GNEDENKO TYPE LIMIT THEOREMS FOR CYCLO-STATIONARY χ^2-PROCESSES

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Abstract: Gnedenko type limit theorems on extremes for the cyclostationary χ^2-processes are proved.

Key words: Gaussian process, cyclostationary process, Berman's inequality, maximum of random process

1. INTRODUCTION AND MAIN RESULTS

During last decades the study of probabilities of large extremes of Gaussian random processes has been intensively progressed. Several powerful methods were developed to get both asymptotic behaviors of the probabilities and uniform boundaries for them, see for details and further references [1], [7], [2], [10], [3]. It turned out that a part of the methods, in particular, the double sum method could be applied to Gaussian random fields too, so that asymptotic methods for Gaussian random fields were developed in parallel with the methods for Gaussian processes, see [2], [10]. But, in spite of increasing interest to behaviors of large deviations of Gaussian vector processes, see [8], [2], [12] one can see some backlog in developments of corresponding asymptotic methods for the vector case. The present paper is in abreast of works on generalizations of the double sum method to Gaussian vector processes, see [9], [4], [10]. In line with [5] we consider the norm of a Gaussian vector cyclostationary processes with independent identically distributed components. We call this process a χ-process, which is also cyclostationary. The random process $X(t),\ t \in \mathbb{R}$ is called cyclostationary with period τ if for every $v \in \mathbb{R}$ its mean $\mathbf{E}X(t)$ and covariance function $r_t(v) = r(t, t+v)$ are periodic functions in t with period τ.

Results and references relating to spectral properties of this class of random processes can be found in [13], see also [5]. A spectral representation and conditions for

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the harmonizability of the process are also provided there. The evolutionary spectral density can be estimated by means of the periodogram. Similar problems where considered in [11] for stationary χ-processes. Here we extend methods and approaches from [9], [4], [10], [11], to get results for cyclostationary processes.

Let $X_1(t), ..., \ X_n(t), \ t \in \mathbb{R}$, be independent copies of a Gaussian zero mean cyclostationary processes $X(t)$, denote $\mathbf{X}(\mathbf{t}) := (X_1(t), X_2(t), ..., X_n(t))$ and $\chi^2(t) := \|\mathbf{X}(\mathbf{t})\|^2, \ t \in \mathbb{R}$, the cyclostationary χ^2-process. We also write $\chi(t) := \|\mathbf{X}(\mathbf{t})\|$. Hereinafter we assume smoothness and non-degeneracy of $X(t)$:

Assumption 1 *The process $X(t)$ is square mean differentiable, a.s. continuous and centered.*

Assumption 2 *The correlation function of $X(t)$ equals one only at coincided points.*

We use the following notations for maximum distributions,

$$P_X(u, W) = \mathbf{P}\left(\max_{t \in W} X(\mathbf{t}) \leq u\right) \text{ and } Q_X(u, W) = 1 - P_X(u, W),$$

where $X(\mathbf{t}), \mathbf{t} \in W$, is a random process or field.

First we formulate results concerning behaviors of the maximum distributions of the norm of a cyclostaionary χ-process on its period. These results constitute the base of our main considerations. The corresponding proofs are known or simple, we postpone them to the section 3.

Proposition 1 *[9] Suppose that there exists a unique $t_0 \in (0, \tau)$ such that*

$$\sup_{t \in [0,T]} \sigma^2(t) = \sigma^2(t_0) = 1,$$

where the variance $\sigma^2(t) = \operatorname{var} X(t)$ is twice continuously differentiable at the point t_0 with $\sigma'(t_0) = 0, \ \sigma''(t_0) < 0, \ \mathbf{E}X'(t_0)^2 > 0$. Then

$$Q_\chi(u, [0, \tau]) = \mathcal{C} \ u^{n-1} e^{-\frac{u^2}{2}} (1 + o(1))$$

as $u \to \infty$, where

$$\mathcal{C} = \frac{2^{1-\frac{n}{2}}}{\Gamma(n/2)} \sqrt{1 - \frac{\mathbf{E}X'(t_0)^2}{\sigma''(t_0)}}$$

and Γ is the Gamma function.

Proposition 2 *Suppose that at some set $T_0 \subset [0,\tau]$ with a positive measure, the following conditions hold,*

$$\sigma^2(t) = 1,\ t \in T_0,\ \sigma^2(t) < 1,\ t \in [0,\tau]\backslash T_0,\ \mathbf{E}X'(t)^2 > 0,\ \forall t \in T_0.$$

Then

$$Q_\chi(u,[0,\tau]) = \mathcal{D}\ u^{n-1}e^{-u^2/2}(1+o(1))$$

as $u \to \infty$, where

$$\mathcal{D} = \frac{\int\limits_{T_0} \sqrt{\mathbf{E}X'(t)^2}dt}{\sqrt{\pi}\Gamma(n/2)}.$$

Proposition 3 *Suppose that function $\sigma^2(t),\ t \in [0,\tau]$, attains maximum at interior points $t_i, i = 1,2,...,k,$ of $[0,\tau]$, and*

$$\sigma^2(t_i) = 1,\ \sigma^{'}(t_i) = 0,\ \sigma''(t_i) < 0,\ \mathbf{E}X^{'}(t_i)^2 > 0.$$

Suppose that function $\sigma^2(t)$ is twice continuously differentiable at the points $t_i, i = 1,2,...,k$. Then

$$Q_\chi(u,[0,T]) = \sum_{i=1}^{k} \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}\alpha_i u^{n-1}e^{-\frac{u^2}{2}}(1+o(1)),$$

where $u \to \infty$ and $\alpha_i = \sqrt{1 - \mathbf{E}X'(t_i)^2/\sigma''(t_i)}$.

Now formulate main results of the paper.

Theorem 1 *Let Assumptions 1 and 2 for the component $X_1(t)$ of the Gaussian vector cyclostationary process with period τ, $\tau > 0$, be fulfilled. Let the assumption of Proposition 1 be fulfilled on the interval $[0,\tau]$. Suppose that the covariance function $r(t_1,t_2)$ of the process $X(t)$ satisfies*

$$\max_{t \geqslant 0} r(t,t+s)\log s \to 0, s \to \infty. \tag{1}$$

Then, for all $x \in \mathbb{R}$,

$$\lim_{T\to\infty} \mathbf{P}\left(\max_{t\in[0,T]} \chi(t) < \frac{x}{a_T} + b_T\right) = e^{-e^{-x}},$$

where

$$a_T = \sqrt{2\ln\frac{T}{\tau}},\ b_T = \sqrt{2\ln\frac{T}{\tau}} + \frac{\ln\mathcal{C} + \frac{n-1}{2}\ln\left(2\ln\frac{T}{\tau}\right)}{\sqrt{2\ln\frac{T}{\tau}}}.$$

Theorem 2 *Let Assumptions 1 and 2 for the component $X_1(t)$ of the Gaussian vector cyclostationary process with period τ, $\tau > 0$, be fulfilled. Let the assumption of Proposition 2 be fulfilled on the interval $[0,\tau]$. Suppose that (1) is fulfilled. Then, for all $x \in \mathbb{R}$*

$$\lim_{T\to\infty} \mathbf{P}\left(\max_{t\in[0,T]} \chi(t) < \frac{x}{a_T} + b_T \right) = e^{-e^{-x}},$$

where

$$a_T = \sqrt{2\ln\frac{T}{\tau}}, b_T = \sqrt{2\ln\frac{T}{\tau}} + \frac{\ln\mathcal{D} + \frac{n-1}{2}\ln(2\ln\frac{T}{\tau})}{\sqrt{2\ln\frac{T}{\tau}}}.$$

Theorem 3 *Let Assumptions 1 and 2 for the component $X_1(t)$ of the Gaussian vector cyclostationary process with period τ, $\tau > 0$, be fulfilled. Let the assumption of Proposition 3 be fulfilled on the interval $[0,\tau]$. Suppose that (1) is fulfilled. Then, for all $x \in \mathbb{R}$*

$$\lim_{T\to\infty} \mathbf{P}\left(\max_{t\in[0,T]} \chi(t) < \frac{x}{a_T} + b_T \right) = e^{-e^{-x}},$$

where

$$a_T = \sqrt{2\ln\frac{T}{\tau}},\ b_T = \sqrt{2\ln\frac{T}{\tau}} + \frac{\ln\mathcal{E} + \frac{n-1}{2}\ln(2\ln\frac{T}{\tau})}{\sqrt{2\ln\frac{T}{\tau}}},$$

and $\mathcal{E} = \sum_{i=1}^{k} \frac{2^{1-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\alpha_i$.

2. PROOF OF MAIN RESULTS.

We prove Theorem 1. The proofs of the followed theorems are similar and based on the Propositions 2 and 3, respectively, instead of the Proposition 1. Introduce the Gaussian field

$$Y(t,\mathbf{v}) = X_1(t)v_1 + X_2(t)v_2 + ... + X_n(t)v_n,\ t \in [0,T],\ \mathbf{v} =(v_1,...,v_n).$$

Note that for any $S \subset [0,T]$,

$$\sup_{t\in S} \chi(t) = \sup_{(t,\mathbf{v})\in S\times S_{n-1}} Y(t,\mathbf{v}),$$

where $S_{n-1} := \{(v_1, v_2, ..., v_n) : v_1^2 + v_2^2 + ... + v_n^2 = 1\}$. The covariance function of $Y(t, \mathbf{v})$ equals

$$r((t_1, \mathbf{v}_1), (t_2, \mathbf{v}_2) = r(t_1, t_2)(\mathbf{v}_1, \mathbf{v}_2),$$

where $(\mathbf{v}_1, \mathbf{v}_2)$ is the scalar product. Under the assumptions of Theorem 1, for the correlation function of the field $Y(t, \mathbf{v})$ we have, ([9], Lemma 9),

$$\rho_Y((t_1, \mathbf{v}_1), (t_2, \mathbf{v}_2)) = 1 - \left(D(t_0)|t_1 - t_2|^2 + \frac{1}{2}||\mathbf{v}_1 - \mathbf{v}_2||^2 \right) (1 + o(1)), \tag{2}$$

as $t_1, t_2 \to t_0$, where $D(t_0) = \frac{1}{2}EX'(t_0)^2 > 0$. For the variance of the field $Y(t, \mathbf{v})$ we have,

$$\sqrt{\operatorname{var} Y(t, \mathbf{v})} = \sigma(t) = 1 - \frac{1}{2}\sigma''(t_0)(t - t_0)^2(1 + o(1)), \tag{3}$$

as $t \to t_0$. We partition the interval $[0, T]$ with long intervals of length L intermitted short intervals and show that short intervals play negligible role in the limit distribution. To show this we will use Berman's inequality for a time discretization of the process χ, making sure that the discrete time approximation is precise enough. These ideas are discussed in the following lemmata.

Lemma 1 *Denote $L_u = Lu^{1-n}e^{u^2/2}$, L is a constant. For any $L > 0$ and any $\epsilon > 0$ one can find $b > 0$, $u_0 > 0$, $C = C(L) > 0$ and a grid $\mathcal{R}_b = \mathcal{R}_{b,u,\epsilon}$ (a finite set of points) on the cylinder $[0, L_u] \times S_{n-1}$, such that for all $u \geq u_0$,*

$$P_Y(u, ([0, L_u] \times S_{n-1}) \cap \mathcal{R}_b) - P_Y(u, ([0, L_u] \times S_{n-1})) \leq B\epsilon.$$

Proof: We partition the sphere S_{n-1} onto $N(\epsilon)$ parts $A_1, ..., A_{N(\epsilon)}$ in the following way. Consider the polar coordinates on the sphere S_{n-1},

$$(x_1, x_2, ..., x_n) = S(\varphi_1, \varphi_2, ...\varphi_{n-1}),$$

$\varphi_1, \varphi_2, ...\varphi_{n-2} \in [0, \pi)$, $\varphi_{n-1} \in [0, 2\pi)$ and divide the intervals $[0, \pi]$ and $[0, 2\pi]$ on intervals of length ϵ (or less for the last interval). This partition of the parallelepiped $[0, \pi]^{n-2} \times [0, 2\pi]$ generates the partition $A_1, ..., A_{N(\epsilon)}$ of the sphere. Now we construct the grid $\mathcal{R}_{b,u,\epsilon}$. Choose in any A_j an inner point and consider the tangent plane to the cylinder $[0, L_u] \times S_{n-1}$ at this point. Introduce in the tangent plane rectangular coordinates, with origin at the tangent point, where the first coordinate is held by t, so that the plane becomes n-dimensional Euclidean space and consider on it the grids

$$\mathcal{R}^j(b) := \mathcal{R}^{j,P}_{b,u,\epsilon} := (bk_1u^{-1}, bk_2u^{-1}, ..., bk_nu^{-1}), \qquad j = 1, 2, ..., N(\epsilon),$$

where $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Suppose that ϵ is so small that the orthogonal projections of all A_j onto corresponding tangent planes are one-to-one. We denote by A_j^P the projection of A_j at the tangent plane, and by $\mathcal{R}^j_{b,u,\epsilon}$, the prototype of $\mathcal{R}^{j,P}_{b,u,\epsilon}$ under this projection. We show that the grid

$$\mathcal{R}_b := \bigcup_{j=1}^{N(\epsilon)} \mathcal{R}^j_{b,u,\epsilon},$$

with an appropriate choice of its parameters, satisfies the assertion of the lemma. Let us denote by $\mathbf{w}_1, \mathbf{w}_2, ...,$ the projections of points $\mathbf{v}_1, \mathbf{v}_2, ...$ from A_j at the corresponding tangent plane. From the geometry of sphere it follows that for all sufficiently small ϵ,

$$\sup_{\mathbf{v}_1, \mathbf{v}_2 \in A_j,\ j=1,2,...,N(\epsilon)} \frac{\|\mathbf{w_1} - \mathbf{w_2}\|}{\|\mathbf{v_1} - \mathbf{v_2}\|} \leq 1 + 2\epsilon,$$
$$\inf_{\mathbf{v}_1, \mathbf{v}_2 \in A_j,\ j=1,2,...,N(\epsilon)} \frac{\|\mathbf{w_1} - \mathbf{w_2}\|}{\|\mathbf{v_1} - \mathbf{v_2}\|} \geqslant 1 - 2\epsilon.$$

Therefore from (2) it follows that for all sufficiently small ϵ there exist $\delta(\epsilon) > 0$ such that for every $j = 1, 2, ..., N(\epsilon)$, for the covariance function $r_j((t, \mathbf{w}_1), (s, \mathbf{w}_2))$ of the Gaussian field $Z_j(t, \mathbf{w}) = Y(t, \mathbf{v})$, $\mathbf{v} \in A_j$ the following inequalities hold,

$$(1 - 2\epsilon)\left(D(t_0)(t-s)^2 + \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||^2\right) \leq 1 - r_j((t, \mathbf{w}_1), (s, \mathbf{w}_2))$$
$$\leq (1 + 2\epsilon)\left(D(t_0)(t-s)^2 + \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||^2\right),$$

where $t, s \in (t_0 - \delta/2, t_0 + \delta/2)$, $\mathbf{w}_1, \mathbf{w}_2 \in A_j^P$. Since the process $X(t)$ is cyclostationary we have

$$P_Y(u, ([0, L_u] \times S_{n-1}) \cap \mathcal{R}_b) - P_Y(u, [0, L_u] \times S_{n-1})$$
$$= \mathbf{P}\left(\bigcap_{j=1}^{N(\epsilon)} \left(\max_{([0,L_u]\times A_j)\cap \mathcal{R}_b} Y(t, \mathbf{v}) \leq u\right) \cap \bigcup_{j=1}^{N(\epsilon)} \left(\max_{[0,L_u]\times A_j} Y(t, \mathbf{v}) > u\right)\right)$$
$$\leq \left(\left[\frac{L_u}{\tau}\right] + 1\right) \sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{([0,\tau]\times A_j)\cap \mathcal{R}_b} Y(t, \mathbf{v}) \leq u, \max_{[0,\tau]\times A_j} Y(t, \mathbf{v}) > u\right) \leq$$

$$\leq \left(\left[\frac{L_u}{\tau} \right] + 1 \right) \Bigg[Q_Y \left(u, \left([0,\tau] \backslash (t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}) \right) \times S_{n-1} \right)$$
$$+ \sum_{j=1}^{N(\epsilon)} \mathbf{P} \left(\max_{\left((t_0-\delta/2, t_0+\delta/2) \times A_j^P \right) \cap \mathcal{R}^j(b)} Z_j(t,\mathbf{w}) \leq u, \max_{(t_0-\delta/2, t_0+\delta/2) \times A_j^P} Z_j(t,\mathbf{w}) > u \right) \Bigg] . \quad (4)$$

Using Fernique-Landau-Shepp inequality, we get

$$\left(\left[\frac{L_u}{\tau} \right] + 1 \right) Q_Y \left(u, ([0,\tau] \backslash (t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2})) \times S_{n-1} \right) = O \left(\left[\frac{L_u}{\tau} \right] e^{-\frac{u^2}{2\sigma_1^2}} \right) = o(1),$$

as $u \to \infty$, where $\sigma_1^2 < \sigma^2(t_0) = 1$.

To estimate the sum in the right-hand part of (4), we need to prove the following

Lemma 2 *Let all assumptions of the Proposition 1 be fulfilled. Then*

$$Q_Y(u, \left((t_0 - \delta/2, t_0 + \delta/2) \times A_j^P \right) \cap \mathcal{R}^j(b)) = \mathcal{C}_b \, u^{n-1} e^{-\frac{u^2}{2}} (1 + o(1))$$

as $u \to \infty$, *with* $\mathcal{C}_b \to \mathcal{C}$ *as* $b \to 0$.

To prove this Lemma, one has to repeat some parts of the proof of Theorem 1, [9], changing the parametric set in continuous time with discrete one, $\mathcal{R}^j(b)$. We give more detailed explanations in the Appendix how to modify the proof of Theorem 1, [9].

Now we have for some $C > 0$ and all sufficiently small $b > 0$,

$$\mathbf{P} \left(\max_{\left((t_0-\delta/2, t_0+\delta/2) \times A_j^P \right) \cap \mathcal{R}^j(b)} Z_j(t,\mathbf{w}) \leq u, \max_{(t_0-\delta/2, t_0+\delta/2) \times A_j^P} Z_j(t,\mathbf{w}) > u \right)$$
$$= Q_Y(u, \left((t_0 - \delta/2, t_0 + \delta/2) \times A_j^P \right) - Q_Y(u, \left((t_0 - \delta/2, t_0 + \delta/2) \times A_j^P \right) \cap \mathcal{R}^j(b))$$
$$= (\mathcal{C} - \mathcal{C}_b) u^{n-1} e^{-\frac{u^2}{2}} (1 + o(1)) \leq C \epsilon u^{n-1} e^{-\frac{u^2}{2}}.$$

Thus for sufficiently large u and sufficiently small b, the sum in the right-hand part of (4) can be bounded by ϵ, so Lemma follows.

Denote by $K_j,\ j = 1, 2, ...,$ the intervals $\left(\frac{\delta}{2}, \tau - \frac{\delta}{2}\right),\ \left(\tau + \frac{\delta}{2}, 2\tau - \frac{\delta}{2}\right), ...$ We need a partition of the interval $[0, L_u]$ on subintervals which are issued by shifting the interval $(\frac{\delta}{2}, \tau - \frac{\delta}{2})$ with a step τ. Let us denote the union of these subintervals by $t(u, L, \delta)$.

Lemma 3 *For every* $L > 0$, *any* $\epsilon > 0$ *and any* $\delta > 0$ *such that* $t_0 \in \left(\frac{\delta}{2}, \tau - \frac{\delta}{2}\right)$, *and all sufficiently large* u

$$P_Y(u, t(u, L, \delta) \times S_{n-1} \cap \mathcal{R}_b) - P_Y(u, ([0, L_u] \times S_{n-1}) \cap \mathcal{R}_b) \leq \varepsilon.$$

Proof: Since

$$\sup_{(0,\tau)\setminus(\frac{\delta}{2},\tau-\frac{\delta}{2})} \sigma^2(t) = b < 1,$$

one can see that

$$\begin{aligned} &P_Y(u, t(u,L,\delta) \times S_{n-1} \cap \mathcal{R}_b) - P_Y(u, ([0,L_u] \times S_{n-1}) \cap \mathcal{R}_b) \leq \\ &\leq Q\left(u, ([0,L_u] \setminus t(u,L,\delta)) \times S_{n-1}\right) \leq \frac{2}{\tau} L_u u^{n-1} e^{-u^2/2b} = o(1), \end{aligned}$$

as $u \to \infty$.

Consider a sequence of independent copies $Y_j(t,\mathbf{v})$ of the Gaussian field $Y(t,\mathbf{v}),\ (t,\mathbf{v}) \in K_j \times S_{n-1},\ j = 1,2,...$, and introduce the Gaussian random field $Y_0(t,\mathbf{v}) = Y_j(t,\mathbf{v})$, for $(t,\mathbf{v}) \in K_j \times S_{n-1},\ j = 1,2,...$

Lemma 4 *For any* $L > 0$

$$P_{Y_0}\left(u, [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b\right) - P_Y\left(u, [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b\right) \to 0,$$

as $u \to \infty$.

Proof: By Berman's inequality, we have

$$\begin{aligned} &|P_Y(u, [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b) - P_{Y_0}(u, [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b)| = \\ &\left|\mathbf{P}\left(\exists (t,\mathbf{v}) \in [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b : \frac{Y(t,\mathbf{v})}{\sigma(t,\mathbf{v})} \leq \frac{u}{\sigma(t,\mathbf{v})}\right)\right. \\ &\left. - \mathbf{P}\left(\exists (t,\mathbf{v}) \in [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b : \frac{Y_0(t,\mathbf{v})}{\sigma(t,\mathbf{v})} \leq \frac{u}{\sigma(t,\mathbf{v})}\right)\right| \\ &\leq \sum_{\substack{(t,\mathbf{v}_1),(s,\mathbf{v}_2) \in [t(u,L,\delta) \times S_{n-1}] \cap \mathcal{R}_b \\ (t,\mathbf{v}_1) \neq (s,\mathbf{v}_2)}} |\overline{r}_Y((t,\mathbf{v}_1),(s,\mathbf{v}_2)) - \overline{r}_{Y_0}((t,\mathbf{v}_1),(s,\mathbf{v}_2))| \\ &\times \int_0^1 g_h\left(\frac{u}{\sigma(t,\mathbf{v_1})}, \frac{u}{\sigma(t,\mathbf{v_2})}\right) dh \end{aligned}$$

where $\sigma^2(t,\mathbf{v}) = \mathbf{Var}Y(t,\mathbf{v})$ is the variance of Y, $g_h(x,y)$ is the probability density of two-dimensional Gaussian vector

$$\left(\sqrt{h}\frac{Y(t,\mathbf{v_1})}{\sigma(t,\mathbf{v_1})} + \sqrt{1-h}\frac{Y_0(t,\mathbf{v_1})}{\sigma(t,\mathbf{v_1})},\ \sqrt{h}\frac{Y(s,\mathbf{v_2})}{\sigma(s,\mathbf{v_2})} + \sqrt{1-h}\frac{Y_0(s,\mathbf{v_2})}{\sigma(s,\mathbf{v_2})}\right)$$

at the point (x,y) , $\overline{r}_Y((t,\mathbf{v}_1),(s,\mathbf{v}_2))$ is the covariance function of the field $\frac{Y(t,\mathbf{v})}{\sigma(t,\mathbf{v})}$ and $\overline{r}_{Y_0}((t,\mathbf{v}_1),(s,\mathbf{v}_2))$ is the covariance function of the field $\frac{Y_0(t,\mathbf{v})}{\sigma(t,\mathbf{v})}$.

Taking into account the expression for the covariance function of Y we get that the last expression is bounded by

$$\frac{1}{\pi}\sum_{\substack{i\neq j\\ t\in K_i, s\in K_j}}\sum_{K(i,j)}|\rho\left(t,s\right)|\left[1-\frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2\right]\times$$
$$\int_0^1\left(1-h^2\rho^2(t,s)\left[1-\frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2\right]\right)^{-\frac{1}{2}}$$
$$\times\exp\left\{-\left[2\left(1-h^2\rho^2(t,s)\left[1-\frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2\right]\right)\right]^{-1}\times\right.$$
$$\left.\left(\frac{u^2}{\sigma^2(t)}-2h\rho(t,s)\left[1-\frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2\right]\frac{u^2}{\sigma(t)\sigma(s)}+\frac{u^2}{\sigma^2(t)}\right)\right\}dh$$
$$\leq\frac{1}{\pi}\sum_{K(i,j)}|\rho(t,s)|\exp\left(-\frac{u^2}{1+|\rho(t,s)|}\right)$$

where $\rho(t,s)$ is the correlation function of the process $X(t)$ and we have denoted to shorten the formulas

$$K(i,j):=\{(t,\mathbf{v}_1)\in(K_i\times S_{n-1})\cap\mathcal{R}_b\ (s,\mathbf{v}_2)\in(K_j\times S_{n-1})\cap\mathcal{R}_b\}.$$

From (2) it follows that for any $\Delta>0$, $\sup\limits_{|t-s|>\Delta}|\rho(t,s)|<1$. Hence there exists γ_2, $0<\gamma_2<1$, such that $\sup\limits_{|t-s|>\Delta}|\rho(t,s)|<1-\gamma_2$.

Consider first "not too outstanding" t and s, that is, $t\in K_i$, $s\in K_j$, and $d(K_i,K_j):=\sup\{|t-s|:t\in K_i,s\in K_j\}\leq L_u^{\gamma_1}$, where $\gamma_1\in\left(0,\frac{\gamma_2}{2-\gamma_2}\right)$. Denoting by Σ_1 the part of the sum over such $t,s,\mathbf{v}_1,\mathbf{v}_2$ we get,

$$\begin{aligned}\Sigma_1&\leq C_1\sum_{K(i,j)}\exp\left(-\frac{u^2}{1+|\rho(t,s)|}\right)\\&\leq C_2\sum_{\substack{i\neq j\\ d(K_i,K_j)\leq L_u^{\gamma_1}}}\sum_{K(i,j)}\exp\left(-\frac{u^2}{2-\gamma_2}\right)\\&=O\left(L_u^{1+\gamma_1}\epsilon^{n(n-1)/2}u^{2n}\epsilon^{n(n-1)/2}e^{-\frac{u^2}{2-\gamma_2}}\right)=o(1),\end{aligned}$$

as $u \to \infty$ where C_1 and C_2 are constants. We have used above that the volume of every A_j has order $\epsilon^{n(n-1)/2}$, for small enough ϵ.

Turn now to those t,s, $\mathbf{v}_1, \mathbf{v}_2$ for which $d(K_i, K_j) > L_u^{\gamma_1}$, $t \in K_i$, $s \in K_j$. Denote the corresponding part of the sum by Σ_2. From (1) we get in this case,

$$\kappa(u) := \sup_{|t-s| \geq L_u^{\gamma_1}} |\rho(t,s)| = o(u^{-2})$$

as $u \to \infty$, hence

$$\begin{aligned}
\Sigma_2 &\leq C_3 \kappa(u) \sum_{K(i,j)} \exp\left(-\frac{u^2}{1+|\rho(t,s)|}\right) \\
&\leq C_4 \kappa(u) e^{-u^2} \sum_{\substack{i \neq j \\ d(K_i,K_j) > L_u^{\gamma_1}}} \sum_{K(i,j)} \exp\left(\frac{|\rho(t,s)| u^2}{1+|\rho(t,s)|}\right) \\
&= O\left(\left(L_u \epsilon^{n(n-1)/2} u^n\right)^2 \kappa(u) e^{-u^2}\right) = O(u^2 \kappa(u)) = o(1),\ u \to \infty,
\end{aligned}$$

where C_3 and C_4 are constants. Thus Lemma is proven.

Proof of Theorem 4: We denote by $u = u(x,T) = \frac{x}{a_T} + b_T$. It is easy to check that

$$\lim_{T \to \infty} T \frac{\mathcal{C}}{\tau} u^{n-1} e^{-u^2/2} e^x = 1.$$

Note that Lemma 1 holds true also for the field Y_0, with the same grid (we consider the field Y_0 at the set $t(u, L, \delta) \times S_{n-1}$). Combining Lemma 1, Lemma 3, Lemma 4, we get

$$\begin{aligned}
&P\left(\max_{t \in [0,T]} \chi(t) < \frac{x}{a_T} + b_T\right) \\
&= (1+o(1)) \left(1 - P\left(\max_{\left(\frac{\delta}{2}, \tau - \frac{\delta}{2}\right) \times S_{n-1}} Y_0(t, \mathbf{v}) > \frac{x}{a_T} + b_T\right)\right)^{T/\tau} \\
&= (1+o(1)) \exp\left(-\frac{T}{\tau}\, \mathcal{C} u^{n-1} e^{-u^2/2}\right) = (1+o(1)) \exp\left(-e^{-x}\right),\ T \to \infty.
\end{aligned}$$

3. PROOF OF AUXILIARY RESULTS.

Proposition 1 is proved in [9].

Outline of proof of Proposition 2: Denote by U a union of finite numbers of intervals such that $T_0 \subset U$ and $\varepsilon > \operatorname{dist}([0,T] \setminus U, T_0) > 0$. By Landau-Shepp inequality, for some $\delta > 0,\ C > 0$,

$$Q_Y(u, ([0,T]\backslash U) \times S_{n-1}) \leq Ce^{-\frac{u^2}{2(1-\delta)}},$$

Assuming that $T_0 = [t_0, t_1],\ 0 < t_0 < t_1 < T$, one can use Corollary 7.4, [10], to get

$$Q_Y(u, T_0 \times S_{n-1}) = \mathcal{D}u^{n-1}e^{-\frac{u^2}{2}}(1 + o(1)),\ u \to \infty. \tag{5}$$

Limit relation (5) for unions of finite numbers of intervals can be easily proved by standard Double Sum Method. It also follows from Theorem 7.1, [10], by repetition arguments used in the proof of Corollary 7.4.

For general T_0 we take in account that due to differentiability σ, for any ε, one can find two finite unions of non-intersecting intervals, U_- and U_+, such that $U_- \subset T_0 \subset U_+$ with $\nu(U_+ \setminus U_-) \leq \varepsilon$, ν is Lebesgue measure. It can be seen that

$$Q_Y(u, U_- \times S_{n-1}) \leq Q_Y(u, T_0 \times S_{n-1}) \leq Q_{Y_+}(u, U_+ \times S_{n-1}),$$

where $Y_+(t, \mathbf{v})$ generated by

$$X_+(t) = \sigma(t)^{-1}X(t),\ t \in U_+.$$

From here and (5) we have for general T_0,

$$Q_Y(u, T_0 \times S_{n-1}) \geq (\mathcal{D}-\varepsilon)u^{n-1}e^{-\frac{u^2}{2}}(1 + o(1)),\ u \to \infty.$$

Further, we have,

$$\begin{aligned} Q_Y(u, [0,T] \times S_{n-1}) &\leq Q_{Y_+}(u, U \times S_{n-1}) + Q_Y(([0,T] \setminus U) \times S_{n-1}) \\ &\leq Q_{Y_+}(u, U \times S_{n-1}) + Ce^{-\frac{u^2}{2(1-\delta)}}, \end{aligned}$$

Using the above arguments with a Gaussian field generated Gaussian process $X_\varepsilon(t) = \sigma(t)^{-1}X(t),\ t \in U$ we get that for any positive ε and all sufficiently large u,

$$Q_Y(u, [0,T] \times S_{n-1}) \leq (\mathcal{D} + \varepsilon)u^{n-1}e^{-\frac{u^2}{2}} + Ce^{-\frac{u^2}{2(1-\delta(\epsilon))}}.$$

The above inequalities follow The Proposition.

Proposition 3 can be easily derived from Proposition 1 by standard double sum method.

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APPENDIX: OUTLINE OF THE PROOF OF LEMMA 2.

First, Lemma 3 from [9] must be changed to prove the Lemma 2. Let $\xi(\mathbf{t})$, $\mathbf{t}=(t_1,...,t_n)$, be a Gaussian homogeneous centered field with covariance function $r(\mathbf{t})$ such that

$$r(\mathbf{t})=1-|\mathbf{t}|^2+o(|\mathbf{t}|^2)$$

as $\mathbf{t}\to\mathbf{0}$. Consider the standard field,

$$\eta(\mathbf{t})=\frac{\xi(\mathbf{t})}{1+\beta t_1^2}.$$

Lemma 5 *(Lemma 3 [9]) For any $\lambda_i\le 0$, $\mu_i\ge 0$, $i=1,...,n$, and any finite collection of points $\mathcal{R}\subset\bigotimes_{i=1}^n[\lambda_i,\ \mu_i]$,*

$$P\left(\max_{u\mathbf{t}\in\mathcal{R}}\frac{\xi(\mathbf{t})}{1+\beta t_1^2}>u\right)=\Psi(u)H_2^\beta(\lambda,\mu,\mathcal{R})(1+o(1))$$

as $u\to\infty$, where

$$H_2^\beta(\lambda,\mu,\mathcal{R})=E\exp\left[\max_{\mathbf{t}\in\mathcal{R}}\left(\sqrt{2}\sum_{i=1}^n\xi_i t_i-\beta t_1^2-|\mathbf{t}|^2\right)\right]$$

and ξ_i are independent standard Gaussian variables.

The proof of this Lemma repeats the proof of Lemma 3 from [9].

Lemma 6 *Let $\mathcal{R}_N$ be a sequence of grids on $\bigotimes_{i=1}^n[\lambda_i,\ \mu_i]$ (finite collections of points from $\bigotimes_{i=1}^n[\lambda_i,\ \mu_i]$) such that $\bigcup_N\mathcal{R}_N$ is dense in $\bigotimes_{i=1}^n[\lambda_i,\ \mu_i]$. Then*

$$\lim_{N\to\infty}H_2^\beta(\lambda,\mu,\mathcal{R}_N)=H_2^\beta(\lambda_1,\mu_1)\prod_{i=2}^n H_2^0(\lambda_i,\mu_i),$$

where

$$H_2^\beta(\lambda,\mu)=E\exp\left[\max_{t\in[\lambda,\mu]}\left(\sqrt{2}\xi t-(\beta+1)t^2\right)\right].$$

Proof of this Lemma can be easily obtained from the Dominance Convergence Theorem.

Now the proof of Lemma 2 immediately follows from proof of Theorem 1, [9].