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UNIFORM ASYMPTOTICS FOR DISCOUNTED AGGREGATE CLAIMS IN DEPENDENT RISK MODELS

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Abstract

In this paper, we consider some non-standard renewal risk models with some dependent claim sizes and stochastic return, where an insurance company is allowed to invest her/his wealth in financial assets, and the price process of the investment portfolio is described as a geometric Lévy process. When the claim-size distribution belongs to some classes of heavy-tailed distributions and a constraint is imposed on the Lévy process in terms of its Laplace exponent, we obtain some asymptotic formulas for the tail probability of discounted aggregate claims and ruin probabilities holding uniformly for some finite or infinite time horizons.

Keywords: Discounted aggregate claims; dependence; Lévy process; consistently varying tail; dominatedly varying tail; long tail; uniformity

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1. Introduction

We consider a non-standard renewal risk model in which successive claim sizes $\{X_n,\ n \geq 1\}$ form a sequence of identically distributed but not necessarily independent

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nonnegative random variables (r.v.s) with common distribution F and the inter-arrival times $\{\theta_n,\ n \geq 1\}$ form another sequence of independent and identically distributed (i.i.d.) nonnegative r.v.s, which are independent of $\{X_n,\ n \geq 1\}$. We suppose that the claim arrival times $\tau_n = \sum_{k=1}^{n} \theta_k,\ n \geq 1$, constitute a renewal counting process

$$N(t) = \sup\{n \geq 0 \,:\, \tau_n \leq t\}\,,\ t \geq 0\,,$$

which represents the number of claims up to time t, with a finite mean function $\lambda(t) = \mathbb{E}[N(t)] \to \infty$ as $t \to \infty$. Suppose that the insurer is allowed to make risk-free and risky investments. The price process of the investment portfolio is described as a geometric Lévy process $\{\mathrm{e}^{R_t}\,,\ t \geq 0\}$ with $\{R_t\,,\ t \geq 0\}$ being a Lévy process, which starts from zero and has independent and stationary increments. This assumption on price processes is widely used in mathematical finance. The reader is referred to [18], [19], [20], [11], [4], [26], [27], [24] [15] among others.

As usual, we assume that $\{X_n\,,\ \ n \geq 1\}$, $\{\theta_n\,,\ n \geq 1\}$ and $\{R_t\,,\ t \geq 0\}$ are mutually independent. The discounted aggregate claims up to time $t \geq 0$ can be expressed as

$$D(t) = \sum_{k=1}^{\infty} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k \leq t\}}\,, \tag{1.1}$$

where $\mathbf{1}_A$ denotes the indicator function of an event A. Then, for any $t \geq 0$, the discounted value of the surplus process with stochastic return on investments of an insurance company is described as

$$U(t) = x + \int_{0-}^{t} c(s)\,\mathrm{e}^{-R_s}\,\mathrm{d}s - D(t)\,, \tag{1.2}$$

where $x \geq 0$ is the initial risk reserve of the insurance company, and $c(t)$ denotes the density function of premium income at time t. Throughout the paper, we assume that the premium density function $c(t)$ is bounded, i.e. $0 \leq c(t) \leq M$ for some constant $M > 0$ and all $t \geq 0$. In this way, in the above renewal risk model, for any $t \geq 0$, the finite-time ruin probability can be defined as

$$\psi(x,\, t) = \mathbb{P}\left(\inf_{s\in[0,\, t]} U(s) < 0 \,\middle|\, U(0) = x\right)\,,$$

and the infinite-time ruin probability as

$$\psi(x,\, \infty) = \mathbb{P}\left(\inf_{s\geq 0} U(s) < 0 \,\middle|\, U(0) = x\right)\,.$$

In the present paper, we shall investigate the asymptotics for the tail probability of discounted aggregate claims or ruin probabilities, holding uniformly for all t, such that $\lambda(t)$ is positive. For this purpose, as in [22], define $\Lambda = \{t : 0 < \lambda(t) \le \infty\} = \{t : \mathbb{P}(\theta_1 \le t) > 0\}$ with $\underline{t} = \inf\{t : \lambda(t) > 0\} = \inf\{t : \mathbb{P}(\theta_1 \le t) > 0\}$. Clearly,

$$\Lambda = \begin{cases} [\underline{t}, \infty]\,, & \text{if} \quad \mathbb{P}(\theta_1 = \underline{t}) > 0\,, \\ (\underline{t}, \infty]\,, & \text{if} \quad \mathbb{P}(\theta_1 = \underline{t}) = 0\,. \end{cases}$$

For the above risk model with a constant premium rate $c > 0$ (i.e. $c(t) = c$ for all $t > 0$) and a constant interest force $\delta > 0$ (e.g. $R_t = \delta\, t$ for all $t > 0$), if the claim sizes $\{X_n\,,\ n \ge 1\}$ and the inter-arrival times $\{\theta_n\,,\ n \ge 1\}$ are, respectively, i.i.d. r.v.s, some earlier works on ruin probabilities can be found in [1], [12], [10], [13], [21] among others; Tang [22] and Hao and Tang [8] derived some uniform results in Λ. If the claim sizes $\{X_n\,,\ n \ge 1\}$ follow a certain dependence structure, ruin probabilities have also been investigated by many researchers. Chen and Ng [5] considered a dependent risk model, where the claim sizes are pairwise negatively quadrant dependent (NQD, see the definition in Section 2.1) with common distribution F, belonging to the class of extended regularly varying distributions. They obtained the following asymptotics for the infinite-time ruin probability:

$$\psi(x,\,\infty) \sim \int_{0-}^{\infty} \overline{F}(x\,\mathrm{e}^{\delta u})\,\lambda(\mathrm{d}u)\,.$$

Further, assuming some restrictive dependence structure and heavy-tailed claim sizes, Yang and Wang [25] obtained the relation

$$\psi(x,\,t) \sim \int_{0-}^{t} \overline{F}(x\,\mathrm{e}^{\delta u})\,\lambda(\mathrm{d}u)\,, \tag{1.3}$$

holding for each fixed $t \in \Lambda$ or uniformly for all $t \in \Lambda$.

Two recent interesting papers Tang *et al.* [24] and Li [15] investigated the above renewal risk model with stochastic return, allowing the price process of the investment portfolio being a geometric Lévy process under independence and dependence structures, respectively. We remark that the paper [24] investigated the independent renewal risk model, where the claim sizes $\{X_n\,,\ n \ge 1\}$ and the inter-arrival times $\{\theta_n\,,\ n \ge 1\}$ are two sequences of i.i.d. nonnegative r.v.s, respectively, and they are mutually independent; whereas the recent paper [15] considered a more general time-dependent renewal risk model, where $\{(X_n\,,\ \theta_n)\,,\ n \geq 1\}$ are assumed to be i.i.d. random vectors but a certain dependence exists between X_n and θ_n for each fixed n. Motivated by these two papers, the main goal of this paper is to establish some asymptotic formulas of the tail probability of discounted aggregate claims and ruin probabilities, holding uniformly for some finite or infinite time horizons, under the conditions that the claim sizes $\{X_n\,,\ n \geq 1\}$ are dependent and have common distribution belonging to the class of long and dominatedly-varying tailed distributions or to the class of consistently-varying tailed distributions.

The rest of the paper is organized as follows. Section 2 presents the four main results on the uniform asymptotics for the tail probability of discounted aggregate claims and ruin probabilities after introducing some preliminaries; Sections 3 and 4 prove these results, respectively, after preparing a series of lemmas.

2. Preliminaries and main results

Hereafter, all limit relationships hold for x tending to ∞, unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup a(x)/b(x) \leq 1$; $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$; and $a(x) \asymp b(x)$ if $a(x) = O(b(x))$ and $b(x) = O(a(x))$. Furthermore, for two positive bivariate functions $a(x,\,t)$ and $b(x,\,t)$, we write $a(x,\,t) \sim b(x,\,t)$ uniformly for all t in a nonempty set A, if

$$\lim_{x\to\infty} \sup_{t\in A} \left| \frac{a(x,\,t)}{b(x,\,t)} - 1 \right| = 0\,,$$

and write $a(x,\,t) \lesssim b(x,\,t)$ or $b(x,\,t) \gtrsim a(x,\,t)$ uniformly for all $t \in A$, if

$$\limsup_{x\to\infty} \sup_{t\in A} \frac{a(x,\,t)}{b(x,\,t)} \leq 1.$$

2.1. Dependence structures

Since what we are interested in is some actual dependent risk models, we start this section by introducing some dependence structures. A sequence of r.v.s $\{\xi_n\,,\ n \geq 1\}$ is said to be pairwise negatively quadrant dependent (NQD), if, for any $i \neq j \geq 1$ and real x, y,

$$\mathbb{P}(\xi_i > x\,,\ \xi_j > y) \leq \mathbb{P}(\xi_i > x)\mathbb{P}(\xi_j > y)\,.$$

The pairwise NQD structure was introduced by Lehmann [14], and is weaker and more verifiable than the commonly used notions of the upper/lower negative dependence (see [3]), and the negative association (see [9]). A more general dependence structure, namely upper tail asymptotic independence (UTAI) structure, was proposed by Maulik and Resnick [16]. A sequence of r.v.s $\{\xi_n\,,\ n \geq 1\}$ is said to be UTAI, if $\mathbb{P}(\xi_n > x) > 0$ for all $x \in \mathbb{R}\,,\ n \geq 1$, and for any $i \neq j \geq 1$,

$$\lim_{\min\{x,\,y\}\to\infty} \mathbb{P}(\xi_i > x \mid \xi_j > y) = 0\,.$$

Clearly, if a sequence of r.v.s is pairwise NQD, then it is also UTAI.

2.2. Some classes of heavy-tailed distributions

We shall restrict the claim-size distribution F to some class of heavy-tailed distributions, whose moment generating functions do not exist. An important class of heavy-tailed distributions is $\mathscr{D}$, which consists of all distributions $F = 1 - \overline{F}$ with dominated variation. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{D}$, if $\limsup \overline{F}(x\,y)/\overline{F}(x) < \infty$ for any $0 < y < 1$. A slightly smaller class is $\mathscr{C}$ of consistently varying distributions. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{C}$, if $\lim_{y\uparrow 1}\limsup_{x\to\infty} \overline{F}(x\,y)/\overline{F}(x) = 1$. Closely related is a wider class $\mathscr{L}$ of long-tailed distributions. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{L}$, if $\lim \overline{F}(x+y)/\overline{F}(x) = 1$ for any $y \in \mathbb{R}$. There are some other heavy-tailed subclasses, the class $\mathrm{ERV}(-\alpha, -\beta)$ of distributions with extended regularly varying tails, and the class $\mathscr{R}_{-\alpha}$ of distributions with regularly varying tails. A distribution F on $\mathbb{R}$ belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$, if there are some constants $0 < \alpha \leq \beta < \infty$ such that $y^{-\beta} \leq \liminf \overline{F}(x\,y)/\overline{F}(x) \leq \limsup \overline{F}(x\,y)/\overline{F}(x) \leq y^{-\alpha}$ for any $y \geq 1$. If $\alpha = \beta$, F belongs to the class $\mathscr{R}_{-\alpha}$.

For a distribution F on $[0,\,\infty)$, denote its upper Matuszewska index by

$$J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y} \quad \text{with} \quad \overline{F}_*(y) := \liminf_{x\to\infty} \frac{\overline{F}(x\,y)}{\overline{F}(x)} \quad \text{for} \quad y > 1.$$

The presented definitions yield, that the following assertions are equivalent (for details, see [2]):

$$\text{(i) } F \in \mathscr{D}\,, \quad \text{(ii) } \overline{F}_*(y) > 0 \ \text{ for some } \ y > 1\,, \quad \text{(iii) } J_F^+ < \infty\,.$$

The lemma below is from Proposition 2.2.1 of [2].

Lemma 2.1. *For a distribution $F \in \mathscr{D}$ on $[0,\infty)$ and any $p > J_F^+$, there are positive constants C_1 and D_1, such that the inequality*

$$\frac{\overline{F}(y)}{\overline{F}(x)} \le C_1 \left(\frac{y}{x}\right)^{-p},$$

holds for all $x \ge y \ge D_1$.

The next lemma follows immediately from Lemma 2.1 (see Lemma 3.5 of [23]).

Lemma 2.2. *For a distribution $F \in \mathscr{D}$ on $[0,\, \infty)$ and for any $p > J_F^+$, the asymptotics $x^{-p} = o(\overline{F}(x))$ holds.*

2.3. Main results

Suppose that the Lévy process $\{R_t\,,\ t \ge 0\}$ in (1.1) is right continuous with left limit. Let $\mathbb{E}[R_1] > 0$, so that R_t drifts to ∞ almost surely as $t \to \infty$. The Laplace exponent for the Lévy process $\{R_t\,,\ t \ge 0\}$ is defined as

$$\phi(z) = \log \mathbb{E}[\mathrm{e}^{-z\,R_1}]\,,\ z \in \mathbb{R}\,.$$

If $\phi(z)$ is finite, then it holds for any $t \ge 0$ that

$$\mathbb{E}[\mathrm{e}^{-z\,R_t}] = \mathrm{e}^{t\,\phi(z)} < \infty\,,$$

(see, e.g. Proposition 3.14 of [7]).

To simplify the discussion, we assume throughout the paper that $\underline{t} = 0$. For any $T \in \Lambda$ and $\epsilon \in \Lambda$, set $\Lambda^T = [0,\, T]$ and $\Lambda_\epsilon = [\epsilon,\, \infty]$. Note that ϵ can be chosen to be 0 if $\mathbb{P}(\theta_1 = 0) > 0$.

We are now ready to state the main results of this paper. The first two results below consider the UTAI claim sizes, requiring that the Lévy process $\{R_t\,,\ t \ge 0\}$ is almost surely nonnegative, which means that the insurer invests her/his wealth only into a risky-free market. We obtain two uniform asymptotic formulas for the tail probability of discounted aggregate claims and ruin probabilities, when the claim sizes have both long and dominatedly varying tails, or consistently varying tails, respectively.

Theorem 2.1. *In the non-standard renewal risk model, described in Section 1, let assume that the claim sizes $\{X_n\,,\ n \ge 1\}$ are UTAI nonnegative r.v.s with common distribution $F \in \mathscr{L} \cap \mathscr{D}$. If $R_t \ge 0$ a.s. for any $t \ge 0$, then, for any fixed $T > 0$,*

$$\mathbb{P}(D(t) > x) \sim \int_{0-}^{t} \mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)\,, \tag{2.1}$$

holds uniformly for all $t \in \Lambda^T$.

Corollary 2.1. *Under the conditions of Theorem 2.1, if* $F \in \mathscr{C}$*, then, for any fixed* $T > 0$,

$$\psi(x, t) \sim \int_{0-}^{t} \mathbb{P}(X_1 \, \mathrm{e}^{-R_s} > x) \, \lambda(\mathrm{d}s) \,, \tag{2.2}$$

holds uniformly for all $t \in \Lambda^T$.

Our next two results restrict the claim sizes to be pairwise NQD r.v.s with extended regularly varying tails, but allow the insurer to make risk-free and risky investments. This means the Lévy process $\{R_t \,,\ t \geq 0\}$ can be real-valued.

Theorem 2.2. *In the non-standard renewal risk model described in Section 1, let assume that the claim sizes* $\{X_n \,,\ n \geq 1\}$ *are pairwise NQD nonnegative r.v.s with common distribution* $F \in \mathrm{ERV}(-\alpha, -\beta)$, $0 < \alpha \leq \beta < \infty$. *If* $\phi(r) < 0$ *for some* $r > \beta$*, then,* (2.1) *holds uniformly for all* $t \in \Lambda_\epsilon$.

Corollary 2.2. *Under the conditions of Theorem 2.2,* (2.2) *holds uniformly for all* $t \in \Lambda_\epsilon$.

3. Proof of Theorem 2.1

Before proving our first two results, we cite two lemmas. The following lemma is due to [22].

Lemma 3.1. *For the renewal counting process* $\{N(t) \,,\ t \geq 0\}$*, described in Section 1, for any* $v > 0$ *and any fixed* $T > 0$ *it holds*

$$\lim_{x\to\infty} \sup_{t\in\Lambda^T} \lambda^{-1}(t)\mathbb{E}[N^v(t) \, \mathbf{1}_{\{N(t)>x\}}] = 0 \,.$$

The second lemma was proven by Maulik and Zwart [16].

Lemma 3.2. *Consider the exponential functional of the Lévy process* $\{R_t \,,\ t \geq 0\}$ *defined as* $Z = \int_0^\infty \mathrm{e}^{-R_s} \, \mathrm{d}s$. *For every* $v > 0$ *satisfying* $\phi(v) < 0$*, it holds* $\mathbb{E}[Z^v] < \infty$.

Proof of Theorem 2.1. Since the positive Lévy process (i.e. a subordinator) $\{R_t \,,\ t \geq 0\}$ has non-decreasing paths, it holds uniformly for all $t \in \Lambda^T$ that

$$\int_{0-}^{t} \mathbb{P}(X_1 \, \mathrm{e}^{-R_s} > x) \, \lambda(\mathrm{d}s) \geq \mathbb{P}(X_1 \, \mathrm{e}^{-R_T} > x) \, \lambda(t) \asymp \overline{F}(x) \, \lambda(t) \,,$$

where the second step follows from Theorem 3.3 (iv) in [6] and $F \in \mathscr{D}$. Hence, there exists some positive constant C_2, such that for sufficiently large x and all $t \in \Lambda^T$,

$$\int_{0-}^{t} \mathbb{P}(X_1\, \mathrm{e}^{-R_s} > x)\, \lambda(\mathrm{d}s) \geq C_2\, \overline{F}(x)\, \lambda(t)\,. \tag{3.1}$$

For any integer $m \geq 1$, $t \in \Lambda^T$ and $x > 0$,

$$\mathbb{P}(D(t) > x) = \left(\sum_{n=1}^{m} + \sum_{n=m+1}^{\infty} \right) \mathbb{P}\left(\sum_{k=1}^{n} X_k\, \mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t) = n \right) =: I_1 + I_2\,. \tag{3.2}$$

For I_2, by $R_s \geq 0$ a.s. for all $s \geq 0$, Lemma 2.1, Markov's inequality and Lemma 2.2, it holds uniformly for all $t \in \Lambda^T$ that

$$\begin{aligned}
I_2 &\leq \left(\sum_{m<n\leq x/D_1} + \sum_{n>x/D_1} \right) \mathbb{P}\left(\sum_{k=1}^{n} X_k > x \right) \mathbb{P}(N(t) = n) \\
&\leq \sum_{m<n\leq x/D_1} n\, \overline{F}\left(\frac{x}{n}\right) \mathbb{P}(N(t) = n) + \mathbb{P}\left(N(t) > \frac{x}{D_1} \right) \\
&\leq C_1\, \overline{F}(x) \sum_{m<n\leq x/D_1} n^{p+1}\, \mathbb{P}(N(t) = n) + \left(\frac{x}{D_1} \right)^{-(p+1)} \mathbb{E}[N^{p+1}(t)\, \mathbf{1}_{\{N(t)>x/D_1\}}] \\
&\lesssim C_1\, \overline{F}(x)\, \mathbb{E}[N^{p+1}(t)\, \mathbf{1}_{\{N(t)>m\}}]\,,
\end{aligned} \tag{3.3}$$

for some $p > J_F^+$. Hence, from (3.1), (3.3) and Lemma 3.1, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_{x\to\infty} \sup_{t\in\Lambda^T} \frac{I_2}{\int_{0-}^{t} \mathbb{P}(X_1\, \mathrm{e}^{-R_s} > x)\, \lambda(\mathrm{d}s)} \\
&\leq \frac{C_1}{C_2} \lim_{m\to\infty} \sup_{t\in\Lambda^T} \lambda^{-1}(t)\, \mathbb{E}[N^{p+1}(t)\, \mathbf{1}_{\{N(t)>m\}}] = 0\,.
\end{aligned} \tag{3.4}$$

We mainly deal with I_1. Let $H(y_1, \ldots, y_{n+1})$ be the joint distribution of the random vector $(\tau_1, \ldots, \tau_{n+1})$, $n \geq 1$. Clearly, for $1 \leq n \leq m$ and $t \in \Lambda^T$, $x > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k=1}^{n} X_k\, \mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t) = n \right) \\
&= \int_{\{0\leq s_1\leq \ldots \leq s_n\leq t,\, s_{n+1}>t\}} \mathbb{P}\left(\sum_{k=1}^{n} X_k\, \mathrm{e}^{-R_{s_k}} > x \right) H(\mathrm{d}s_1, \ldots, \mathrm{d}s_{n+1})\,.
\end{aligned} \tag{3.5}$$

Similarly to (3.1), there exist some $0 < C_3 < 1$ and large $x_1 > 0$, both depending only on F, such that for all $1 \leq k \leq n$, $0 \leq s_k \leq t \leq T$, when $x \geq x_1$, it holds that

$$\mathbb{P}(X_k\, \mathrm{e}^{-R_{s_k}} > x) \geq C_3\, \overline{F}(x)\,. \tag{3.6}$$

Next, we aim to show, that for some $\Delta > 0$, which can be arbitrarily small, there exists some $\tilde{x}$, depending only on F, , Δ and n, such that for all $x \geq \tilde{x}$ and $0 \leq s_1 \leq \ldots \leq s_n \leq t \leq T$, $1 \leq n \leq m$,

$$\begin{aligned}(1-\Delta)\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{s_k}} > x) &\leq \mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{s_k}} > x\right)\\ &\leq (1+\Delta)\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{s_k}} > x)\,. \end{aligned} \tag{3.7}$$

For any $\varepsilon \in (0, \frac{1}{2})$, by $\mathbb{P}(\sup_{s\in[0,\,T]} R_s = \infty) = 0$, there exists a $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$\mathbb{P}\left(\sup_{s\in[0,\,T]} R_s < -\log\delta\right) \geq 1-\varepsilon\,, \tag{3.8}$$

which implies, that for all $1 \leq k \leq n$, $0 \leq s_k \leq t \leq T$ and $x > 0$,

$$\begin{aligned}\mathbb{P}(X_k\,\mathrm{e}^{-R_{s_k}} > x) &\geq \int_{\delta}^{1}\overline{F}\left(\frac{x}{u}\right)\mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u)\\ &\geq \overline{F}\left(\frac{x}{\delta}\right)\mathbb{P}\left(\sup_{s\in[0,\,T]} R_s < -\log\delta\right) \geq (1-\varepsilon)\,\overline{F}\left(\frac{x}{\delta}\right)\,. \end{aligned} \tag{3.9}$$

By $F \in \mathscr{L}$, there exists some positive, increasing and slowly varying function $l(x) \uparrow \infty$ such that $l(x)/x \to 0$ and for any fixed constant K,

$$\overline{F}(x - K\,l(x)) \sim \overline{F}(x)\,, \tag{3.10}$$

which implies, that for the above $\varepsilon > 0$, there exists some $x_2 \geq x_1$, depending only on F and ε, such that for all $x \geq x_2$,

$$\overline{F}(x - l(x)) \leq \overline{F}\left(x - \frac{l(x)}{\delta}\right) \leq (1+\varepsilon)\,\overline{F}(x)\,. \tag{3.11}$$

Since $X_1, \ldots, X_n$ are UTAI, and by $F \in \mathscr{D}$, there exists some $x_3 \geq x_2$, depending only on F, ε and n, such that for all $x \geq x_3$ and all $1 \leq i \neq j \leq n$ $(1 \leq n \leq m)$,

$$\mathbb{P}(X_i > x\,,\ X_j > x) \leq \mathbb{P}\left(X_i > \frac{l(x)}{n-1}\,,\ X_j > \frac{x}{n}\right) \leq \frac{\varepsilon}{n}\,C_3\,\overline{F}(x)\,. \tag{3.12}$$

Let choose $\tilde{x} = \max\{x_3\,,\ D_1/(1-\delta)\}$, which also depends only on F, ε and n, then for $x \geq \tilde{x}$, it holds that $x - l(x) \geq (1-\delta)x$ and for some $p > J_F^+$,

$$\frac{\overline{F}((1-\delta)x)}{\overline{F}(x)} \leq C_1\,(1-\delta)^{-p}\,, \tag{3.13}$$

together with equations (3.11), (3.12). For the lower bound of (3.7), by Bonferroni inequality, (3.12) and (3.6) we have, that for all $x \geq \tilde{x}$ and $0 \leq s_k \leq t \leq T$, $1 \leq k \leq n$,

$$\begin{aligned}\mathbb{P}\left(\sum_{k=1}^{n} X_k \,\mathrm{e}^{-R_{s_k}} > x\right) &\geq \sum_{k=1}^{n} \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x) - \sum_{1\leq i<j\leq n} \mathbb{P}(X_i > x\,,\ X_j > x)\\ &\geq (1-\varepsilon) \sum_{k=1}^{n} \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x)\,. \qquad (3.14)\end{aligned}$$

For the upper bound of (3.7),

$$\begin{aligned}\mathbb{P}\left(\sum_{k=1}^{n} X_k \,\mathrm{e}^{-R_{s_k}} > x\right) &\leq \sum_{k=1}^{n} \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x - l(x))\\ &\quad + \sum_{1\leq i\neq j\leq n} \mathbb{P}\left(X_i > \frac{l(x)}{n-1}\,,\ X_j > \frac{x}{n}\right), \qquad (3.15)\end{aligned}$$

where $l(x)$ is defined in (3.10). Since $l(x)$ is infinitely increasing, and by (3.11), (3.13), (3.8) and (3.9), we have, that for all $x \geq \tilde{x}$, $0 \leq s_k \leq t \leq T$ and $1 \leq k \leq n$,

$$\begin{aligned}&\mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x - l(x)) = \left(\int_{\delta}^{1} + \int_{0}^{\delta}\right) \overline{F}\left(\frac{x - l(x)}{u}\right) \mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u)\\ &\leq \int_{\delta}^{1} \overline{F}\left(\frac{x}{u} - \frac{l(x/u)}{\delta}\right) \mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u) + \int_{0}^{\delta} \overline{F}\left(\frac{(1-\delta)\,x}{u}\right) \mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u)\\ &\leq (1+\varepsilon) \int_{\delta}^{1} \overline{F}\left(\frac{x}{u}\right) \mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u) + C_1\,(1-\delta)^{-p} \int_{0}^{\delta} \overline{F}\left(\frac{x}{u}\right) \mathbb{P}(\mathrm{e}^{-R_{s_k}} \in \mathrm{d}u)\\ &\leq (1+\varepsilon)\,\mathbb{P}(X_k \mathrm{e}^{-R_{s_k}} > x) + C_1\,(1-\delta)^{-p}\,\overline{F}\left(\frac{x}{\delta}\right) \mathbb{P}\left(\sup_{s\in[0,\,T]} R_s > -\log\delta\right)\\ &\leq \left(1 + \varepsilon + \frac{\varepsilon\, C_1\,(1-\delta)^{-p}}{1-\varepsilon}\right) \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x)\,. \qquad (3.16)\end{aligned}$$

By (3.12) and (3.6), the second sum on the right-hand side of (3.15) can be bounded from above by

$$\varepsilon \sum_{k=1}^{n} \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x)\,. \qquad (3.17)$$

Combining (3.15)–(3.17), we obtain, that for all $x \geq \tilde{x}$ and $0 \leq s_k \leq t \leq T$, $1 \leq k \leq n$,

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k \,\mathrm{e}^{-R_{s_k}} > x\right) \leq \left(1 + \varepsilon\left(2 + \frac{C_1\,(1-\delta)^{-p}}{1-\varepsilon}\right)\right) \sum_{k=1}^{n} \mathbb{P}(X_k \,\mathrm{e}^{-R_{s_k}} > x)\,. \qquad (3.18)$$

Let $\Delta = \varepsilon\,(2 + C_1\,(1-\delta)^{-p}\,(1-\varepsilon)^{-1}) > 0$, which can be arbitrarily small because of the arbitrariness of ε. Therefore, (3.7) follows from (3.14) and (3.18).

By (3.5) and (3.7), we have, that for all $x \geq \tilde{x}$ (depending only on F, Δ and n), $1 \leq n \leq m$ and $t \in \Lambda^T$,

$$
\begin{aligned}
&(1-\Delta)\sum_{k=1}^{n}\mathbb{P}(X_k \mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n) \\
&\leq \mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n\right) \\
&\leq (1+\Delta)\,\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n)\,.
\end{aligned}
$$

This means, that it holds uniformly for all $t \in \Lambda^T$ and $1 \leq n \leq m$

$$
\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n\right) \sim \sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n)\,.
$$

Thus, it holds uniformly for all $t \in \Lambda^T$ that

$$
\begin{aligned}
I_1 \quad &\sim \quad \sum_{n=1}^{m}\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n) \\
&= \quad \left(\sum_{n=1}^{\infty} - \sum_{n=m+1}^{\infty}\right)\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t)=n) =: I_3 - I_4\,, \qquad (3.19)
\end{aligned}
$$

where

$$
I_3 = \sum_{k=1}^{\infty}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}} > x\,,\ N(t) \geq k) = \int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)\,. \qquad (3.20)
$$

Since

$$
I_4 \leq \overline{F}(x)\sum_{n=m+1}^{\infty} n\mathbb{P}(N(t)=n) = \overline{F}(x)\,\mathbb{E}[N(t)\,\mathbf{1}_{\{N(t)>m\}}]\,,
$$

similarly to the proof of (3.4), and by (3.1), Lemma 3.1, we obtain

$$
\begin{aligned}
&\lim_{m\to\infty}\limsup_{x\to\infty}\sup_{t\in\Lambda^T}\frac{I_4}{\int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)} \\
&\leq \frac{1}{C_2}\lim_{m\to\infty}\sup_{t\in\Lambda^T}\lambda^{-1}(t)\,\mathbb{E}[N(t)\,\mathbf{1}_{\{N(t)>m\}}] = 0\,. \qquad (3.21)
\end{aligned}
$$

Therefore, by (3.4) and (3.19)–(3.21) the desired (2.1) holds uniformly for all $t \in \Lambda^T$ and thus we conclude the proof.

Proof of Corollary 2.1. The upper bound of $\psi(x,\,t)$ is trivial. Indeed, by Theorem 2.1 it holds uniformly for all $t \in \Lambda^T$

$$
\psi(x,\,t) \leq \mathbb{P}(D(t) > x) \sim \int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)\,. \qquad (3.22)
$$

We next turn to the asymptotic lower bound of $\psi(x,\,t)$. For any $0<\varepsilon<1$, since the positive Lévy process $\{R_t\,,\ t\geq 0\}$ has non-decreasing paths, and according to $0\leq c(u)\leq M$, (2.1), we have, that for all $t\in\Lambda^T$ and sufficiently large x,

$$\begin{aligned}
\psi(x,\,t) &\geq \mathbb{P}(D(t)>x+MT)\geq \mathbb{P}(D(t)>(1+\varepsilon)\,x)\\
&\sim \int_{0-}^{t}\int_{0}^{1}\overline{F}\left(\frac{(1+\varepsilon)\,x}{u}\right)\mathbb{P}(\mathrm{e}^{-R_s}\in \mathrm{d}u)\lambda(\mathrm{d}s)\\
&\geq \inf_{u\in(0,\,1]}\frac{\overline{F}\left(\frac{(1+\varepsilon)\,x}{u}\right)}{\overline{F}\left(\frac{x}{u}\right)}\int_{0-}^{t}\int_{0}^{1}\overline{F}\left(\frac{x}{u}\right)\mathbb{P}(\mathrm{e}^{-R_s}\in \mathrm{d}u)\,\lambda(\mathrm{d}s)\\
&\gtrsim \overline{F}_*(1+\varepsilon)\int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s}>x)\,\lambda(\mathrm{d}s)\,. \qquad (3.23)
\end{aligned}$$

Noting $\overline{F}_*(1+\varepsilon)\to 1$ as $\varepsilon\downarrow 0$ because of $F\in\mathscr{C}$, the desired lower bound follows from (3.23). This completes the proof of Corollary 2.1.

4. Proof of Theorem 2.2

We start this section by a series of lemmas. The first two lemmas are special cases of Lemmas 3.2 and 3.3 in [15] under the independence structure between $\{X_n\,,\ n\geq 1\}$ and $\{\theta_n\,,\ n\geq 1\}$.

Lemma 4.1. *Under the conditions of Theorem 2.2, for each $k\geq 1$, it holds uniformly for all $t\in\Lambda$*

$$\overline{F}(x)\,\mathbb{P}(\tau_k\leq t)=O(1)\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\leq t\}}>x)\,.$$

Lemma 4.2. *Under the conditions of Theorem 2.2, for any positive function $a(x)=x/l(x)\to\infty$, with the slowly varying function $l(x)\to\infty$, and each $k\geq 1$, it holds uniformly for all $t\in\Lambda$*

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\leq t\}}>a(x))=o(1)\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\leq t\}}>x)\,.$$

The next lemma plays an important role in the proof of Theorem 2.2.

Lemma 4.3. *Under the conditions of Theorem 2.2, for any fixed $0<y<1$ and each $k\geq 1$, it holds uniformly for all $t\in\Lambda_\epsilon$*

$$\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\leq t\}}>xy)\lesssim y^{-\beta}\,\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\leq t\}}>x)\,.$$

Proof. We firstly show that there exists some $\Delta > 0$, not depending on t, such that

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > \Delta) > 0 \tag{4.1}$$

holds for all $t \in \Lambda_\epsilon$. Indeed, if $\mathbb{P}(\theta_1 = 0) > 0$, then for any $\Delta \in (0,\, 1)$ and all $t \in \Lambda$,

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > \Delta) \ge \mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k = 0\}} > \Delta) = (\mathbb{P}(\theta_1 = 0))^k > 0\,.$$

If $\mathbb{P}(\theta_1 = 0) = 0$, then $\epsilon/k \in \Lambda$. Hence,

$$\mathbb{P}(\tau_k \le \epsilon) \ge \left(\mathbb{P}\left(\theta_1 \le \frac{\epsilon}{k}\right)\right)^k > 0\,. \tag{4.2}$$

For all $t \in \Lambda_\epsilon$ and $x \ge 0$,

$$\begin{aligned}\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > x) \quad &\ge \quad \int_{0-}^{\epsilon} \mathbb{P}(\mathrm{e}^{-R_s} > x)\, \mathbb{P}(\tau_k \in \mathrm{d}s) \\ &\ge \quad \int_{0-}^{\epsilon} \mathbb{P}\left(\sup_{s \in [0,\epsilon]} R_s < -\log x\right) \mathbb{P}(\tau_k \in \mathrm{d}s)\,. \end{aligned} \tag{4.3}$$

Noting $\mathbb{P}(\sup_{[0,\,\epsilon]} R_s = \infty) = 0$, there exists some $\Delta > 0$, not depending on t, such that

$$C_4 := \mathbb{P}\left(\sup_{s \in [0,\epsilon]} R_s < -\log \Delta\right) > 0\,.$$

Plugging this into (4.3), and by (4.2), we have that for all $t \in \Lambda_\epsilon$,

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > \Delta) \ge C_4\, \mathbb{P}(\tau_k \le \epsilon) \ge C_4 \left(\mathbb{P}\left(\theta_1 \le \frac{\epsilon}{k}\right)\right)^k > 0\,.$$

We secondly prove that

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > x) = o(\overline{F}(x))\,, \tag{4.4}$$

holds uniformly for all $t \in \Lambda$. Indeed, for all $t \in \Lambda$, by Markov's inequality, $\phi(r) < 0$ and Lemma 2.2, we have

$$\begin{aligned}\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > x) \quad &\le \quad \int_{0-}^{\infty} \mathbb{P}(\mathrm{e}^{-R_s} > x)\, \mathbb{P}(\tau_k \in \mathrm{d}s) \\ &\le \quad x^{-r} \int_{0-}^{\infty} \mathrm{e}^{s\,\phi(r)} \mathbb{P}(\tau_k \in \mathrm{d}s) = o(\overline{F}(x))\,. \end{aligned}$$

The claim (4.4) implies, that there exists some increasing function $l(x) \uparrow \infty$, not depending on t, such that $x/l(x) \to \infty$ and

$$\mathbb{P}(\mathrm{e}^{-R_{\tau_k}} \mathbf{1}_{\{\tau_k \le t\}} > x/l(x)) = o(\overline{F}(x))\,, \tag{4.5}$$

holds uniformly for all $t \in \Lambda$. Then, we have that for all $x > \Delta$, any fixed $y \in (0, 1)$ and all $t \in \Lambda_\epsilon$,

$$\begin{aligned}\frac{\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > xy)}{\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > x)} &\le \frac{\int_{0-}^{x/l(x)} \overline{F}(x\,y/u)\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}\in \mathrm{d}u)}{\int_{0-}^{x/l(x)} \overline{F}(x/u)\,\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}\in \mathrm{d}u)} \\ &\quad + \frac{\int_{x/l(x)}^{\infty} \overline{F}(x\,y/u)\,\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}\in \mathrm{d}u)}{\int_{\Delta}^{\infty} \overline{F}(x/u)\,\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}\in \mathrm{d}u)} \\ &\le \sup_{z\ge l(x)} \frac{\overline{F}(y\,z)}{\overline{F}(z)} + \frac{\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > x/l(x))}{\overline{F}(x/\Delta)\,\mathbb{P}(\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > \Delta)}\,,\end{aligned}$$

which, combined with (4.1), (4.5), leads to the desired result.

Lemma 4.4. *Under the conditions of Theorem 2.2, for each $j > i \ge 1$, it holds uniformly for all $t \in \Lambda$*

$$\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}} > x\,,\ X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}} > x) = o(1)\sum_{k=i,j} \mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > x)\,.$$

Proof. By Lemma 4.2, we have that uniformly for all $t \in \Lambda$,

$$\begin{aligned}&\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}} > x\,,\ X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}} > x) \\ &\quad \le \mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}} > x\,,\ X_j\,\mathrm{e}^{-R_{\tau_j}} > x\,,\ \tau_i \le t\,,\ \mathrm{e}^{-R_{\tau_j}} \le a(x)) \\ &\quad + \mathbb{P}(\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}} > a(x)) =: J_1 + o(1)\mathbb{P}(X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}} > x)\,, \end{aligned} \tag{4.6}$$

where $a(x)$ is the function defined in Lemma 4.2. For J_1, since $\{X_n\,,\ n \ge 1\}$ are pairwise NQD, it holds uniformly for all $t \in \Lambda$

$$\begin{aligned} J_1 &\le \mathbb{P}\left(X_i\,\mathrm{e}^{-R_{\tau_i}} > x\,,\ \tau_i \le t\,,\ X_j > \frac{x}{a(x)}\right) \\ &= \int_{0-}^{t}\int_{0}^{\infty} \mathbb{P}\left(X_i > \frac{x}{u}\,,\ X_j > \frac{x}{a(x)}\right)\mathbb{P}(\mathrm{e}^{-R_s}\in \mathrm{d}u)\,\mathbb{P}(\tau_i\in \mathrm{d}s) \\ &\le \int_{0-}^{t}\int_{0}^{\infty} \overline{F}\left(\frac{x}{u}\right)\overline{F}\left(\frac{x}{a(x)}\right)\mathbb{P}(\mathrm{e}^{-R_s}\in \mathrm{d}u)\,\mathbb{P}(\tau_i\in \mathrm{d}s) \\ &= \overline{F}\left(\frac{x}{a(x)}\right)\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}} > x) = o(1)\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}} > x)\,.\end{aligned}$$

Plugging this estimate into (4.6), we finish the proof of the lemma.

Lemma 4.5. *Under the conditions of Theorem 2.2, for each $n \ge 1$, it holds uniformly for all $t \in \Lambda_\epsilon$*

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > x\right) \sim \sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}} > x)\,.$$

Proof. For any $0<\varepsilon<1$, we have

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right)\le\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>(1-\varepsilon)\,x) \tag{4.7}$$
$$+\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\,,\ \max_{1\le k\le n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}\le(1-\varepsilon)\,x\right)=:J_2+J_3\,.$$

We firstly deal with J_2. By Lemma 4.3 we have

$$J_2\lesssim(1-\varepsilon)^{-\beta}\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\,, \tag{4.8}$$

holds uniformly for all $t\in\Lambda_\epsilon$. For J_3, by Lemmas 4.4 and 4.3, we have

$$\begin{aligned}
J_3 &\le \mathbb{P}\left(\bigcup_{i=1}^{n}\left\{X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}}>\frac{x}{n}\,,\ \sum_{1\le j\le n,\,j\ne i} X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}}>\varepsilon\,x\right\}\right)\\
&\le \sum_{i=1}^{n}\sum_{1\le j\le n,\,j\ne i}\mathbb{P}\left(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}}>\frac{\varepsilon\,x}{n}\,,\ X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}}>\frac{\varepsilon x}{n}\right)\\
&= o(1)\sum_{i=1}^{n}\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}}>x)\,,
\end{aligned} \tag{4.9}$$

holds uniformly for all $t\in\Lambda_\epsilon$. From (4.8)–(4.9), it holds uniformly for all $t\in\Lambda_\epsilon$

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right)\lesssim(1-\varepsilon)^{-\beta}\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\,. \tag{4.10}$$

Then, the upper bound follows from (4.10) and the arbitrariness of ε.

For the lower bound, according to Bonferroni inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k=1}^{n} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right)\ge\mathbb{P}\left(\bigcup_{k=1}^{n}\left\{X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right\}\right)\\
&\ge\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\\
&\quad-\sum_{1\le i<j\le n}\mathbb{P}(X_i\,\mathrm{e}^{-R_{\tau_i}}\,\mathbf{1}_{\{\tau_i\le t\}}>x\,,\ X_j\,\mathrm{e}^{-R_{\tau_j}}\,\mathbf{1}_{\{\tau_j\le t\}}>x)\\
&\sim\sum_{k=1}^{n}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\,,
\end{aligned}$$

holds uniformly for all $t\in\Lambda$, where we used Lemma 4.4 in the last step.

The last lemma can be found in [15].

Lemma 4.6. *Under the conditions of Theorem 2.2, it holds*

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{x\to\infty}\sup_{t\in\Lambda}\frac{\mathbb{P}\left(\sum_{k=n+1}^{\infty}X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right)}{\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1\le t\}}>x)}\\
&\quad=\lim_{n\to\infty}\limsup_{x\to\infty}\sup_{t\in\Lambda}\frac{\sum_{k=n+1}^{\infty}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)}{\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1\le t\}}>x)}=0\,.
\end{aligned}$$

Proof of Theorem 2.2. For any $\varepsilon>0$, according to Lemma 4.6, there exists some sufficiently large integer n_0 such that uniformly for all $t\in\Lambda$,

$$\begin{aligned}
&\max\left\{\mathbb{P}\left(\sum_{k=n_0+1}^{\infty}X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right),\sum_{k=n_0+1}^{\infty}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\right\}\\
&\lesssim\varepsilon\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1\le t\}}>x)\,.
\end{aligned}\tag{4.11}$$

For the upper bound, by Lemma 4.5, (4.11) and Lemma 4.3, we have that for any $0<\delta<1$,

$$\begin{aligned}
\mathbb{P}(D(t)>x)&\le\mathbb{P}\left(\sum_{k=1}^{n_0}X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>(1-\delta)\,x\right)\\
&\quad+\mathbb{P}\left(\sum_{k=n_0+1}^{\infty}X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>\delta\,x\right)\\
&\lesssim\sum_{k=1}^{n_0}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>(1-\delta)\,x)+\varepsilon\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1\le t\}}>\delta\,x)\\
&\lesssim(1-\delta)^{-\beta}\sum_{k=1}^{n_0}\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)+\varepsilon\,\delta^{-\beta}\,\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1\le t\}}>x)\\
&\le\left((1-\delta)^{-\beta}+\varepsilon\,\delta^{-\beta}\right)\int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s}>x)\,\lambda(\mathrm{d}s)\,,
\end{aligned}\tag{4.12}$$

holds uniformly for all $t\in\Lambda_\epsilon$. Thus, the upper bound of (2.1) follows from (4.12) by letting firstly $\varepsilon\downarrow 0$, then $\delta\downarrow 0$.

For the lower bound, we obtain from Lemma 4.5 and (4.11)

$$\begin{aligned}
\mathbb{P}(D(t)>x)&\ge\mathbb{P}\left(\sum_{k=1}^{n_0}X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x\right)\\
&\sim\left(\sum_{k=1}^{\infty}-\sum_{k=n_0+1}^{\infty}\right)\mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k\le t\}}>x)\\
&\ge(1-\varepsilon)\int_{0-}^{t}\mathbb{P}(X_1\,\mathrm{e}^{-R_s}>x)\,\lambda(\mathrm{d}s)\,,
\end{aligned}\tag{4.13}$$

holds uniformly for all $t\in\Lambda_\epsilon$. It ends the proof of the theorem.

Proof of Corollary 2.2. Clearly, Theorem 2.2 implies, that the upper bound (3.22) holds uniformly for all $t \in \Lambda_\epsilon$.

We estimate the lower bound for the ruin probability. For any $\varepsilon > 0$ and the integer n_0 defined in the proof of Theorem 2.2, we have

$$\begin{aligned} \psi(x,\,t) &\geq \mathbb{P}(D(t) - M\,Z > x) \qquad (4.14)\\ &\geq \mathbb{P}\left(\sum_{k=1}^{n_0} X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k \leq t\}} > (1+\varepsilon)\,x\right) - \mathbb{P}\left(Z > \frac{\varepsilon\,x}{M}\right) =: J_4 - J_5\,, \end{aligned}$$

where $Z = \int_{0-}^{\infty} \mathrm{e}^{-R_s}\,\mathrm{d}s$ and $M > 0$ is the upper bound of the intensity of premium payments. As done in (4.13), by Lemmas 4.5, 4.3 and (4.11), we get that

$$\begin{aligned} J_4 &\gtrsim \sum_{k=1}^{n_0} \mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k \leq t\}} > (1+\varepsilon)\,x)\\ &\gtrsim (1+\varepsilon)^{-\beta} \sum_{k=1}^{n_0} \mathbb{P}(X_k\,\mathrm{e}^{-R_{\tau_k}}\,\mathbf{1}_{\{\tau_k \leq t\}} > x)\\ &\geq (1+\varepsilon)^{-\beta}\,(1-\varepsilon) \int_{0-}^{t} \mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)\,, \qquad (4.15) \end{aligned}$$

holds uniformly for all $t \in \Lambda_\epsilon$. For J_5, by Markov's inequality and Lemmas 3.2, 2.2, we have

$$J_5 \leq \left(\frac{M}{\varepsilon}\right)^r \mathbb{E}[Z^r]\,x^{-r} = o(\overline{F}(x))\,. \qquad (4.16)$$

If $\mathbb{P}(\theta_1 = 0) > 0$, then by Lemma 4.1, it holds that uniformly for all $t \in \Lambda$

$$\overline{F}(x) \leq (\mathbb{P}(\theta_1 = 0))^{-1}\overline{F}(x)\,\mathbb{P}(\theta_1 \leq t) = O(1)\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1 \leq t\}} > x)\,,$$

which, together with (4.16), yields that uniformly for all $t \in \Lambda$,

$$J_5 = o(1)\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1 \leq t\}} > x) = o(1)\int_{0-}^{t} \mathbb{P}(X_1\,\mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)\,. \qquad (4.17)$$

Similarly, if $\mathbb{P}(\theta_1 = 0) = 0$, also by Lemma 4.1, it holds uniformly for all $t \in \Lambda_\epsilon$

$$\overline{F}(x) \leq (\mathbb{P}(\theta_1 \leq \epsilon))^{-1}\overline{F}(x)\,\mathbb{P}(\theta_1 \leq t) = O(1)\mathbb{P}(X_1\,\mathrm{e}^{-R_{\tau_1}}\,\mathbf{1}_{\{\tau_1 \leq t\}} > x)\,,$$

which implies, that (4.17) holds uniformly for all $t \in \Lambda_\epsilon$. Combining (4.15)-(4.17) and noticing the arbitrariness of ε, we can derive the lower bound.

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