Insurance: Mathematics and Economics 27 (2000) 145–149
www.elsevier.nl/locate/econbase

Ruin under interest force and subexponential claims: a simple treatment☆

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Received 1 December 1999; received in revised form 1 February 2000; accepted 22 February 2000

Abstract

A simple proof of the asymptotic formula for the ruin probability of a risk process with a positive constant interest force [derived earlier by Asmussen (Asmussen, S., 1998. The Annals of Applied Probability 8, 354–374)] is given. The proof is based on a formula obtained by Sundt and Teugels (Sundt, B., Teugels, J.L., 1995. Insurance: Mathematics and Economics 16, 7–22). © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Ruin probability; Risk model; Subexponential distribution; Interest force

1. Introduction

In this note we deal with the classical risk model with a Poisson claims occurrence process, constant premium rate, and a constant interest force. Such a model was considered by Sundt and Teugels (1995, 1997) where the probability of ultimate ruin was examined under an analogue of the Cramér condition. Klüppelberg and Stadtmüller (1998) used sophisticated analytical arguments to derive, asymptotic formula for the ruin probability in the presence of claims having a distribution with a regularly varying tail. This result was refined by Asmussen (1998) who used the reflected random walk theory for obtaining asymptotic formulae for the ruin probability in the presence of subexponentially distributed claim sizes. The purpose of this paper is to give an elementary proof of the Asmussen result, using equations derived in Sundt and Teugels (1995, 1997).

Let $u > 0$ be the initial capital of an insurance company. The claim sizes $(Y_k)_{k\in N}$ are positive i.i.d. random variables with the common distribution B with the mean $b_1 = \mathbf{E}Y_1$. Denote by

$$F(x) = \frac{1}{b_1}\int_0^x (1 - B(z))\,\mathrm{d}z$$

the integrated tail of the claim size distribution. In this paper, we will only assume that F is a subexponential distribution (see Embrechts et al., 1997). Let the claims occur at random times forming a homogeneous Poisson process with the intensity $\lambda > 0$ that is independent of $\{Y_k\}$. Let $c > 0$ be the constant gross premium rate (not necessarily positive) and denote by $X(t)$ a compound Poisson process representing the total claim amount

☆ This work was partly supported by the Russian Foundation for Basic Research (grant 98-01-00855) and INTAS (grant 98-1625).
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PII: S0167-6687(00)00045-7

accumulated until time t. Assume also that a constant interest force $\delta > 0$ affects the risk process. In this case the risk reserve $(U_\delta(t))_{t\geq 0}$ satisfies the equation (see Sundt and Teugels, 1995)

$$U_\delta(t) = u\mathrm{e}^{\delta t} + c\int_0^t \mathrm{e}^{\delta y}\,\mathrm{d}y - \int_0^t \mathrm{e}^{\delta(t-y)}\,\mathrm{d}X(y).$$

Let

$$\psi_\delta(u) = \mathbf{P}\left(\inf_{t\geq 0} U_\delta(t) < 0 | U_\delta(0) = u\right)$$

be the ultimate ruin probability for this risk process.

Our goal is to give a simple proof of the following Asmussen's result (see Asmussen, 1998, Section 4).

Theorem 1. *If F is a subexponential distribution, then*

$$\psi_\delta(u) \sim \frac{\lambda}{\delta}\int_u^\infty \frac{1-B(z)}{z}\mathrm{d}z. \tag{1}$$

We show, in particular, that this result follows immediately from Eq. (2) derived in Sundt and Teugels (1995). Hopefully, our approach can be applied to more general models.

2. Auxiliary relations

Let us use the following notation:

$\rho = \lambda b_1$, the expected claim amount per time unit;

$\phi_\delta(u) = 1 - \psi_\delta(u)$, the survival probability;

$G_\delta(u) = 1 - \psi_\delta(u)/\psi_\delta(0)$, an auxiliary distribution function;

$$k_\delta(u) = \int_u^\infty z\,\mathrm{d}G_\delta(z).$$

The Laplace–Stieltjes transform will be denoted by equipping the corresponding original function with hat and using s as its argument. For example,

$$\hat{k}_\delta(s) = \int_0^\infty \mathrm{e}^{-su}\,\mathrm{d}k_\delta(u).$$

Let us assume, for a while, that the positive safety loading condition $\rho < c$ takes place (later we will get rid of it). Then the following equation is valid (see Sundt and Teugels, 1995, p. 14):

$$G_\delta(u) = -\frac{\delta}{c}\sum_{n=0}^\infty \left(\frac{\rho}{c}\right)^n F^{n*} * (k_\delta(0) - k_\delta(u)) + \frac{K_\delta}{c}\sum_{n=0}^\infty \left(\frac{\rho}{c}\right)^n F^{(n+1)*}(u), \tag{2}$$

where $*$ stands for the Stieltjes convolution,

$$K_\delta = \frac{\rho(1-\psi_\delta(0))}{\psi_\delta(0)}, \tag{3}$$

$$k_\delta(0) = \frac{K_\delta - c + \rho}{\delta}.$$

Eq. (2) was used by Teugels and Sundt in order to investigate the behaviour of the ruin probability via behaviour of the auxiliary function $G_\delta(u)$ (see Sundt and Teugels, 1995). One of the difficulties they encountered was the fact that $1 - G_\delta(u) = \mathrm{o}(k_\delta(u))$. We propose to use (2) in order to investigate $k_\delta(u)$ and, via this function, the ruin probability. This allows us to obtain the desired results directly, without complex analytical arguments.

By the Beekman convolution formula (or, which is the same, by the Pollaczek–Khinchine formula), the probability of ruin $\psi_0(u)$ (without interest rate) can be expressed as

$$\psi_0(u) = \left(1 - \frac{\rho}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\rho}{c}\right)^n (1 - F^{n*}(u)). \tag{4}$$

Using (4), we can rewrite (2) in the form

$$1 - G_\delta(u) = \left(1 + \frac{K_\delta}{\rho}\right) \psi_0(u) - \frac{\delta}{c - \rho} \phi_0 * k_\delta(u). \tag{5}$$

The following equation is a corollary of (5):

$$k_\delta(u) = \frac{\rho + K_\delta}{\delta}(1 - F(u)) - \frac{c}{\delta}(1 - G_\delta(u)) + \frac{\rho}{\delta} F * (1 - G_\delta)(u). \tag{6}$$

Actually, taking the Laplace–Stieltjes transform of (5) and using the representation (4), we arrive at

$$(1 - \hat{G}_\delta(s)) \left(1 - \frac{\rho \hat{F}(s)}{c}\right) = \frac{\rho + K_\delta}{c}(1 - \hat{F}(s)) - \frac{\delta \hat{k}_\delta(s)}{c}.$$

Upon inverting, this relation yields (6).

3. Basic results

Firstly, let us estimate $k_\delta(u)$. We will not tend to give estimates as accurate as possible, but restrict ourselves to bounds that are sufficient to get the desired asymptotic result.

Lemma 1. *If $\rho < c$, then*

$$k_*(u) \le k_\delta(u) \le k^*(u), \tag{7}$$

where

$$k_*(u) = \left(1 + \frac{c}{\delta u}\right)^{-1} \frac{\rho + K_\delta}{\delta}(1 - F(u)), \tag{8}$$

$$k^*(u) = \left(\frac{c - \rho}{\rho}\right) \frac{(\rho + K_\delta)\psi_0(u)}{\delta \phi_0(u)}. \tag{9}$$

Proof. The evident inequality

$$k_\delta(u) \ge u(1 - G_\delta(u))$$

and the relation (6) yield the lower bound in (7).

Eq. (5) and the inequality

$$\phi_0 * k_\delta(u) \ge \phi_0(u) k_\delta(u)$$

yield the upper bound in (7). □

Lemma 2. *For subexponential F and* $\rho < c$,

$$k_\delta(u) \sim \frac{\rho + K_\delta}{\delta}(1 - F(u)), \quad u \to \infty. \tag{10}$$

Proof. Evidently,

$$k_*(u) \sim \frac{\rho + K_\delta}{\delta}(1 - F(u)).$$

As F is a subexponential distribution,

$$\psi_0(u) \sim \frac{\rho}{c - \rho}(1 - F(u))$$

(see Theorem 1.3.8 in Embrechts et al., 1997). Therefore,

$$k^*(u) \sim \frac{\rho + K_\delta}{\delta}(1 - F(u)),$$

which completes the proof. □

The following two lemmas prove Theorem 1.

Lemma 3. *If* $\rho < c$ *and F is subexponential, then* (1) *holds true*.

Proof. We have

$$\begin{aligned}\frac{\psi_\delta(u)}{\psi_\delta(0)} &= -\int_u^\infty \frac{\mathrm{d}k_\delta(z)}{z} = \frac{k_\delta(u)}{u} - \int_u^\infty \frac{k_\delta(z)}{z^2}\,\mathrm{d}z \quad \text{(by integration by parts)} \\ &\sim \frac{\rho + K_\delta}{\delta}\left(\frac{1 - F(u)}{u} - \int_u^\infty \frac{1 - F(z)}{z^2}\,\mathrm{d}z\right) \qquad \text{(by Lemma 2)} \\ &= \frac{\rho + K_\delta}{\delta}\int_u^\infty \frac{\mathrm{d}F(z)}{z} = \frac{\rho + K_\delta}{\delta b_1}\int_u^\infty \frac{1 - B(z)}{z}\,\mathrm{d}z.\end{aligned}$$

Using the definition (3) of K_δ, we get (1). □

The following remark was made by the referee and we reproduce it with gratitude.

Remark. *If one has information on the difference in* k_δ *values rather than on the values themselves, then it is neater to use the following steps in the proof above*:

$$\begin{aligned}\frac{\psi_\delta(u)}{\psi_\delta(0)} &= \int_u^\infty \frac{k_\delta(u) - k_\delta(z)}{z^2}\,\mathrm{d}z = \frac{1}{u}\int_1^\infty \frac{k_\delta(u) - k_\delta(uv)}{v^2}\,\mathrm{d}v \sim \frac{\rho + K_\delta}{\delta u}\int_1^\infty \frac{F(uv) - F(u)}{v^2}\,\mathrm{d}v \\ &= \frac{\rho + K_\delta}{\delta b_1}\int_1^\infty \frac{\mathrm{d}v}{uv^2}\int_u^{uv}(1 - B(z))\,\mathrm{d}z.\end{aligned}$$

The last term is, evidently, equal to the right-hand side of (1).

Let us now get rid of the assumption $\rho < c$. Note that we impose no further restriction on the premium rate c, which may even be non-positive.

Lemma 4. *If* $\rho \geq c$ *and F is subexponential, relation* (1) *still holds true*.

Proof. Let us re-denote the ruin probability as follows:

$$\psi_\delta(u) = \psi_\delta(u, c).$$

That is, we indicate the value of the premium rate in the notation regarding λ and $B(z)$ as fixed. Then

$$c \le c' \quad\Rightarrow\quad \psi_\delta(u, c) \ge \psi_\delta(u, c').$$

Take c' such that $\rho < c'$. Then, by Lemmas 1–3,

$$\psi_\delta(u, c') \sim \frac{\lambda}{\delta}\int_u^\infty \frac{1 - B(z)}{z}\,\mathrm{d}z.$$

Furthermore, for any $u^* > 0$ and $u \ge u^*$,

$$\psi_\delta(u, c) \le \psi_\delta(u - u^*, c + \delta u^*) = \mathbf{P}\left(\inf_{t\ge 0} U_\delta(t) < u^* | U_\delta(0) = u\right).$$

Take u^* such that $c + \delta u^* > \rho$. Then, by Lemmas 1–3,

$$\psi_\delta(u - u^*, c + \delta u^*) \sim \frac{\lambda}{\delta}\int_{u-u^*}^\infty \frac{1 - B(z)}{z}\,\mathrm{d}z \sim \frac{\lambda}{\delta}\int_u^\infty \frac{1 - B(z)}{z}\,\mathrm{d}z. \tag{11}$$

The second equivalence in (11) is due to the subexponentiality of F. Thus, upper and lower bounds of $\psi_\delta(u)$ are equivalent to the same function. Therefore, (1) is true. □

Note that Lemma 1 can help in bounding of the ruin probability provided that one can estimate $\psi_0(u)$, but this subject is out of the scope of this communication.

Acknowledgements

We appreciate greatly the attention given to this work by Professors Søren Asmussen, Josef Teugels, and Ragnar Norberg and their valuable advises.

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