = 0 < x_2 < \cdots < x_{m+1}$ essential failures occur. For the sake of definiteness and without loss of generality, let $1, 2, \cdots, m+1$ be the numbers of the elements of these failures. Obviously, the probability $q'_1(\pi_0)$ is less than the probability of the following event. In a busy period failures of the elements with numbers $1, 2, \cdots, m+1$ occurred (the first of them at the opening of the busy period) and in the interval between the first and last failure more (at least one) failures occurred. The probability of the last event can increase only if we change all the intensities $\lambda_i(e)$ to $\bar{\lambda}_i$ and, apart from that, require that every failed element is immediately replaced by a new one. Then

$$q'_1(\pi_0) \leqq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \int \cdots \int_{\Delta_m} (1 - \exp(-\bar{\lambda} x_{m+1})) \bar{G}_1(x_{m+1}) \bar{G}_2(x_{m+1} - x_2) \cdots \bar{G}_m(x_{m+1} - x_m) dx_2 \cdots dx_{m+1},$$

where

$$\bar{\Lambda}(\pi_0) = \bar{\lambda}_1 \bar{\lambda}_2 \cdots \bar{\lambda}_{m+1}, \qquad \bar{\Delta}_m = \{(x_2, \cdots, x_{m+1}) : 0 < x_2 < x_3 < \cdots < x_{m+1}\}.$$

After changing the variables $u_1 = x_{m+1}$, $u_k = x_{m+1} - x_k$, $k = 2, \cdots, m$ we have

$$q'_1(\pi_0) \leqq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \int \cdots \int_{\Delta_m} (1 - \exp(-\bar{\lambda} u_1)) \bar{G}_1(u_1) \cdots \bar{G}_m(u_m) du_1 \cdots du_m.$$

As in Lemma 1, we now sum the last inequality over all the monotonic paths π_0, which are taken from the permutations of the numbers $1, 2, \cdots, m$ with fixed last number $m+1$. Then using the notation of Lemma 1 we have:

$$\sum_{\pi_0 \in \Pi_0(e, m+1)} q'_1(\pi_0) \equiv q'_1(e, m+1)$$

$$\leqq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \int_0^\infty \cdots \int_0^\infty (1 - \exp(-\bar{\lambda} u_{\max})) \bar{G}_1(u_1) \cdots \bar{G}_m(u_m) du_1, \cdots du_m.$$

On the other hand, as was proved in Lemma 1, $q_0(e, m+1) \approx \bar{q}_0(e, m+1)$ and

$$\frac{q_1'(e, m+1)}{q_0(e, m+1)}$$
$$\approx \frac{q_1'(e, m+1)}{\bar{q}_0(e, m+1)}$$
$$\leq \frac{\bar{\Lambda}(\pi_0)}{\Lambda(\pi_0)} \frac{\int_0^\infty \cdots \int_0^\infty (1-\exp(-\bar{\lambda} u_{\max}))\bar{G}_1(u_1)\cdots \bar{G}_m(u_m)du_1\cdots du_m}{\int_0^\infty \cdots \int_0^\infty \bar{G}_1(u_1)\cdots \bar{G}_m(u_m)du_1\cdots du_m}$$
$$\leq c^{m+1} \sum_{k=1}^{m} \frac{1}{T_k} \int_0^\infty (1-\exp(-\bar{\lambda} u_k))\bar{G}_k(u_k)du_k \to 0,$$

where $\Lambda(\pi_0) = \lambda_1 \lambda_2 \cdots \lambda_m$ and the last inequality is proved in Lemma 1. Thus $q_1'(e, m+1) = o[\bar{q}_0(e, m+1)]$. After summing the last relation, as in Lemma 1, we take $q_1' = o(\bar{q}_0) = o(q_0)$. The lemma is proved.

Lemma 3. If all the parameters and distributions change in such a way that the following conditions are fulfilled:

$$\frac{\bar{\lambda}_i}{\lambda_i} \leq c < \infty, \quad \bar{\lambda} T_i \to 0, \qquad i = 1, \cdots, n,$$

then

$$q_1'' \equiv o(q_0). \tag{4}$$

Proof. We examine a monotonic path π_0, the element failures of which correspond to the numbers $1, 2, \cdots, m+1$. It is easy to remark that the probability $q_1''(\pi_0)$ does not exceed the probability of the following event: at moments $x_1, x_2, \cdots, x_{m+1}$ $(0 < x_1 < x_2 < \cdots < x_{m+1})$ elements fail with numbers $1, 2, \cdots, m+1$ and the busy period, which began at moment 0, does not finish until the moment x_1. This probability, in its turn, increases only if, as in Lemma 2, we change all the $\lambda_i(e)$ to $\bar{\lambda}_i$ and require that every element at the moment of its failure is replaced by a standby. Then

$$q_1''(\pi_0) \leq \bar{\Lambda}(\pi_0) \int \cdots \int_{\bar{\Delta}} \bar{\Phi}(x_1)\bar{G}_1(x_{m+1}-x_1)\cdots \bar{G}_m(x_{m+1}-x_m)dx_1\cdots dx_{m+1},$$

where $\bar{\Delta} = \{(x_1, \cdots, x_{m+1}): 0 < x_1 < \cdots < x_{m+1}\}$, and $\Phi(x_1)$, as we can easily see, is the distribution function of the busy period for the queueing system $M/G/\infty$ with input intensity $\bar{\lambda} = \Sigma_{i=1}^n \lambda_i$ and the distribution function of the service time $G_0(x) = \Sigma_{i=1}^n (\lambda_k/\bar{\lambda})G_k(x)$. We change the variables $u_1 = x_1$, $u_k = x_{m+1} - x_k$, $k = 2, \cdots, m+1$. Then the last inequality takes the form:

$$q_1''(\pi_0) \leq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \int \cdots \int_{\Delta_{m+1}} \bar{\Phi}(u_1)\bar{G}_1(u_2)\cdots \bar{G}_m(u_{m+1})du_1\cdots du_{m+1},$$

where $\Delta_{m+1} = \{(u_1, \cdots, u_{m+1}): u_1 > 0, 0 < u_2 < \cdots < u_{m+1}\}$. The integral $\int_0^\infty \Phi(x)dx = \hat{T}$ is the mean length of the busy period in the system $M/G/\infty$. We know (see Klimov (1966)) that $\hat{T} = (1/\hat{\lambda})(\exp(\hat{\lambda}T_0) - 1)$, where $T_0 = (1/\hat{\lambda})\Sigma_{k=1}^n \hat{\lambda}_k T_k$, and it is the mean service time in the system $M/G/\infty$. Integrating the last inequality over u_1, we take

$$q''_1(\pi_0) \leqq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \hat{T} \int \cdots \int_{\Delta_m} \bar{G}_1(u_1) \cdots \bar{G}_m(u_m) du_1 \cdots du_m.$$

If we sum this inequality over all the montonic paths from the class $\Pi_0(e, m+1)$, we take

$$q''_1(e, m+1) \leqq \frac{\bar{\Lambda}(\pi_0)}{\lambda(\bar{0})} \hat{T} \prod_{k=1}^m T.$$

On the other hand, as proved,

$$\bar{q}_0(e, m+1) \leqq \frac{\Lambda(\pi_0)}{\lambda(\bar{0})} \prod_{k=1}^m T_k,$$

whence

$$\frac{q''_1(e, m+1)}{\bar{q}_0(e, m+1)} \leqq \hat{T} c^{m+1}.$$

Since $\hat{\lambda} T_0 = \Sigma_{k=1}^n \hat{\lambda}_k T_k \leqq \hat{\lambda} \Sigma_{k=1}^n T_k \to 0$, then $\lambda(\bar{0})\hat{T} \leqq \hat{\lambda}\hat{T} = e^{\hat{\lambda}T_0} - 1 \to 0$, that is

$$q'_1(e, m+1) = o[\bar{q}_0(e, m+1)] = o[q_0(e, m+1)].$$

After summing this relation over all $m+1$ and over all boundary states e, we take $q''_1 = o(\bar{q}_0) = o(q_0)$. The lemma is proved.

Corollary 1. Under the conditions of Lemma 2 the following applies:

$$q_1 = o(\bar{q}_0) = o(q_0). \tag{5}$$

Indeed, from the conditions of Lemma 2 it follows (see Barzilovich et al. (1983)) that $\hat{\lambda} T_i \to 0$, and this means that the conditions of Lemma 3 are fulfilled. Then, adding the relations (3) and (4) we obtain (5).

From Theorem 1 and the last corollary we get the following result.

Theorem 2. If the parameters $\lambda_i(e)$ and the distribution functions $G_i(x)$ change so that

$$\frac{\bar{\lambda}_i}{\lambda_i} \leqq c < \infty, \quad \frac{1}{T_i} \int_0^\infty (1 - \exp(-\bar{\lambda} x)) \bar{G}_i(x) dx \to 0, \qquad i = 1, \cdots, n,$$

then

$$\lim P\{\lambda(\bar{0}) \bar{q}_0 \tau > x\} = e^{-x}. \tag{6}$$

Remark 1. Let ξ be a non-negative random variable with distribution function $F(x)$ and mean T. We say that ξ tends to zero (by Khinchin) and we write $\xi \overset{\text{Kh}}{\to} 0$, if

$(1/T)\int_0^\infty (1-e^{-x})\bar{F}(x)dx \to 0$. Then the main condition of Theorem 2 is written as $\bar{\lambda}\eta_i \overset{\text{Kh}}{\to} 0$, $i = 1, \cdots, n$.

Remark 2. The condition $\bar{\lambda}\eta_i \overset{\text{Kh}}{\to} 0$ seems to be minimal. For the simpler model $M/G/\infty$, which is included in our model (as failure we understand the fall of the process to the state $m+1$), Yu. A. Veretenikov provided a counterexample (coursework, Moscow State University, 1972), which shows that for a slightly weaker condition $\bar{\lambda}T_i \to 0$, $i = 1, \cdots, n$ the principle of monotonic paths (4) does not apply; in other words, $q_1 \neq o(q_0)$.

3. Estimation of the probability q_0 in a parametric model

We now examine a parametric model of a complex renewable system with an unbounded number of repair units, where we can asymptotically simplify the expression for q_0.

We assume that the failure intensities have the form $\lambda_i(e)\varepsilon^{\alpha_i}$ and the distribution functions of the repair times have the form $G_i(t/\varepsilon^{\beta_i})$, where $\lambda_i(e)$, $G_i(t)$, $\alpha_i \geqq 0$ and $\beta_i > 0$ are fixed and $\varepsilon \to 0$. We note that by choosing a proper time unit we can always have that at least one α_i, for example α_1, is equal to zero. A model of this kind was introduced and studied on a semiheuristic level by I. N. Kovalenko, who estimated the stationary probabilities of the process $e(t)$.

For every monotonic path π_0, where failed elements have numbers $1, 2, \cdots, m+1$, the quantity $\bar{q}(\pi_0)$ has the form

$$(7)\qquad \bar{q}(\pi_0) = \frac{\Lambda(\pi_0)}{\lambda(\bar{0})}\varepsilon^{\alpha_1+\alpha_2+\cdots+\alpha_{m+1}} \int\cdots\int_{\Delta_m} \bar{G}_1\left(\frac{u_1}{\varepsilon^{\beta_1}}\right)\cdots \bar{G}_m\left(\frac{u_m}{\varepsilon^{\beta_m}}\right) du_1 du_2 \cdots du_m,$$

where $\Lambda(\pi_0) = \lambda_1(\bar{0})\lambda_2(e^{(1)})\cdots\lambda_{m+1}(e^{(m)})$, and Δ_m is defined in the first paragraph. Now we shall show that the main part is included in only a small proportion of the whole class of monotonic paths.

We study first the paths belonging to the class $\Pi_0(e, m+1)$. Obviously, without loss of generality, we can consider $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_m$. We divide the class $\Pi_0(e, m+1)$, which is taken from the permutations of the numbers $1, 2, \cdots, m$, into two subclasses: $\Pi_{00}(e, m+1)$ denotes the subclass of paths which are given by any permutation $(k_1, k_2, \cdots, k_m)$ for which $\beta_{k_1} \leqq \beta_{k_2} \leqq \cdots \leqq \beta_{k_m}$ applies and $\Pi_{01}(e, m+1) = \Pi_0(e, m+1) \setminus \Pi_{00}(e, m+1)$. For this subclass, for at least one s, $\beta_{k_s} > \beta_{k_{s+1}}$ applies. Let us correspondingly denote:

$$q_{00}(e, m+1) = \sum_{\pi_0 \in \Pi_{00}(e, m+1)} q(\pi_0), \qquad q_{01}(e, m+1) = \sum_{\pi_0 \in \Pi_{01}(e, m+1)} q(\pi_0).$$

Lemma 4. If the means $T_i = \int_0^\infty \bar{G}_i(x)dx$, $i = 1, \cdots, n$ exist, then for $\varepsilon \to 0$,

$$(8)\qquad q_{01}(e, m+1) = o[q_{00}(e, m+1)].$$

Proof. We denote by $I(A) = I(A, x_1, \cdots, x_m)$ the indicator (characteristic function) of the set A. The set A is given by inequalities in regard to arguments $x_1, \cdots, x_m$. We examine the integral in the right member of (7) for an arbitrary path from $\Pi_0(e, m+1)$, given by permutation of the $(k_1, k_2, \cdots, k_m)$:

$$J(\pi_0) = \int_{\Delta_m}\cdots\int \bar{G}_{k_1}\left(\frac{x_1}{\varepsilon^{\beta k_1}}\right)\cdots \bar{G}_{k_m}\left(\frac{x_m}{\varepsilon^{\beta k_m}}\right) dx_1\cdots dx_m$$

$$= \int_0^\infty \cdots \int_0^\infty I(x_1 > x_2 > \cdots > x_m)\bar{G}_{k_1}\left(\frac{x_i}{\varepsilon^{\beta k_1}}\right)\cdots \bar{G}_{k_m}\left(\frac{x_m}{\varepsilon^{\beta k_m}}\right) dx_1\cdots dx_m,$$

and we change the variables $x_i/\varepsilon^{\beta k_i} = u_i$. Then

$$J(\pi_0) = \varepsilon^{\beta_1+\cdots+\beta_m}\int_0^\infty \cdots \int_0^\infty I(\varepsilon^{\beta k_1}u_1 > \cdots > \varepsilon^{\beta k_m}u_m)\bar{G}_{k_1}(u_1)\cdots\bar{G}_{k_m}(u_m)du_1\cdots du_m$$

(9)

$$= \varepsilon^{\beta_1+\cdots+\beta_m}A_\varepsilon(\pi_0).$$

We note that the indicator, which stays under the integral, can be written as a product:

$$I(\varepsilon^{\beta k_1}u_1 > \varepsilon^{\beta k_2}u_2 > \cdots > \varepsilon^{\beta k_m}u_m) = \prod_{s=1}^{m-1} I(\varepsilon^{\beta k_s}u_s > \varepsilon^{\beta k_{s+1}}u_{s+1}).$$

If $\pi_0 \in \Pi_{00}(e, m+1)$, then for $\varepsilon \to 0$,

$$\lim I(\varepsilon^{\beta k_s}u_s > \varepsilon^{\beta k_{s+1}}u_{s+1}) = \begin{cases} I(u_s > u_{s+1}), & \text{if } \beta_{k_s} = \beta_{k_{s+1}}, \\ 1, & \text{if } \beta_{k_s} < \beta_{k_{s+1}}. \end{cases} \tag{10}$$

Since $\bar{G}_i(x) > 0$ in a neighbourhood of zero, the limit is positive $\lim_{\varepsilon\to 0} A_\varepsilon(\pi_0) = A(\pi_0) > 0$. Further, since

$$A_\varepsilon(\pi_0) \leqq \int_0^\infty \cdots \int_0^\infty \bar{G}_{k_1}(u_1)\cdots\bar{G}_{k_m}(u_m)du_1\cdots du_m = T_{k_1}T_{k_2}\cdots T_{k_m} = T_1T_2\cdots T_m,$$

then this limit is bounded. If the path $\pi_0 \in \Pi_{01}(e, m+1)$, then for at least one s $\beta_{k_s} > \beta_{k_{s+1}}$ and so the limit of the indicator is equal to zero:

$$\lim_{\varepsilon\to 0} I(\varepsilon^{\beta k_s}u_s > \varepsilon^{\beta k_{s+1}}u_{s+1}) = 0,$$

over a set of complete measure in $\mathbb{R}_m^+$. Then $\lim_{\varepsilon\to 0} A_\varepsilon(\pi_0) = 0$. The lemma is proved.

Remark 3. Let $\pi_0 \in \Pi_{00}(e, m+1)$, that is $\beta_{k_1} \leqq \beta_{k_2} \leqq \cdots \leqq \beta k_m$ and $\beta k_1 = \beta k_2 = \cdots = \beta k_{s_1} < \beta k_{s_1+1} = \cdots = \beta k_{s_2} < \cdots < \beta k_{s_l+1} = \cdots = \beta k_m$. Then the limit integral can be transformed by force (10) to the product:

$$A(\pi_0) = \lim_{\varepsilon\to 0} A_\varepsilon(\pi_0)$$

$$= \prod_{i=0}^{l} \int_{\Delta_{(s_{i+1}-s_i)}}\cdots\int \bar{G}_{k_{s_i+1}}(u_1)\cdots\bar{G}_{k_{s_{i+1}}}(u_{s_{i+1}-s_i})du_1\cdots du_{s_{i+1}-s_i},$$

where $s_0 = 0$, $s_{l+1} = m$. By force of the initial condition $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_m$ every ith group of indexes $(k_{s_i+1}, \cdots, k_{s_{i+1}})$ presents a certain permutation of the indexes $(s_i + 1, \cdots, s_{i+1})$. The $A(\pi_0)$ has an especially simple expression in the case of $\beta_1 < \beta_2 < \cdots < \beta_m$. Then the class $\Pi_{00}(e, m + 1)$ includes one path $\pi_0 = (1, 2, \cdots, m, m + 1)$ and $A(\pi_0) = T_1 T_2 \cdots T_m$.

In this way we have shown that

$$q_0(e, m + 1) \approx A(e, m + 1)\varepsilon^{\alpha_1 + \cdots + \alpha_{m+1} + \beta_1 + \cdots + \beta_m},$$

where $A(e, m + 1)$ is easily found from the expressions (7) and (9) (we must additionally consider that since $\alpha_i = 0$ applies for a certain i, it follows that $\lambda(\bar{0}) \approx \lambda_i(\bar{0}) > 0$). Hence for the arbitrary path π_0, given by the numbers of failures $1, 2, \cdots, m + 1$ for $\varepsilon \to 0$, $q_0(e, m + 1) \approx A(e, m + 1)\varepsilon^{\gamma(\pi_0)}$, where $\gamma(\pi_0) = \alpha_1 + \cdots + \alpha_{m+1} + \beta_1 + \cdots + \beta_m$. We write $\gamma_0 \equiv \min_{\pi_0 \in \Pi_0} \gamma(\pi_0)$ and examine the subclass of the monotonic paths Π_{00} for which $\gamma(\pi_0) = \gamma_0$ and $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_m$, and we write $q_{00} = \Sigma_{\pi_0 \in \Pi_{00}} q_0(\pi_0)$. The above reasoning finally leads to the following assertion.

Theorem 3. If the means $T_i = \int_0^\infty \bar{G}_i(x)dx$, $i = 1, \cdots, n$ exist, then for $\varepsilon \to 0$, $q_0 \approx q_{00}$.

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