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Asymptotics for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks*

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Abstract

Let us consider a discrete-time insurance risk model with insurance and financial risks, where the net insurance loss within period i and the stochastic discount factor from time i to time $i-1$ follow some certain dependence structure for each fixed $i \geq 1$. Under the assumption that the distribution of net insurance loss within one time period is consistently-varying-tailed, precise estimates for finite and infinite time ruin probabilities are derived. Furthermore, these estimates are uniform with respect to the time horizon.

Keywords: Asymptotics; consistent variation; financial and insurance risks; finite and infinite time ruin probabilities; dependence

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1 Introduction

In this paper, we consider a discrete-time insurance risk model with financial and insurance risks in a stochastic economic environment, which was proposed by Nyrhinen (1999, 2001). In this environment, within period i, $i \geq 1$, the insurer's net insurance loss (i.e. the total claim amount minus the total premium income) is denoted by a real-valued random variable (r.v.) X_i. Suppose that the insurer invests his/her surplus into both risk-free and risky assets, which lead to an overall positive stochastic discount factor Y_i from time i to time $i-1$. In the terminology of Norberg (1999), r.v.s $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are called insurance risks and financial risks, respectively. For each positive integer n, the sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \,, \tag{1.1}$$

represents the stochastic discounted value of aggregate net losses up to time n. In this paper we are interested in the finite time ruin probability by time n and the infinite time ruin probability,

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which are defined, respectively, by

$$\psi(x,\,n) = \mathbb{P}\left(\max_{1\le k\le n} S_k > x\right), \tag{1.2}$$

$$\psi(x) = \lim_{n\to\infty} \psi(x,\,n) = \mathbb{P}\left(\sup_{n\ge 1} S_n > x\right), \tag{1.3}$$

where $x \ge 0$ is interpreted as the initial wealth of the insurer.

Many works have been devoted to the investigation of the asymptotic behavior of ruin probabilities in the above discrete-time insurance risk model, assuming that both $\{X_i,\ i\ge 1\}$ and $\{Y_i,\ i\ge 1\}$ are sequences of independent and identically distributed (i.i.d.) r.v.s and they are independent of each other as well. One can be referred to Tang and Tsitsiashvili (2003a, 2004), Hashorva et al. (2010) among others. Clearly, if we denote the product $\prod_{j=1}^{i} Y_j$ in (1.1) by a weight r.v. θ_i, then the investigation on ruin probabilities $\psi(x,\,n)$ and $\psi(x)$ boils down to the study of the asymptotics for the tail probabilities of the maximum of randomly weighted sums. Following the work of Tang and Tsitsiashvili (2003b), there has been a vast amount of literature in this aspect, such as Goovaerts et al. (2005), Wang et al. (2005), Wang and Tang (2006). We remark that the assumption of complete independence is for the mathematical convenience, but appears unrealistic in most practical situations. A recent new trend of the study is to introduce various dependence structures to the risk model. One direction is to allow some dependent $\{X_i,\ i\ge 1\}$ and arbitrarily dependent $\{\theta_i,\ i\ge 1\}$, but keep the independence between $\{X_i,\ i\ge 1\}$ and $\{\theta_i,\ i\ge 1\}$, see Zhang et al. (2009), Chen and Yuen (2009), Shen et al. (2009), Weng et al. (2009), Gao and Wang (2010), Yi et al. (2011) among others. In the presence of heavy-tailed insurance risks, they established the asymptotic formula

$$\psi(x,\,n) \sim \sum_{i=1}^{n} \mathbb{P}(X_i\,\theta_i > x) = \sum_{i=1}^{n} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j > x\Big),\ x\to\infty, \tag{1.4}$$

holds for each fixed n, or

$$\psi(x) \sim \sum_{i=1}^{\infty} \mathbb{P}(X_i\,\theta_i > x),\ x\to\infty. \tag{1.5}$$

In such a type of dependent risk model, Shen et al. (2009) obtained the asymptotics for the finite time ruin probability is uniform with respect to the time horizon (i.e. relation (1.4) holds uniformly for all $n \ge 1$), hence apply for the case of infinite time ruin (i.e. relation (1.5) holds).

Recently, Chen (2011) considered another type of the discrete-time risk model by allowing the dependence between insurance and financial risks. Precisely speaking, she assumed that $\{(X_i,\,Y_i),\ i\ge 1\}$ form a sequence of i.i.d. random vectors with generic random vector $(X,\,Y)$ following a bivariate Farlie-Gumbel-Morgenstern distribution. Under the condition that the net loss (insurance risk) distribution is subexponential, she derived the asymptotic formula (1.4) for the finite time ruin probability. Some related results can be found in Yang et al. (2012a) and Yang and Hashorva (2013).

Motivated by Chen (2011), in the present paper we aim to allow the generic random vector $(X,\,Y)$ following a more general dependence structure. We firstly derive two general asymptotic formulas for the ruin probabilities in (1.2) and (1.3) under the condition that the loss distribution is consistently-varying-tailed (see the definition in Section 2). In doing so, each pair of the net loss (insurance risk) and stochastic discount factor (financial risk) follows Assumptions A_1–A_3 in Section 2. Then we investigate the uniformly asymptotic behavior of the ruin probabilities with respect to $n \ge 1$. Finally, we pursue some refinements of the general formulas by restricting the loss distribution to be regularly-varying-tailed. The obtained formulas are further refined to be completely transparent. All estimates obtained for ruin probabilities successfully capture the impact of the underlying dependence structure of (X, Y).

The rest of this paper is organized as follows. Section 2 prepares some preliminaries of heavy-tailed distributions and dependence structures, Section 3 presents the main results of the present paper and Section 4 gives their proofs.

2 Preliminaries

Throughout the paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) = O[b(x)]$ if $\limsup a(x)/b(x) < \infty$; write $a(x) \asymp b(x)$ if $a(x) = O[b(x)]$ and $b(x) = O[a(x)]$; and $a(x) = o[b(x)]$ if $\lim a(x)/b(x) = 0$. Furthermore, for two positive bivariate functions $a(x, t)$ and $b(x, t)$, we write $a(x, t) \sim b(x, t)$ uniformly for all t in a nonempty set A, if

$$\lim_{x\to\infty} \sup_{t\in A} \left| \frac{a(x, t)}{b(x, t)} - 1 \right| = 0 .$$

The indicator function of an event A is denoted by $\mathbb{1}_A$. For real x and y, denote by $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and $x^+ = x \vee 0$. C represents a positive constant, whose value may vary from place to place.

We shall restrict the net loss (insurance risk) distribution to some classes of heavy-tailed distributions. An important class of heavy-tailed distributions is $\mathscr{D}$, which consists of all distributions with dominated variation. A distribution $F = 1 - \overline{F}$ on $\mathbb{R}$ belongs to the class $\mathscr{D}$, if $\limsup \overline{F}(x\,y)/\overline{F}(x) < \infty$ for any $0 < y < 1$. A slightly smaller class is $\mathscr{C}$ of consistently-varying-tailed distributions. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{C}$, if $\lim_{y\downarrow 1} \liminf \overline{F}(x\,y)/\overline{F}(x) = 1$. Closely related distribution class is the class $\mathscr{L}$ of long-tailed distributions. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{L}$, if $\overline{F}(x + y) \sim \overline{F}(x)$ for any $y > 0$. A useful subclass of the class $\mathscr{C}$ is the class of regularly varying tailed distributions, denoted by $\mathscr{R}$. A distribution F on $\mathbb{R}$ belongs to the class $\mathscr{R}_{-\alpha}$, if $\lim \overline{F}(x\,y)/\overline{F}(x) = y^{-\alpha}$ for some $\alpha \geq 0$ and all $y > 0$. It is well known that the following inclusion relationships hold:

$$\mathscr{R}_{-\alpha} \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L}$$

(see, e.g., Cline and Samorodnitsky (1994)). Furthermore, for a distribution F on $\mathbb{R}$, denote its upper and lower Matuszewska indices, respectively, by

$$J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y} \quad \text{with} \quad \overline{F}_*(y) := \liminf \frac{\overline{F}(x\,y)}{\overline{F}(x)} \quad \text{for} \quad y > 1 ,$$

$$J_F^- = -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y} \quad \text{with} \quad \overline{F}^*(y) := \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} \quad \text{for} \quad y > 1 .$$

Clearly, $F \in \mathscr{D}$ is equivalent to $J_F^+ < \infty$. For more details, see Bingham et al. (1987, Chapter 2.1). The following property of the class $\mathscr{D}$ is due to Lemma 3.5 of Tang and Tsitsiashvili (2003a).

Lemma 2.1. *For a distribution $F \in \mathscr{D}$ on $\mathbb{R}$, $x^{-p} = o[\overline{F}(x)]$ holds for any $p > J_F^+$.*

We next introduce the dependence structure between the net loss and the stochastic discount factor via some restriction on their copula function. Let (X, Y) be a random vector with continuous marginal distributions $F(x)$ and $G(y)$, then the dependence structure of X and Y is characterized in terms of a bivariate copula function $C(u, v)$ by the Sklar representation (see Nelsen (1998) or Joe (1997)), and the joint distribution of X and Y can be given by $C[F(x), G(y)]$. Define the corresponding survival copula function by

$$\hat{C}(u, v) = u + v - 1 + C(u, v), \ (u, v) \in [0, 1]^2 .$$

Assume that the copula function $C(u, v)$ is absolutely continuous. Denote by $C_1(u, v) = \frac{\partial}{\partial u} C(u, v)$, $C_2(u, v) = \frac{\partial}{\partial v} C(u, v)$, $C_{12}(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v)$, $\hat{C}_2(u, v) = \frac{\partial}{\partial v} \hat{C}(u, v) = 1 - C_2(1 - u, 1 - v)$ and $\hat{C}_{12}(u, v) = \frac{\partial^2}{\partial u \partial v} \hat{C}(u, v) = C_{12}(1 - u, 1 - v)$.

Recently, in the study of the closure and tail behavior of the product of two dependent r.v.s, Yang and Sun (2013) utilized the following two assumptions on the absolutely continuous copula function $C(u, v)$. We remark that Assumption A_1 can be attributed to Albrecher et al. (2006) and A_2 to Asimit and Badescu (2010).

Assumption $\mathbf{A}_1$. *There exists a positive constant M such that*

$$\limsup_{v \uparrow 1} \limsup_{u \uparrow 1} C_{12}(u, v) < M .$$

Assumption $\mathbf{A}_2$. *The relation*

$$\hat{C}_2(u, v) \sim u \, \hat{C}_{12}(0+, v), \ u \downarrow 0 ,$$

holds uniformly for all $v \in (0, 1]$.

Remark 2.1. *Clearly, Assumption A_2 is equivalent to*

$$1 - C_2(u, v) \sim (1 - u) \, C_{12}(u, v), \ u \uparrow 1 ,$$

holds uniformly for all $v \in [0, 1)$. Thus, if the copula $C(u, v)$ of random vector (X, Y) satisfies Assumptions A_1 and A_2, then the copula of (X^+, Y), denoted by $C^+(u, v)$, satisfies these two assumptions as well.

Similar assumptions related to Assumption A_1 can be found in Ko and Tang (2008). Pointed out by Yang and Sun (2013), some commonly used copula functions satisfy Assumptions A_1 and A_2 such as the Ali-Mikhail-Haq copula of the form

$$C(u, v) = \frac{u \, v}{1 - r \, (1 - u) \, (1 - v)} , \ r \in (-1, 1) ,$$

the Farlie-Gumbel-Morgenstern copula

$$C(u, v) = u \, v + r \, u \, v \, (1 - u) \, (1 - v), \ r \in (-1, 1) ,$$

and the Frank copula

$$C(u, v) = -\frac{1}{r} \log \Big(1 + \frac{(e^{ru} - 1) \, (e^{rv} - 1)}{e^{-r} - 1} \Big), \ r \neq 0 .$$

In order to derive our main results, we further need the third assumption below.

Assumption $\mathbf{A}_3$. *The relation*

$$C_2(u, v) \to 0, \ u \downarrow 0 ,$$

holds uniformly for all $v \in [0, 1]$.

It is easy to verify that Assumption A_3 is satisfied by the above three copula functions. We remark that such an assumption is equivalent to the fact that

$$\mathbb{P}(X \le x \mid Y = y) = C_2[F(x), G(y)] \to 0 , \ x \to -\infty , \tag{2.1}$$

holds uniformly for all $y \in \mathbb{R}$.

3 Main results

Recall the discrete-time insurance risk model described in Section 1. In the sequel, denote by F on $\mathbb{R}$, G on $\mathbb{R}^+$ and H on $\mathbb{R}$ the distributions of the net loss X, the stochastic discount factor Y and their product XY, respectively. For each $i \geq 1$, denote by H_i the distribution of $X_i \prod_{j=1}^{i} Y_j,\ j \geq 1$. Clearly, $H_1 = H$.

Now we state the main results of the paper. The first two theorems derive some asymptotics for the finite and infinite time ruin probabilities, respectively.

Theorem 3.1. *In the discrete-time risk model described in Section 1, assume that* $\{(X_i, Y_i),\ i \geq 1\}$ *are i.i.d. random vectors with generic random vector* (X, Y) *satisfying Assumptions* A_1*–*A_3*. If* $F \in \mathscr{C}$ *and* $\mathbb{E}Y^p < \infty$ *for some* $p > J_F^+$*, then, for each fixed* $n \geq 1$*, it holds that*

$$\psi(x, n) \sim \sum_{i=1}^{n} \overline{H}_i(x)\,. \tag{3.1}$$

Theorem 3.2. *Under the conditions of Theorem 3.1, if* $J_F^- > 0$ *and* $\mathbb{E}Y^p < 1$ *for some* $p > J_F^+$*, then it holds that*

$$\psi(x) \sim \sum_{i=1}^{\infty} \overline{H}_i(x)\,. \tag{3.2}$$

The third result below shows that the asymptotic relation (3.1) for the finite time probability is uniform with respect to $n \geq 1$.

Theorem 3.3. *Under the conditions of Theorem 3.2, (3.1) holds uniformly for all* $n \geq 1$*.*

We now return to an important special case of the above three theorems in which we assume that $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \geq 0$. By using the well-known Breiman theorem (see Breiman (1965) and Cline and Samorodnitsky (1994)), we focus on improving the asymptotic formulas given in (3.1) and (3.2) to be completely transparent. Define a nonnegative r.v. Y_c, independent of $\{X, X_i,\ i \geq 1\}$ and $\{Y, Y_i,\ i \geq 1\}$, with the distribution G_c, given by

$$G_c(\mathrm{d}y) = C_1[1-, G(\mathrm{d}y)] = C_{12}[1-, G(y)]G(\mathrm{d}y)\,. \tag{3.3}$$

Since Y is a positive r.v., by Assumption A_2 we know that Y_c is not degenerate at zero.

Corollary 3.1. *(1) Under the conditions of Theorem 3.1, if* $F \in \mathscr{R}_{-\alpha}$ *for some* $\alpha \geq 0$*, then, for each fixed* $n \geq 1$*, it holds that*

$$\psi(x, n) \sim \mathbb{E}Y_c^\alpha \, \frac{1 - (\mathbb{E}Y^\alpha)^n}{1 - \mathbb{E}Y^\alpha}\, \overline{F}(x)\,, \tag{3.4}$$

by convention, $(1 - (\mathbb{E}Y^\alpha)^n)/(1 - \mathbb{E}Y^\alpha) = n$ *if* $\mathbb{E}Y^\alpha = 1$*.*

(2) Under the conditions of Theorem 3.2, if $F \in \mathscr{R}_{-\alpha}$ *for some* $\alpha > 0$*, then*

$$\psi(x) \sim \frac{\mathbb{E}Y_c^\alpha}{1 - \mathbb{E}Y^\alpha}\, \overline{F}(x)\,. \tag{3.5}$$

Moreover, (3.4) *holds uniformly for all* $n \geq 1$*.*

4 Proofs

We start with some lemmas. The first one presents a property of the distribution with dominated variation, which is due to Zhou et al. (2012) (see Prop. 3.1).

Lemma 4.1. *A distribution $F \in \mathscr{D}$ if and only if for any distribution G on $\mathbb{R}^+$ satisfying $\overline{G}(x) = o[\overline{F}(x)]$, there exists a positive function $g(\cdot)$ such that*

$$g(x) \downarrow 0, \quad x\,g(x) \uparrow \infty \quad \text{and} \quad \overline{G}[x\,g(x)] = o[\overline{F}(x)]\,.$$

The next two lemmas are related to the product of two r.v.s.

Lemma 4.2. *Let X be a real-valued r.v. with distribution F, and Y be a nonnegative and nondegenerate at zero r.v., independent of X, with distribution G. If $F \in \mathscr{C}$ and $\overline{G}(x) = o[\overline{F}(x)]$, then the distribution H of the product $X\,Y$ belongs to the class $\mathscr{C}$ as well.*

Proof. By Lemma 3.3 (iii) of Yang et al. (2012b), we have that for any fixed $y \in (0,\,1]$

$$1 \le \limsup \frac{\overline{H}(x\,y)}{\overline{H}(x)} \le \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)}\,,$$

which, combined with $F \in \mathscr{C}$, implies $H \in \mathscr{C}$. □

Lemma 4.3. *Let X be a real-valued r.v. with distribution F, and Y be a positive r.v. with distribution G. Assume that X and Y are dependent according to the copula function $C(u,\,v)$ satisfying Assumptions A_1 and A_2. (1) If there exists a positive function $g(\cdot)$ such that*

$$g(x) \downarrow 0\,, \quad x\,g(x) \uparrow \infty \quad \text{and} \quad \overline{G}[x\,g(x)] = o[\overline{H}(x)]\,, \tag{4.1}$$

then

$$\overline{H}(x) \sim \mathbb{P}(X\,Y_c > x)\,, \tag{4.2}$$

where Y_c is a nonnegative and non-degenerate at zero r.v., independent of X, with distribution defined in (3.3).

(2) If $F \in \mathscr{D}$ and $\overline{G}(x) = o[\overline{F}(x)]$, then (4.2) *holds.*

Proof. (1) The proof of this part of the lemma can be found in Yang and Sun (2013), see Lemma 4.2 of that paper.

(2) Note that Y_c is a nonnegative and non-degenerate at zero r.v., then there exists a positive constant a such that $\mathbb{P}(Y_c > a) > 0$. By Assumption A_2 we have that

$$\begin{aligned}
\overline{H}(x) &\ge \mathbb{P}(X\,Y > x,\, Y > a) \ge \int_a^\infty \mathbb{P}\Big(X > \frac{x}{a}\;\Big|\; Y = u\Big) G(\mathrm{d}u) \\
&= \int_a^\infty \hat{C}_2\Big[\overline{F}\Big(\frac{x}{a}\Big),\, \overline{G}(u)\Big]\, G(\mathrm{d}u) \\
&\sim \overline{F}\Big(\frac{x}{a}\Big) \int_a^\infty C_{12}[1-,\, G(u)]\, G(\mathrm{d}u) = \mathbb{P}(Y_c > a)\, \overline{F}\Big(\frac{x}{a}\Big)\,,
\end{aligned}$$

which, combined with $F \in \mathscr{D}$, implies $\overline{F}(x) = O[\overline{H}(x)]$. According to Lemma 4.1, there exists a positive function $g(\cdot)$ such that (4.1) is satisfied. Therefore, the desired (4.2) follows from statement (1). □

In the following lemma from Pakes (2007), we find a sufficient condition on the equivalence of two differences.

Lemma 4.4. *Let $f_i(\cdot)$, $i = 1, 2, 3, 4$, be four positive functions. If $f_1(x) = O[f_3(x)]$, $f_2(x) = O[f_4(x)]$, $f_4(x) = O[f_3(x)]$ and $f_1(x) + f_2(x) \sim f_3(x) + f_4(x)$, then it holds that $f_1(x) \sim f_3(x)$.*

PROOF OF THEOREM 3.1. Clearly, for each $n \geq 1$

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x\Big) \leq \psi(x, n) \leq \mathbb{P}\Big(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\Big),$$

which indicates that the desired (3.1) holds if we can establish

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x\Big) \sim \sum_{i=1}^{n} \overline{H}_i(x), \tag{4.3}$$

and

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\Big) \sim \sum_{i=1}^{n} \overline{H}_i(x). \tag{4.4}$$

We deal with relation (4.3). Since $\{(X_i, Y_i),\ i \geq 1\}$ are i.i.d. random vectors, it holds that

$$\sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j =: T_n, \ n \geq 1,$$

where $\stackrel{\mathrm{d}}{=}$ stands for equality in distribution. Thus, for (4.3), it is sufficient to prove

$$\mathbb{P}(T_n > x) \sim \sum_{i=1}^{n} \overline{H}_i(x). \tag{4.5}$$

We will prove $F_{T_n} \in \mathscr{C}$ and (4.5) by induction on n. Since $\mathbb{E}Y^p < \infty$ for some $p > J_F^+$, then by Markov inequality and Lemma 2.1 we know

$$\overline{G}(x) \leq x^{-p}\, \mathbb{E}Y^p = o[\overline{F}(x)], \tag{4.6}$$

which, according to Lemma 4.3, implies that (4.2) holds. Recall that Y_c is a nonnegative and non-degenerate at zero r.v., independent of $\{X, X_i,\ i \geq 1\}$ and $\{Y, Y_i,\ i \geq 1\}$, with distribution defined in (3.3). As done in the proof of Theorem 3.1 of Yang and Sun (2013), by Assumption A_1,

$$\limsup \frac{\overline{G_c}(x)}{\overline{G}(x)} = \limsup \frac{1 - C_1[1-, 1 - \overline{G}(x)]}{\overline{G}(x)} = \limsup_{u\uparrow 1} \limsup_{v \uparrow 1} C_{12}(u, v) < M, \tag{4.7}$$

which, combined with (4.6), implies $\overline{G_c}(x) = o[\overline{F}(x)]$. Thus, by (4.2), $F \in \mathscr{C}$ and Lemma 4.2, $T_1 = X_1 Y_1$ has the distribution $H \in \mathscr{C}$ and (4.5) trivially holds for $n = 1$. We aim to prove $F_{T_{n+1}} \in \mathscr{C}$ and (4.5) holds for $n + 1$, by assuming that $F_{T_n} \in \mathscr{C}$ and (4.5) holds for n. For each $i \geq 2$, since $X_i Y_i$ is independent of $Y_1, \ldots, Y_{i-1}$, by $\mathbb{E}(\prod_{j=1}^{i-1} Y_j)^p = (\mathbb{E}Y^p)^{i-1} < \infty$ for some $p > J_F^+$ and $H \in \mathscr{C}$, as done in (4.6), we can get that $H_i \in \mathscr{C}$. In addition, by (4.2), $F \in \mathscr{C} \subset \mathscr{D}$ and $\mathbb{E}Y^p < \infty$ (hence, $\mathbb{E}Y_c^p < \infty$ by (4.7)) for some $p > J_F^+$, according to Theorem 3.3 (iv) of Cline and Samorodnitsky (1994), we have $\overline{H}(x) \asymp \overline{F}(x)$. This, together with (4.6) and Lemma 4.1, yields that there exists a positive function $g(\cdot)$ such that (4.1) holds. By $H_i \in \mathscr{C} \subset \mathscr{L}$, $i = 1, \ldots, n$, there exists a positive function $h(x) \uparrow \infty$ such that

$$g^{-1}(x) - h(x) \to \infty \text{ and } \overline{H}_i[g^{-1}(x) - u] \sim \overline{H}_i[g^{-1}(x)] \tag{4.8}$$

holds uniformly for all $|u| \le h(x)$. For each $n \ge 1$, since (X_{n+1}, Y_{n+1}) is independent of T_n, we divide the tail probability $\mathbb{P}(T_{n+1} > x)$ into four parts:

$$\begin{aligned}\mathbb{P}(T_{n+1} > x) &= \mathbb{P}((T_n + X_{n+1})\, Y_{n+1} > x) \\ &= \left(\int_{-h(x)}^{h(x)} \int_0^{xg(x)} + \int_{h(x)}^{\infty} \int_0^{xg(x)} + \int_{-\infty}^{\infty} \int_{xg(x)}^{\infty} + \int_{-\infty}^{-h(x)} \int_0^{xg(x)}\right) \\ &\quad \mathbb{P}\Big(T_n > \frac{x}{v} - u\Big)\, \mathbb{P}(X_{n+1} \in \mathrm{d}u,\, Y_{n+1} \in \mathrm{d}v) =: I_1 + I_2 + I_3 + I_4. \qquad (4.9)\end{aligned}$$

Clearly, by (4.1),

$$I_3 \le \overline{G}[x\, g(x)] = o[\overline{H}(x)]\,. \qquad (4.10)$$

By the inductive assumption (4.5), (4.8) and (4.1), we have

$$\begin{aligned} I_1 &\sim \sum_{i=1}^{n} \int_{-h(x)}^{h(x)} \int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{v} - u\Big)\, \mathbb{P}(X_{n+1} \in \mathrm{d}u,\, Y_{n+1} \in \mathrm{d}v) \\ &\sim \sum_{i=1}^{n} \int_{-h(x)}^{h(x)} \int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{v}\Big)\, \mathbb{P}(X_{n+1} \in \mathrm{d}u,\, Y_{n+1} \in \mathrm{d}v) \\ &= \sum_{i=1}^{n} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j \cdot Y_{i+1} > x,\, -h(x) < X_{i+1} \le h(x),\, Y_{i+1} \le x\, g(x)\Big) \\ &= \sum_{i=2}^{n+1} \overline{H}_i(x) - \sum_{i=1}^{n} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j \cdot Y_{i+1} > x,\, X_{i+1} > h(x),\, Y_{i+1} \le xg(x)\Big) \\ &\quad - \sum_{i=1}^{n} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j \cdot Y_{i+1} > x,\, X_{i+1} \le -h(x),\, Y_{i+1} \le x\, g(x)\Big) + o[\overline{H}(x)] \\ &=: I_{11} - I_{12} - I_{13} + o[\overline{H}(x)]\,. \qquad (4.11)\end{aligned}$$

By Assumption A_3 (i.e. (2.1)), we have

$$\begin{aligned} I_{13} &= \sum_{i=1}^{n} \int_0^{xg(x)} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j > \frac{x}{u}\Big)\, \mathbb{P}(X_{i+1} \le -h(x) \mid Y_{i+1} = u)\, G(\mathrm{d}u) \\ &= o(1) \sum_{i=1}^{n} \int_0^{x\, g(x)} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j > \frac{x}{u}\Big)\, G(\mathrm{d}u) = o\left[\sum_{i=2}^{n+1} \overline{H}_i(x)\right]. \qquad (4.12)\end{aligned}$$

As for I_2, Since $h(x) \uparrow \infty$, by Assumptions A_2, A_1 and (4.1), we obtain

$$\begin{aligned}
I_2 &= \mathbb{P}((T_n + X_{n+1})\, Y_{n+1} > x,\, X_{n+1} > h(x),\, Y_{n+1} \le x g(x)) \\
&= \int_0^\infty \int_0^{xg(x)} \mathbb{P}\Big(X_{n+1} > h(x) \vee (\frac{x}{v} - u) \;\Big|\; Y_{n+1} = v\Big)\, G(\mathrm{d}v)\, \mathbb{P}(T_n \in \mathrm{d}u) \\
&= \int_0^\infty \int_0^{xg(x)} \hat{C}_2\Big[\overline{F}\Big(h(x) \vee (\frac{x}{v} - u)\Big),\, \overline{G}(v)\Big]\, G(\mathrm{d}v)\, \mathbb{P}(T_n \in \mathrm{d}u) \\
&\sim \int_0^\infty \int_0^{xg(x)} \overline{F}\Big(h(x) \vee (\frac{x}{v} - u)\Big)\, C_{12}[1-, G(v)]\, G(\mathrm{d}v)\, \mathbb{P}(T_n \in \mathrm{d}u) \\
&= \int_0^\infty \int_0^{\infty} \overline{F}\Big(h(x) \vee (\frac{x}{v} - u)\Big)\, C_{12}[1-, G(v)]\, G(\mathrm{d}v)\, \mathbb{P}(T_n \in \mathrm{d}u) + o[\overline{H}(x)] \\
&= \mathbb{P}((X_{n+1} + T_n)\, Y_c > x,\, X_{n+1} > h(x)) + o[\overline{H}(x)] \\
&= \mathbb{P}((X_{n+1} + T_n)\, Y_c > x) - \mathbb{P}((X_{n+1} + T_n)\, Y_c > x,\, -h(x) < X_{n+1} \le h(x)) \qquad (4.13) \\
&\quad -\mathbb{P}((X_{n+1} + T_n)\, Y_c > x,\, X_{n+1} \le -h(x)) + o[\overline{H}(x)] =: I_{21} - I_{22} - I_{23} + o[\overline{H}(x)]\,.
\end{aligned}$$

For each $i \ge 2$, since $X_i\, Y_i$ is independent of $Y_1, \ldots, Y_{i-1}$, by Lemma 3.3 (iii) of Yang et al. (2012b), it holds that for any $y > 1$,

$$[\overline{H}_i]_*(y) = \liminf \frac{\overline{H}_i(x\,y)}{\overline{H}_i(x)} \ge \liminf \frac{\overline{H}(x\,y)}{\overline{H}(x)} = \liminf \frac{\mathbb{P}(X\,Y_c > x\,y)}{\mathbb{P}(X\,Y_c > x)} \ge \liminf \frac{\overline{F}(x\,y)}{\overline{F}(x)} = \overline{F}_*(y)\,,$$

which implies that for each $i \ge 1$, $J^+_{H_i} \le J^+_F$. For each $i \ge 1$, this, together with $H_i \in \mathscr{C}$ and $\mathbb{E}Y_c^p < \infty$ (by $\mathbb{E}Y^p < \infty$ and (4.7)) for some $p > J^+_F \ge J^+_{H_i}$, yields that

$$\mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j \cdot Y_c > x\Big) \asymp \overline{H}_i(x)\,, \qquad (4.14)$$

according to Theorem 3.3 (iv) of Cline and Samorodnitsky (1994). By Assumption A_1 and (4.1) we have

$$\overline{G_c}(x\,g(x)) = \int_{xg(x)}^\infty C_{12}[1-, G(u)]\, G(\mathrm{d}u) = o[\overline{H}(x)]\,. \qquad (4.15)$$

Since X_{n+1}, T_n and Y_c are mutually independent, then, by the inductive assumption (4.5), (4.15), (4.8) and (4.14), we can get

$$\begin{aligned}
I_{23} &= \int_{-\infty}^{-h(x)} \int_0^{xg(x)} \mathbb{P}\Big(T_n > \frac{x}{v} - u\Big)\, G_c(\mathrm{d}v)\, F(\mathrm{d}u) + O[\overline{G_c}(x\,g(x))] \\
&\sim \sum_{i=1}^n \int_{-\infty}^{-h(x)} \int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{v} - u\Big)\, G_c(\mathrm{d}v)\, F(\mathrm{d}u) + o[\overline{H}(x)] \\
&\le \sum_{i=1}^n \int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{v} + h(x)\Big)\, G_c(\mathrm{d}v)\, F(-h(x)) + o[\overline{H}(x)] \\
&\sim \sum_{i=1}^n \int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{v}\Big)\, G_c(\mathrm{d}v)\, F(-h(x)) + o[\overline{H}(x)] \\
&= o(1) \sum_{i=1}^n \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j \cdot Y_c > x\Big) + o[\overline{H}(x)] = o\left[\sum_{i=1}^{n+1} \overline{H}_i(x)\right]. \qquad (4.16)
\end{aligned}$$

We next deal with I_4. Similarly to (4.12), by Assumption A_3, the inductive assumption (4.5) and (4.8), we have

$$\begin{aligned} I_4 &= \mathbb{P}((T_n + X_{n+1})\,Y_{n+1} > x,\ X_{n+1} \le -h(x),\ Y_{n+1} \le xg(x)) \\ &\le \mathbb{P}((T_n - h(x))\,Y_{n+1} > x,\ X_{n+1} \le -h(x),\ Y_{n+1} \le x\,g(x)) \\ &= \int_0^{xg(x)} \mathbb{P}\Big(T_n > \frac{x}{u} + h(x)\Big)\,\mathbb{P}(X_{n+1} \le -h(x) \mid Y_{n+1} = u)\,G(\mathrm{d}u) \\ &= o(1)\sum_{i=1}^{n}\int_0^{xg(x)} \overline{H}_i\Big(\frac{x}{u}\Big)\,G(\mathrm{d}u) = o\left[\sum_{i=2}^{n+1}\overline{H}_i(x)\right]. \end{aligned} \tag{4.17}$$

Plugging (4.10)–(4.14), (4.16) and (4.17) into (4.9), we obtain

$$\mathbb{P}(T_{n+1} > x) \sim (I_{11} + I_{21}) - (I_{12} + I_{22}) + o\left[\sum_{i=1}^{n+1}\overline{H}_i(x)\right] =: J_1 - J_2 + o\left[\sum_{i=1}^{n+1}\overline{H}_i(x)\right] \tag{4.18}$$

Since X_{n+1} and T_n are independent, by using Theorem 2.2 of Cai and Tang (2004), together with $F \in \mathscr{C}$, the inductive assumptions $F_{T_n} \in \mathscr{C}$ and (4.5), (4.2) and (4.15), we obtain

$$\begin{aligned} J_1 &= \sum_{i=2}^{n+1}\overline{H}_i(x) + \left(\int_0^{xg(x)} + \int_{xg(x)}^{\infty}\right)\mathbb{P}\Big(X_{n+1} + T_n > \frac{x}{u}\Big)\,G_c(\mathrm{d}u) \\ &\sim \sum_{i=2}^{n+1}\overline{H}_i(x) + \int_0^{xg(x)}\left(\mathbb{P}\Big(X_{n+1} > \frac{x}{u}\Big) + \sum_{i=1}^{n}\overline{H}_i\Big(\frac{x}{u}\Big)\right)G_c(\mathrm{d}u) + o[\overline{H}(x)] \\ &\sim \sum_{i=1}^{n+1}\overline{H}_i(x) + \sum_{i=1}^{n}\mathbb{P}\Big(X_i\prod_{j=1}^{i}Y_j\cdot Y_c > x\Big). \end{aligned} \tag{4.19}$$

To estimate J_2, as done in (4.14), by (4.15), Assumption A_2, the inductive assumption (4.5), (4.8) and (4.14), we derive

$$\begin{aligned} J_2 &= \sum_{i=1}^{n}\int_0^{\infty}\mathbb{P}\Big(X_i\prod_{j=1}^{i}Y_j > \frac{x}{u}\Big)\,\mathbb{P}(X_{i+1} > h(x) \mid Y_{i+1} = u)\,G(\mathrm{d}u) \\ &\quad + \int_{-h(x)}^{h(x)}\int_0^{xg(x)}\mathbb{P}\Big(T_n > \frac{x}{v} - u\Big)\,G_c(\mathrm{d}v)\,F(\mathrm{d}u) + o[\overline{H}(x)] \\ &\sim \sum_{i=1}^{n}\int_0^{\infty}\mathbb{P}\Big(X_i\prod_{j=1}^{i}Y_j > \frac{x}{u}\Big)\overline{F}[h(x)]\,C_{12}[1-,\,G(u)]\,G(\mathrm{d}u) \\ &\quad + \sum_{i=1}^{n}\int_{-h(x)}^{h(x)}\int_0^{xg(x)}\overline{H}_i\Big(\frac{x}{v}\Big)\,G_c(\mathrm{d}v)\,F(\mathrm{d}u) + o[\overline{H}(x)] \\ &= \overline{F}[-h(x)]\sum_{i=1}^{n}\mathbb{P}\Big(X_i\prod_{j=1}^{i}Y_j\cdot Y_c > x\Big) + o[\overline{H}(x)] \sim \sum_{i=1}^{n}\mathbb{P}\Big(X_i\prod_{j=1}^{i}Y_j\cdot Y_c > x\Big). \end{aligned} \tag{4.20}$$

In Lemma 4.4, let consider $f_1(x) = J_1 - J_2$, $f_2(x) = J_2$, $f_3(x) = \sum_{i=1}^{n+1}\overline{H}_i(x)$ and $f_4(x) = \sum_{i=1}^{n}\mathbb{P}(X_i\prod_{j=1}^{i}Y_j\cdot Y_c > x)$. Then, by (4.19) and (4.21), clearly, $f_1(x) + f_2(x) = J_1 \sim f_3(x) +$

$f_4(x)$ and $f_2(x) \sim f_4(x)$; and by (4.14),

$$\begin{aligned}\limsup \frac{f_4(x)}{f_3(x)} &\le \limsup \frac{\sum_{i=1}^n \mathbb{P}(X_i \prod_{j=1}^i Y_j \cdot Y_c > x)}{\sum_{i=1}^n \overline{H}_i(x)} \\ &\le \limsup \bigvee_{i=1}^n \frac{\mathbb{P}(X_i \prod_{j=1}^i Y_j \cdot Y_c > x)}{\overline{H}_i(x)} < \infty\,, \qquad (4.21)\end{aligned}$$

which implies $f_4(x) = O(f_3(x))$. Next, we find that $f_1(x) = O(f_3(x))$. Indeed, by (4.19) and (4.21), we have

$$\limsup \frac{f_1(x)}{f_3(x)} \le \limsup \frac{J_1}{f_3(x)} = 1 + \limsup \frac{f_4(x)}{f_3(x)} < \infty\,.$$

Thus, all conditions in Lemma 4.4 are satisfied, hence, combined with (4.18), it holds that

$$\mathbb{P}(T_{n+1} > x) \sim J_1 - J_2 + o\left[\sum_{i=1}^{n+1} \overline{H}_i(x)\right] \sim \sum_{i=1}^{n+1} \overline{H}_i(x)\,.$$

This shows that the desired (4.5) holds for $n+1$. And by $H_i \in \mathscr{C}$, $i \ge 1$, it is easy to see $F_{T_{n+1}} \in \mathscr{C}$. It ends the proof of (4.3).

Finally, we turn to relation (4.4). Denote by $T_n^+ = \sum_{i=1}^n X_i^+ \prod_{j=i}^n Y_j$. The proof is same to that of (4.3) by noting Remark 2.1, and replacing (4.9), (4.11), (4.14), respectively, by

$$\begin{aligned}\mathbb{P}(T_{n+1}^+ > x) &= \left(\int_0^{h(x)} \int_0^{xg(x)} + \int_{h(x)}^{\infty} \int_0^{xg(x)} + \int_0^{\infty} \int_{xg(x)}^{\infty}\right) \\ &\quad \mathbb{P}\Big(T_n^+ > \frac{x}{v} - u\Big)\, \mathbb{P}(X_{n+1}^+ \in \mathrm{d}u, Y_{n+1} \in \mathrm{d}v) =: I_1 + I_2 + I_3\,,\end{aligned}$$

where

$$\begin{aligned} I_1 &\sim \sum_{i=2}^{n+1} \overline{H}_i(x) - \sum_{i=1}^{n} \mathbb{P}\Big(X_i^+ \prod_{j=1}^{i} Y_j \cdot Y_{i+1} > x,\, X_{i+1}^+ > h(x),\, Y_{i+1} \le x\, g(x)\Big) + o[\overline{H}(x)] \\ &=: I_{11} - I_{12} + o[\overline{H}(x)]\,, \\ I_2 &\sim \mathbb{P}((X_{n+1}^+ + T_n^+)\, Y_c > x) - \mathbb{P}((X_{n+1}^+ + T_n^+)\, Y_c > x\,, X_{n+1}^+ \le h(x)) + o[\overline{H}(x)] \\ &=: I_{21} - I_{22} + o[\overline{H}(x)]\,,\end{aligned}$$

dropping the redundant (4.12), (4.16), (4.17); and substituting X_i, T_i with X_i^+, T_i^+, respectively, $i \ge 1$, in all relations in the above proof. □

The next two lemmas will be needed in the proof of Theorem 3.2, and they are found in Yi et al. (2011).

Lemma 4.5. *Let X be a real-valued r.v. with distribution $F \in \mathscr{D}$ and $J_F^- > 0$, and Y be a nonnegative and non-degenerate at zero r.v., independent of X. Then, for any fixed constants $0 < p_1 < J_F^- \le J_F^+ < p_2 < \infty$, there exist two positive constants C and x_0, which are irrespective of Y, such that for all $x \ge x_0$,*

$$\mathbb{P}(X\,Y > x) \le C\,\overline{F}(x)\,(\mathbb{E}Y^{p_1} \vee \mathbb{E}Y^{p_2})\,.$$

Lemma 4.6. *Let X and Y be two independent nonnegative r.v.s. If $F \in \mathscr{D}$, the distribution of X, and Y is not degenerate at zero, then, for any fixed $p > J_F^+$, there exists a positive constant C, which is irrespective of Y, such that for all $x \ge 0$,*

$$\mathbb{E}(X\,Y)^p\, \mathbb{1}_{(X\,Y \le x)} \le C\, x^p\, \mathbb{P}(X\,Y > x)$$

Proof of Theorem 3.2. For any $0 < p' < J_F^- \le J_F^+ < p < \infty$, denote by $\mu_p = \mathbb{E}Y^{p'} \vee \mathbb{E}Y^p$. Then, by Jensen inequality, $\mathbb{E}Y^p < 1$ implies that $\mu_p < 1$.

We will prove relation (3.2) by using the method of the tail asymptotics for randomly weighted sums. Let $Z_i = X_i Y_i$, $i \ge 1$, $\Theta_1 = 1$ and $\Theta_i = \prod_{j=1}^{i-1} Y_j$, $i \ge 2$. Clearly, for each $i \ge 1$, Z_i is independent of Θ_i. By Lemma 4.5, there exist two positive constants C and x_0, irrespective of Θ_i, $i \ge 1$, such that for each $i \ge 1$ and all $x \ge x_0$,

$$\overline{H}_i(x) = \mathbb{P}(Z_i \Theta_i > x) \le C\,\overline{H}(x)\left(\mathbb{E}\Theta_i^{p'} \vee \mathbb{E}\Theta_i^p\right) = C\,\mu_p^{i-1}\,\overline{H}(x)\,. \tag{4.22}$$

For the lower bound of (3.2), by Theorem 3.1 and (4.22), we have that for any fixed $n \ge 1$,

$$\begin{aligned}\liminf \frac{\psi(x)}{\sum_{i=1}^{\infty}\overline{H}_i(x)} &\ge \liminf \frac{\psi(x,n)}{\sum_{i=1}^{n}\overline{H}_i(x)}\left(1 - \limsup \sum_{i=n+1}^{\infty}\frac{\overline{H}_i(x)}{\overline{H}(x)}\right)\\ &\ge 1 - C\sum_{i=n+1}^{\infty}\mu_p^{i-1} \to 1 \quad \text{as} \quad n \to \infty\,. \end{aligned}\tag{4.23}$$

Now we deal with the upper bound of (3.2). For any $0 < \delta < 1$ and any $n \ge 1$, we have

$$\begin{aligned}\psi(x) &\le \mathbb{P}\Big(\sum_{i=1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j > x\Big)\\ &\le \mathbb{P}\Big(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > (1-\delta)\,x\Big) + \mathbb{P}\Big(\sum_{i=n+1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j > \delta\,x\Big)\\ &=: K_1(x) + K_2(\delta\,x)\,. \end{aligned}\tag{4.24}$$

From (4.4) and $H_i \in \mathscr{C}$, $i \ge 1$, we obtain that for any $n \ge 1$

$$\begin{aligned}\limsup \frac{K_1(x)}{\sum_{i=1}^{\infty}\overline{H}_i(x)} &\le \limsup \frac{\sum_{i=1}^{n}\overline{H}_i((1-\delta)\,x)}{\sum_{i=1}^{n}\overline{H}_i(x)}\\ &\le \limsup \bigvee_{i=1}^{n}\frac{\overline{H}_i((1-\delta)\,x)}{\overline{H}_i(x)} \to 1 \quad \text{as} \quad \delta \downarrow 0\,. \end{aligned}\tag{4.25}$$

For each $i \ge 2$, define two independent r.v.s $Z_i^+ = X_i^+ Y_i$ and $\Theta_i = \prod_{j=1}^{i-1} Y_j$. Then, by Markov inequality we have

$$K_2(x) \le \sum_{i=n+1}^{\infty}\mathbb{P}(Z_i^+\Theta_i > x) + \mathbb{P}\Big(\sum_{i=n+1}^{\infty} Z_i^+\Theta_i\,\mathbb{1}_{\{Z_i^+\Theta_i \le x\}} > x\Big) \tag{4.26}$$

$$\le \sum_{i=n+1}^{\infty}\overline{H}_i(x) + x^{-p}\,\mathbb{E}\Big(\sum_{i=n+1}^{\infty} Z_i^+\Theta_i\,\mathbb{1}_{\{Z_i^+\Theta_i \le x\}}\Big)^p =: K_{21}(x) + K_{22}(x)\,.$$

If $p \le 1$, then, by using the inequality $\sum_{j=1}^{\infty} x_j \le (\sum_{j=1}^{\infty} x_j^q)^{1/q}$ with $x_j \ge 0$, $j \ge 1$ and $0 < q \le 1$, and according to Lemma 4.6, there exists a positive constant C, irrespective of Θ_i, $i \ge 1$, such that for all $x > 0$

$$\begin{aligned}K_{22}(x) &\le x^{-p}\sum_{i=n+1}^{\infty}\mathbb{E}(Z_i^+\Theta_i)^p\,\mathbb{1}_{\{Z_i^+\Theta_i \le x\}} \le C\sum_{i=n+1}^{\infty}\mathbb{P}(Z_i^+\Theta_i > x)\\ &= C\,K_{21}(x)\,. \end{aligned}\tag{4.27}$$

If $p > 1$, then by Minkowski inequality and Lemma 4.6, we have

$$\begin{aligned} K_{22}(x) &\leq x^{-p}\left(\sum_{i=n+1}^{\infty}\left(\mathbb{E}(Z_i^+\Theta_i)^p\,\mathbb{I}_{\{Z_i^+\Theta_i\leq x\}}\right)^{\frac{1}{p}}\right)^p \\ &\leq C\left(\sum_{i=n+1}^{\infty}(\overline{H}_i(x))^{\frac{1}{p}}\right)^p. \end{aligned} \tag{4.28}$$

Combining (4.27)–(4.28) and by (4.22), $\mu_p < 1$, we obtain

$$\limsup \frac{K_2(x)}{\overline{H}(x)} \leq C\left(\sum_{i=n+1}^{\infty}\mu_p^{i-1} + \left(\sum_{i=n+1}^{\infty}\mu_p^{\frac{i-1}{p}}\right)^p\right) \to 0 \quad \text{as} \quad n\to\infty\,,$$

which, together with $H \in \mathscr{C} \subset \mathscr{D}$, implies

$$\limsup \frac{K_2(\delta\, x)}{\sum_{i=1}^{\infty}\overline{H}_i(x)} \leq \limsup \frac{K_2(\delta\, x)}{\overline{H}(\delta\, x)}\cdot \limsup \frac{\overline{H}(\delta\, x)}{\overline{H}(x)} = 0\,. \tag{4.29}$$

Plugging (4.25) and (4.29) into (4.24), we can derive

$$\limsup \frac{\psi(x)}{\sum_{i=1}^{\infty}\overline{H}_i(x)} \leq 1\,. \tag{4.30}$$

Therefore, the desired (3.2) follows from (4.23) and (4.30). □

Proof of Theorem 3.3. We firstly estimate the upper bound. For any positive x and any $N \geq 2$

$$\sup_{n\geq 1}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)} \leq \max\left\{\bigvee_{n=1}^{N}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)},\ \bigvee_{n=N+1}^{\infty}\frac{\psi(x)}{\sum_{i=1}^{n}\overline{H}_i(x)}\right\}. \tag{4.31}$$

By Theorem 3.1 we have

$$\lim \bigvee_{n=1}^{N}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)} = 1\,. \tag{4.32}$$

By Theorem 3.2, (4.22) and $\mu_p < 1$ we have

$$\begin{aligned} \limsup \bigvee_{n=N+1}^{\infty}\frac{\psi(x)}{\sum_{i=1}^{n}\overline{H}_i(x)} &\leq \lim \frac{\psi(x)}{\sum_{i=1}^{\infty}\overline{H}_i(x)}\cdot \limsup \frac{1+\sum_{i=N+2}^{\infty}\frac{\overline{H}_i(x)}{\overline{H}(x)}}{1-\left(\sum_{i=N+2}^{\infty}\frac{\overline{H}_i(x)}{\overline{H}(x)}\right)^2} \\ &\leq \frac{1+C\sum_{i=N+2}^{\infty}\mu_p^{i-1}}{1-C^2\left(\sum_{i=N+2}^{\infty}\mu_p^{i-1}\right)^2} \to 1 \quad \text{as} \quad N\to\infty\,. \end{aligned} \tag{4.33}$$

Then, the upper bound follows from (4.31)–(4.33).

For the lower bound, similarly, for any positive x and any $N \geq 2$,

$$\inf_{n\geq 1}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)} \geq \min\left\{\bigwedge_{n=1}^{N}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)},\ \bigwedge_{n=N+1}^{\infty}\frac{\psi(x,\,N)}{\sum_{i=1}^{n}\overline{H}_i(x)}\right\}. \tag{4.34}$$

By Theorem 3.1 we have

$$\lim \bigwedge_{n=1}^{N}\frac{\psi(x,\,n)}{\sum_{i=1}^{n}\overline{H}_i(x)} = 1\,. \tag{4.35}$$

By Theorem 3.2, (4.22) and $\mu_p < 1$ we have

$$\begin{aligned}\liminf \bigwedge_{n=N+1}^{\infty} \frac{\psi(x,\,N)}{\sum_{i=1}^{n} \overline{H}_i(x)} \;&\geq\; \lim \frac{\psi(x,\,N)}{\sum_{i=1}^{N} \overline{H}_i(x)} \liminf \left(1 - \sum_{i=N+1}^{\infty} \frac{\overline{H}_i(x)}{\overline{H}(x)}\right) \\ &\geq\; 1 - C \sum_{i=N+1}^{\infty} \mu_p^{i-1} \to 1 \quad \text{as} \quad N \to \infty\,. \qquad (4.36)\end{aligned}$$

Therefore, the lower bound is obtained from (4.34)–(4.36). □

PROOF OF COROLLARY 3.1. We firstly prove part (1). If $F \in \mathscr{R}_{-\alpha}$ and $\mathbb{E}Y^p < \infty$ (hence, by (4.7) holds $\mathbb{E}Y_c^p < \infty$) for some $p > J_F^+ = \alpha$, then, by (4.2) in Lemma 4.3 and Breiman theorem, we have

$$\overline{H}(x) \sim \mathbb{P}(X\,Y_c > x) \sim \mathbb{E}Y_c^{\alpha}\,\overline{F}(x)\,,$$

which implies $H \in \mathscr{R}_{-\alpha}$. Again by Breiman theorem, we obtain that for each $i \geq 1$

$$\overline{H}_i(x) \sim \mathbb{E}Y_c^{\alpha}\,(\mathbb{E}Y^{\alpha})^{i-1}\,\overline{F}(x)\,. \qquad (4.37)$$

Therefore, the desired (3.4) follows from (3.1) and (4.37).

The proof of part (2) is similar to that of part (1). The claims follow from (4.37) and by using Theorems 3.2 and 3.3, respectively. □

References

[1] Albrecher, H., Asmussen, S. and Kortschak, D., 2006. Tail asymptotics for the sum of two heavy-tailed dependent risks. Extremes 9, 107–130.

[2] Asimit, A. V. and Badescu, A. L., 2010. Extremes on the discounted aggregate claims in a time dependent risk model. Scand. Actuar. J. 2, 93–104.

[3] Bingham, N. H., Goldie, C. M. and Teugels, J. L., 1987. Regular Variation. Cambridge University Press, Cambridge.

[4] Breiman, L., 1965. On some limit theorems similar to the arc-sin law. Theory Probab. Appl. 10, 323–331.

[5] Cai, J. and Tang, Q., 2004. On max-sum equivalence and convolution closure of heavy-tailed distributions and their applications. J. Appl. Probab. 41, 117–130.

[6] Chen, Y., 2011. The finite-time ruin probability with dependent insurance and financial risks. J. Appl. Probab. 48, 1035–1048.

[7] Chen, Y. and Yuen, K. C., 2009. Sums of pairwise quasi-asymptotically independent random variables with consistent variation. Stochastic Models 25, 76–89.

[8] Cline, D. B. H. and Samorodnitsky, G., 1994. Subexponentiality of the product of independent random variables. Stochastic Process. Appl. 49, 75–98.

[9] Gao, Q. and Wang, Y., 2010. Randomly weighted sums with dominated varying-tailed increments and application to risk theory. J. Korean Stat. Society 39, 305–314.

[10] Goovaerts, M. J., Kaas, R., Laeven, R. J. A., Tang, Q. and Vernic, R., 2005. The tail probability of discounted sums of Pareto-like losses in insurance. Scand. Actuar. J. 6, 446–461.

[11] Hashorva, E., Pakes, A. G. and Tang, Q., 2010. Asymptotics of random contractions. Insurance Math. Econom. 47, 405–414.

[12] Joe, H., 1997. Multivariate models and dependence concepts. Chapman and Hall, London.

[13] Ko, B. and Tang, Q., 2008. Sums of dependent nonnegative random variables with subexponential tails. J. Appl. Probab. 45, 85–94.

[14] Nelsen, R. B., 1998. An introduction to copulas. Springer.

[15] Norberg, R., 1999. Ruin problems with assets and liabilities of difusion type. Stoch. Process. Appl. 81, 255-269.

[16] Nyrhinen, H., 1999. On the ruin probabilities in a general economic environment. Stoch. Process. Appl. 83, 319–330.

[17] Nyrhinen, H., 2001. Finite and in.nite time ruin probabilities in a stochastic economic environment. Stoch. Process. Appl. 92, 265–285.

[18] Pakes, A. G., 2007. Convolution equivalence and infinite divisibility: corrections and corollaries. J. Appl. Probab. 44, 295–305.

[19] Shen, X., Lin, Z. and Zhang, Y., 2009. Uniform estimate for maximum of randomly weighted sums with applications to ruin theory. Methodol. Comput. Appl. Probab. 11, 669–685.

[20] Tang, Q. and Tsitsiashvili, G., 2003a. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stoch. Process. Appl. 108, 299–325.

[21] Tang, Q. and Tsitsiashvili, G., 2003b. Randomly weighted sums of subexponential random variables with application to ruin theory. Extremes 6, 171–188.

[22] Tang, Q. and Tsitsiashvili, G., 2004. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. Adv. Appl. Probab. 36, 1278–1299.

[23] Wang, D., Su, C. and Zeng, Y., 2005. Uniform estimate for maximum of randomly weighted sums with applications to insurance risk theory. Science in China: Series A 48, 1379–1394.

[24] Wang, D. and Tang, Q., 2006. Tail probabilities of randomly weighted sums of random variables with dominated variation. Stoch. Models 22, 253–272.

[25] Weng, C., Zhang, Y. and Tan, K. S., 2009. Ruin probabilities in a discrete time risk model with dependent risks of heavy tail. Scand. Actuar. J. 3, 205–218.

[26] Yang, Y. and Hashorva, E., 2013. Extremes and products of multivariate AC-product risks. Insurance Math. Econom. 52, 312–319.

[27] Yang, Y., Leipus, R. and Šiaulys J., 2012a. Tail probability of randomly weighted sums of subexponential random variables under a dependence structure. Stat. Probab. Lett. 82, 1727–1736.

[28] Yang, Y., Leipus, R. and Šiaulys J., 2012b. On the ruin probability in a dependent discrete time risk model with insurance and financial risks. J. Comput. Appl. Math. 236, 3286–3295.

[29] Yang, H. and Sun, S., 2013. Subexponentiality of the product of dependent random variables. Stat. Probab. Lett. 83, 2039–2044.

[30] Yi, L., Chen, Y. and Su, C., 2011. Approximation of the tail probability of randomly weighted sums of dependent random variables with dominated variation. J. Math. Anal. Appl. 376, 365–372.

[31] Zhang, Y., Shen, X. and Weng, C., 2009. Approximation of the tail probability of randomly weighted sums and applications. Stoch. Process. Appl. 119, 655–675.

[32] Zhou, M., Wang, K. and Wang, Y., 2012. Estimates for the finite-time ruin probability with insurance and financial risks. Acta Math. Appl. Sin., Engl. Ser. 28, 795–806.