*Journal of Mathematical Sciences, Vol. 123, No. 1, 2004*

# TWO-SIDED BOUNDS FOR RUIN PROBABILITY UNDER CONSTANT INTEREST FORCE

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UDC 519.2

## 1. Introduction

In the present work, two-sided bounds for ruin probability in the classical risk model under constant interest force are found. This is done by reduction from well-known inequalities in the model without interest force. The idea of the reduction is not new but only recently appeared the critical accumulation of the necessary tools for its realization. This approach permits a comparison of the rates of convergence at the asymptotic estimates of the ruin probability, as recently demonstrated in [5, 10]. We shall still draw our interests on the case of heavy-tailed claim distributions. Finally, we present the numerics to show the applicability of the method.

We assume throughout the paper that for a nonnegative random variable (r.v.) $Z$ with a finite mean $\beta = \mathbf{E}Z$, its distribution function (d.f.) is denoted by $B(u) = P(Z \le u)$ and its tail by $\overline{B} = 1 - B$. We will use the notation of the equilibrium distribution function (e.d.f.) of $B$, which is denoted by

$$F(u) = \frac{1}{\beta} \int_0^u \overline{B}(z)\, dz. \tag{1.1}$$

Let $u > 0$ be the initial capital of an insurance company and the claim sizes $(Z_k)_{k\ge1}$ be nonnegative independent identically distributed random variables (i.i.d.r.v.'s) distributed as a common d.f. $B$ with a finite mean $\beta$ and an equilibrium d.f. $F$. The claims occur at random times $(T_k)_{k\ge1}$, which form a homogeneous Poisson process $N(t)$, independent of $(Z_k)_{k\ge1}$ with an intensity $\lambda > 0$. Let $\rho = \lambda\beta$ and $c$ be the constant gross premium rate. We denote by

$$X(t) = \sum_{k=1}^{N(t)} Z_k$$

the compound Poisson process representing the total claim amount accumulated up to time $t$. Furthermore, we assume that a constant interest force $r > 0$ affects the risk process. Then the risk reserve $U_r(t)$ satisfies the equation (see [11, 12])

$$U_r(t) = ue^{rt} + c\int_0^t e^{ry}\, dy - \int_0^t e^{r(t-y)}\, dX(y).$$

We define the ultimate ruin probability for this risk process as

$$\psi_r(u) = \mathbf{P}(\inf_{t\ge0} U_r(t) < 0 \mid U_r(0) = u).$$

We will use the notation

$$G_r(u) = 1 - \frac{\psi_r(u)}{\psi_r(0)},$$

$$k_r(u) = \int_u^\infty z\, dG_r(z),$$

in order to take the following representation for the ruin probability:

$$\frac{\psi_r(u)}{\psi_r(0)} = \frac{k_r(u)}{u} - \int_u^\infty k_r(y)\, \frac{dy}{y^2}. \tag{1.2}$$

Proceedings of the International Seminar on Applied Stochastic Models and Information Processes, Petrozavodsk, Russia, 2002.

 1072-3374/04/1231-3824 

The following two-sided bounds for the expression $k_r(u)$ with $K_r = \rho(\overline{\psi}_r(0)/\psi_r(0))$ can be found in [4]:

$$\frac{\rho + K_r}{c + ru} u\overline{F}(u) \le k_r(u) \le (c - \rho)\frac{\rho + K_r}{r\rho}\frac{\psi_0(u)}{1 - \psi_0(u)}. \tag{1.3}$$

We can see here that this relation can be used to reduce the problem of the estimation under interest force to the application of known estimates for the ruin probability in a noninterest force model.

The famous subexponential class of distributions is defined below:

$$\mathcal{S} = \left\{ F \text{ d.f. on } (0, \infty)\colon \lim_{u\to\infty} \frac{\overline{F}^{n*}(u)}{\overline{F}(u)} = n \quad \text{for any} \quad n \in \mathbf{N} \right\},$$

where $F^{*n}$ denotes the $n$th convolution of the distribution $F$. Suppose that $F \in \mathcal{S}$ and that the safety loading condition $\rho < c$ holds; then the classical Embrechts–Veraverbeke asymptote for the ruin probability

$$\psi_0(u) \sim \frac{\rho}{c - \rho}\overline{F}(u) \tag{1.4}$$

as $u \to \infty$ is true (see [2, 3]). Relation (1.4) together with (1.3) gives the asymptote

$$k_r(u) \sim \frac{\rho + K_r}{r}\overline{F}(u) \tag{1.5}$$

as $u \to \infty$. Further, plugging (1.5) into (1.2), it seems tempting for us to take the following type of ruin probability asymptote (see [1, 4, 9]):

$$\frac{\psi_\delta(u)}{\psi_\delta(0)} \sim \frac{\rho + K_\delta}{\delta}\left(\frac{\overline{F}(u)}{u} - \int_u^\infty \frac{\overline{F}(y)}{y^2}\,dy\right) = \frac{\rho + K_\delta}{\delta b_1}\int_u^\infty \frac{\overline{B}(y)}{y}\,dy \tag{1.6}$$

as $u \to \infty$. Therefore,

$$\psi_r(u) \sim \frac{\lambda}{r}\int_u^\infty \frac{\overline{B}(y)}{y}\,dy \tag{1.7}$$

as $u \to \infty$, which first appears in [1]. In fact, it is easily seen that the asymptote (1.6) and (1.7) are invalid under the restriction that

$$\limsup_{u\to\infty} \frac{u}{\overline{F}(u)}\int_u^\infty \frac{\overline{F}(y)}{y^2}\,dy < 1. \tag{1.8}$$

Let us note that (1.8) is equivalent to

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1, \quad \forall v > 1. \tag{1.9}$$

**Theorem 1.** *In the classical risk model with a constant interest force $r > 0$, if the e.d.f. $F \in \mathcal{S}$ and* (1.9) *holds, then the asymptotics* (1.7) *is true.*

**Proof.** We assume temporarily that the safety loading condition $\rho < c$ holds. Therefore, (1.4) is valid. Inequalities (1.3) together with (1.4) give the relation

$$\limsup_{u\to\infty} \frac{u}{k_r(u)}\int_u^\infty k_r(z)\,\frac{dz}{z^2} = \limsup_{u\to\infty} \frac{u}{\overline{F}(u)}\int_u^\infty \overline{F}(z)\,\frac{dz}{z^2}.$$

It is easy to see that if (1.8) holds, then we are allowed to substitute (1.5) into (1.2) on the way to the asymptotic relationship (1.4):

$$\psi_r(u) \sim \frac{\rho}{r}\left(\frac{1}{u}\overline{F}(u) - \int_u^\infty \overline{F}(z)\,\frac{dz}{z^2}\right) = \frac{\lambda}{r}\int_u^\infty \overline{B}(z)\,\frac{dz}{z}$$

as $u \to \infty$, which first appears in [1]. But (1.8) is equivalent to the assertion that (1.9) holds for some $v > 1$, which is implied by the membership of $F$ in $\mathcal{A}$. This proves Theorem 2.1 for the case where $\rho < c$.

If $\rho \geq c$, then, by the same argument as the proof of Lemma 4 in [4], we can still obtain (1.7). Hence, the validity of (1.7) is independent of the safety loading condition. This ends the proof.

Motivated by this background, we introduce a new subclass in the subexponential distributions as below:

$$\mathcal{A} = \left\{ F \in \mathcal{S} \colon \limsup_{u\to\infty} \frac{\overline{F}(vu)}{\overline{F}(u)} < 1 \ \forall v > 1 \right\}. \tag{1.10}$$

Therefore, for $F \in \mathcal{A}$ the classical Embrechts–Veraverbeke asymptote (1.4) has become improved from the noninterest case to the case with constant interest force by (1.7). As can be easily examined, it is a very wide class.

The paper is organized as follows. In Sec. 2, two-sided bounds of the ruin probability $\psi_r(u)$ for the case $F \in \mathcal{A}$ are found. In Sec. 3, some concrete examples connected with the corresponding estimation procedure are studied. Finally, in Sec. 4, some numerical results on the examples are provided.

## 2. Two-Sided Estimates

Let us assume that (1.9) holds for some $v > 1$, or, equivalently, that (1.9) holds for any $v > 1$. Thus, for any $v > 1$ there exists some $l(v) = l_F(v) > 0$ such that

$$d(v) = \sup_{x > l(v)} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1.$$

Conventionally we write $d(1) = 1$. With the $v$, $l(v)$, and $d(v)$ given above, we introduce

$$d(v,k) = \sup_{x>l(v)} \frac{\overline{F}(v^k x)}{\overline{F}(x)}, \qquad \sigma(v) = \sum_{k=1}^{\infty} \frac{d(v,k-1) - d(v,k)}{v^k}.$$

Obviously, it holds for any $v > 1$ that $d(v,0) = d(1) = 1$ and $d(v,1) = d(v)$. With the notation

$$D(x) = 1 - \frac{x}{\overline{F}(x)} \int_x^{\infty} \overline{F}(z)\, \frac{dz}{z^2}, \quad x \geq 0,$$

we immediately obtain the following.

**Lemma 1.** *Let $F$ be a d.f. supported on $[0,\infty)$. If (1.9) holds for some $v > 1$, then for all $x \geq l(v)$,*

$$D(x) \geq \sigma(v) \geq \frac{1 - d(v)}{v} > 0. \tag{2.1}$$

**Proof.** We have that the quantity $d(v,k)$ is nondecreasing for $k \geq 0$. Therefore,

$$\sigma(v) \geq \frac{d(v,0) - d(v,1)}{v} = \frac{1 - d(v)}{v} > 0, \quad v > 1.$$

As for the first inequality in (2.1), we have

$$D(x) \geq 1 - \frac{x}{\overline{F}(x)} \sum_{k=1}^{\infty} \overline{F}(v^{k-1}x) \int_{v^{k-1}x}^{v^k x} \frac{dz}{z^2} \geq 1 - \sum_{k=1}^{\infty} d(v^{k-1}) \left( \frac{1}{v^{k-1}} - \frac{1}{v^k} \right) = \sigma(v). \tag{2.2}$$

This ends the proof.

From (1.6) we obtain an asymptote for $\psi_r(u)$ such that

$$\psi_r(u) \sim \frac{\rho}{r} D(u) \frac{\overline{F}(u)}{u} \tag{2.3}$$

as $u \to \infty$ if $F \in \mathcal{A}$. The relative error of the approximation below

$$\Gamma(u) = \frac{\psi_r(u)}{(\rho/ru)\overline{F}(u)D(u)} - 1$$

can be used as a measure of the distance between $\psi_r(u)$ and its asymptote $(\rho/ru)\overline{F}(u)D(u)$. Let us consider the following positive auxiliary function:

$$\Delta(u) = \sup_{y \geq u} \left| \frac{\psi_0(y)}{(\rho/(c-\rho))\overline{F}(y)} - 1 \right| \longrightarrow 0, \tag{2.4}$$

introduced in [5] (see also [6, 7]).

**Theorem 2.** *In the classical risk model, if $\rho < c$, then for any $u > 0$ we have that*

$$|\Gamma(u)| \leq \frac{1}{D(u)}\left(h_\Delta(u) + \frac{c}{c+ru}\right), \tag{2.5}$$

*where*

$$h_\Delta(u) = \frac{\Delta(u) + (\rho/(c-\rho))\overline{F}(u)[1+\Delta(u)]}{1-(\rho/(c-\rho))\overline{F}(u)[1+\Delta(u)]}$$

*and $\Delta(u)$ is defined in* (2.4).

**Proof.** From (1.2) and (1.3), and noting that the functions $\Delta(z)$ and $\psi_0(z)$ are nonincreasing in $z \geq 0$, we derive the following lower bound of $\psi_r(u)$:

$$\begin{aligned} \psi_r(u) &\geq \frac{\rho}{c+ru}\overline{F}(u) - \frac{c-\rho}{r}\int_u^\infty \frac{\psi_0(z)}{1-\psi_0(z)}\,\frac{dz}{z^2} \\ &\geq \frac{\rho}{ru}\overline{F}(u)\left(\frac{ru}{c+ru} - \frac{u}{[1-\psi_0(u)]\overline{F}(u)}\int_u^\infty [1+\Delta(z)]\overline{F}(z)\,\frac{dz}{z^2}\right) \geq \frac{\rho\overline{F}(u)}{ru}\left(\frac{ru}{c+ru} - \frac{1+\Delta(u)}{1-\psi_0(u)}[1-D(u)]\right) \\ &= \frac{\rho}{ru}\overline{F}(u)D(u)\left[1 - \frac{1}{D(u)}\left(\frac{c}{c+ru} + \frac{\Delta(u)+\psi_0(u)}{1-\psi_0(u)}[1-D(u)]\right)\right]. \end{aligned} \tag{2.6}$$

Similarly, we obtain the upper bound of $\psi_r(u)$ as follows:

$$\begin{aligned} \psi_r(u) &\leq \frac{c-\rho}{ru}\,\frac{\psi_0(u)}{1-\psi_0(u)} - \rho\int_u^\infty \overline{F}(z)\,\frac{dz}{z(c+rz)} \leq \frac{\rho}{ru}\overline{F}(u)\left(\frac{\psi_0(u)}{(\rho/(c-\rho))\overline{F}(u)[1-\psi_0(u)]} - \frac{ru}{c+ru}[1-D(u)]\right) \\ &= \frac{\rho}{ru}\overline{F}(u)D(u)\left(\frac{ru}{c+ru} + \frac{1}{D(u)}\left[\frac{c}{c+ru} + \frac{\Delta(u)+\psi_0(u)}{1-\psi_0(u)}\right]\right). \end{aligned} \tag{2.7}$$

By these two bounds in (2.6) and (2.7), we get the proof of (2.5).

## 3. Examples

**Example 3.1.** Now we put forward some concrete examples to illustrate how to determine the values of $l(v)$, $d(v)$, $\sigma(v)$, and $D(u)$ involved in our main results.

(1) Let $F$ be the Pareto distribution with $a > 1$, i.e., the tail of $F$ satisfies

$$\overline{F}(x) = \begin{cases} 1, & \text{when } x \leq \kappa, \\ (\kappa/x)^a, & \text{when } x > \kappa, \end{cases} \qquad \kappa > 0. \tag{3.1}$$

Clearly, for any $v > 1$ we have

$$d(v) = v^{-a}, \qquad \sigma(v) = \frac{v^a - 1}{v^{a+1}-1}, \qquad l(v) = \kappa.$$

(2) Let $F$ be the Burr distribution with tail

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x^s}\right)^a, \quad a, \kappa, s > 0, \quad x \geq 0.$$

Modeling the proof of (2.2) with slight adjustment yields that for any $v > 1$ and $u > 0$

$$D(u) \geq \frac{1}{v}\left[1 - \frac{\overline{F}(vu)}{\overline{F}(u)}\right] = \frac{1}{v}\left[1 - \left(\frac{\kappa + u^s}{\kappa + (vu)^s}\right)^a\right] \sim \frac{v^{sa} - 1}{v^{sa+1}} > 0$$

as $u \to \infty$.

(3) Let $F$ be the Weibull distribution with tail

$$\overline{F}(x) = \exp\{-\lambda x^b\}, \quad \lambda > 0, \quad b \in (0, 1), \quad x \geq 0. \tag{3.2}$$

We analogously obtain that for any $v > 1$ and $u > 0$

$$D(u) \geq \frac{1}{v}\left[1 - \frac{\overline{F}(vu)}{\overline{F}(u)}\right] = \frac{1 - \exp\{-\lambda(v^b - 1)u^b\}}{v} \longrightarrow \frac{1}{v} > 0$$

as $u \to \infty$.

**Example 3.2.** Next, we study the convergence rate of the deviation $\Gamma(u)$ to 0 as $u \to \infty$.

(1) Clearly, under the assumption that (1.9) holds for some $v > 1$, (2.5) implies

$$\Gamma(u) = O(u^{-1} + \Delta(u) + \overline{F}(u)) \tag{3.3}$$

as $u \to \infty$.

(2) Recently, Mikosch and Nagaev [10] examined the case where the e.d.f. $F$ of claim size belongs to the class $\mathcal{D}$ of d.f.'s with dominated varying tails. We say a d.f. $F$ supported on $[0, \infty)$ belongs to the class $\mathcal{D}$ if and only if for some (or equivalently for any) $0 < c < 1$ it holds that

$$\limsup_{x\to\infty} \frac{\overline{F}(cx)}{\overline{F}(x)} < \infty.$$

This work indicates that if the e.d.f. $F \in \mathcal{D}$ with a finite mean, then $\Delta(u) = O(u^{-1})$ and, therefore, $\Delta(u) = O(u^{-1})$ as $u \to \infty$. From this and (4.3) we immediately obtain:

**Lemma 2.** *In the classical risk model with a constant interest force $r \geq 0$, if the e.d.f. $F$ of the claim size belongs to $\mathcal{A} \cap \mathcal{D}$ and has a finite mean, then $\Gamma(u) = O(u^{-1})$ as $u \to \infty$.*

We note that the intersection $\mathcal{A} \cap \mathcal{D}$ is a large subclass of heavy-tailed distributions. For example, if the e.d.f. $F \in ERV(-\alpha, -\beta)$ for $1 < \alpha \leq \beta < \infty$, i.e.,

$$v^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq v^{-\alpha}$$

for any $v > 1$, then $F$ satisfies all the conditions in the proposition. See [13] for details. Clearly, if $\alpha = \beta$, then the class $ERV(-\alpha, -\beta)$ coincides the class $\mathcal{R}_{-\alpha}$.

(3) We note that in the previous lemma a finite mean of the e.d.f. $F$ should be assumed in order for us to apply the related result in [10]. Thus, it excludes the case where $F$ follows a Pareto distribution with $0 < a < 1$ (recall (4.1)), which is of interest in insurance practice. In this case, from the relation (1.17) of [8] and the value of $\beta_F$ given there we can obtain the asymptote $\Delta(u) = O(u^{-a/(a+1)})$ as $u \to \infty$. This formula, in combination with (4.3), yields $\Gamma(u) = O(u^{-a/(a+1)})$ as $u \to \infty$. The convergence rate is now worse than in the lemma.

(4) If $F$ follows the Weibull distribution given in (4.2), then, by the same approach and using the expression of $\beta_F$ given in [8, p. 269], we obtain $\Delta(u) = O((\log u)^{1/b} u^{b-1})$ as $u \to \infty$. This formula, together with (4.3), yields $\Gamma(u) = O((\log u)^{1/b} u^{b-1})$ as $u \to \infty$.

(5) If $F$ follows the Lognormal distribution with the density function

$$f(x) = \frac{1}{\sqrt{2\pi}\, x} \exp\left\{-\frac{\log^2 x}{2}\right\}, \quad x > 0,$$

then the same approach together with the value of $\beta_F$ given in [8, p. 270] gives $\Delta(u) = O(u^{-1}(\log u)\exp\{\sqrt{2\log u}\})$ as $u \to \infty$, from which we similarly obtain $\Gamma(u) = O(u^{-1}(\log u)\exp\{\sqrt{2\log u}\})$ as $u \to \infty$.

## 4. Numerical Results

For $u > 0$ we write

$$\psi_r^-(u) = \max\left\{[1+\Gamma_-(u)]\frac{\rho}{ru}D(u)\overline{F}(u), 0\right\}, \qquad \psi_r^+(u) = [1+\Gamma_+(u)]\frac{\rho}{ru}D(u)\overline{F}(u),$$

and

$$\psi_r^*(u) = \frac{\rho}{ru}D(u)\overline{F}(u).$$

If the e.d.f. $F \in \mathcal{A}$, we know that $\psi_r(u) \sim \psi_r^*(u)$ as $u \to \infty$; under the conditions of the theorem, we know that $\psi_r^-(u) \le \psi_r(u) \le \psi_r^+(u)$ for $u > 0$. We proceed to the calculation of these estimates in the Pareto and Weibull cases. The upper bounds for the ruin probability $\psi_0(u)$ involved are found in [8]. We take the numerical results produced in the package Mathematica. In each table we vary the interest force from 0.01 to 0.31 by step 0.1. For simplicity we assume that $c = 1$.

In the first case, we assume that the e.d.f. $F$ follows a Pareto distribution, the tail of which has the form

$$\overline{F}(u) = (1+bu)^{-a}, \quad a, b, u > 0.$$

We take the parameters above with values $a = 3$ and $b = 0.5$. In Tables 1–3, we take the numerical results for $\rho = 0.1$ and $u$ equal to 9, 100, 1000.

TABLE 1. Pareto case with $a = 3$, $b = 0.5$, $\rho = 0.1$, $u = 9$, $\psi_0(u) = 0.000725$, $\Delta(u) = 0.0855969$, $D(u) = 0.718924$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.00480123 | 0 | 0.00710027 |
| 0.11 | 0.000436475 | 0.000116647 | 0.000574674 |
| 0.21 | 0.00022863 | 0.000110868 | 0.000287031 |
| 0.31 | 0.000154878 | 0.0000928057 | 0.000189465 |

TABLE 2. Pareto case with $a = 3$, $b = 0.5$, $\rho = 0.1$, $u = 100$, $\psi_0(u) = 8.44 \cdot 10^{-7}$, $\Delta(u) = 0.007617$, $D(u) = 0.74702$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.63147 \cdot 10^{-7}$ | $1.84765 \cdot 10^{-7}$ | $6.64245 \cdot 10^{-7}$ |
| 0.11 | $5.11952 \cdot 10^{-8}$ | $4.5352 \cdot 10^{-8}$ | $5.3162 \cdot 10^{-8}$ |
| 0.21 | $2.68165 \cdot 10^{-8}$ | $2.51156 \cdot 10^{-8}$ | $2.75028 \cdot 10^{-8}$ |
| 0.31 | $1.8166 \cdot 10^{-8}$ | $1.73592 \cdot 10^{-8}$ | $1.85435 \cdot 10^{-8}$ |

TABLE 3. Pareto case with $a = 3$, $b = 0.5$, $\rho = 0.1$, $u = 1000$, $\psi_0(u) = 8.85 \cdot 10^{-10}$, $\Delta(u) = 0.00161071$, $D(u) = 0.749592$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.9609 \cdot 10^{-11}$ | $5.23476 \cdot 10^{-11}$ | $6.15473 \cdot 10^{-11}$ |
| 0.11 | $5.419 \cdot 10^{-12}$ | $5.35095 \cdot 10^{-12}$ | $5.44695 \cdot 10^{-12}$ |
| 0.21 | $2.83852 \cdot 10^{-12}$ | $2.81905 \cdot 10^{-12}$ | $2.84911 \cdot 10^{-12}$ |
| 0.31 | $1.92287 \cdot 10^{-12}$ | $1.91359 \cdot 10^{-12}$ | $1.92907 \cdot 10^{-12}$ |

In Tables 4–6, we take $\rho = 0.9$ and $u = 9$, 100, 1000.

TABLE 4. Pareto-like case with $a = 3$, $b = 0.5$, $\rho = 0.9$, $u = 9$, $\psi_0(u) = 0.356$, $\Delta(u) = 5.58106$, $D(u) = 0.718924$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.0432111 | 0 | 0.612822 |
| 0.11 | 0.00392828 | 0 | 0.0550738 |
| 0.21 | 0.00205767 | 0 | 0.0287223 |
| 0.31 | 0.0013939 | 0 | 0.0194123 |

TABLE 5. Pareto case with $a = 3$, $b = 0.5$, $\rho = 0.9$, $u = 100$, $\psi_0(u) = 0.000348$, $\Delta(u) = 4.12917$, $D(u) = 0.74702$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.06832 \cdot 10^{-6}$ | 0 | 0.0000339539 |
| 0.11 | $4.60756 \cdot 10^{-7}$ | 0 | $3.0217 \cdot 10^{-6}$ |
| 0.21 | $2.41349 \cdot 10^{-7}$ | 0 | $1.5797 \cdot 10^{-6}$ |
| 0.31 | $1.63494 \cdot 10^{-7}$ | 0 | $1.06933 \cdot 10^{-6}$ |

TABLE 6. Pareto case with $a = 3$, $b = 0.5$, $\rho = 0.9$, $u = 1000$, $\psi_0(u) = 8.24 \cdot 10^{-8}$, $\Delta(u) = 0.151325$, $D(u) = 0.749592$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.36481 \cdot 10^{-10}$ | $4.44297 \cdot 10^{-10}$ | $6.61076 \cdot 10^{-10}$ |
| 0.11 | $4.8771 \cdot 10^{-11}$ | $4.57194 \cdot 10^{-11}$ | $5.87635 \cdot 10^{-11}$ |
| 0.21 | $2.55467 \cdot 10^{-11}$ | $2.40937 \cdot 10^{-11}$ | $3.07444 \cdot 10^{-11}$ |
| 0.31 | $1.73058 \cdot 10^{-11}$ | $1.63568 \cdot 10^{-11}$ | $2.08181 \cdot 10^{-11}$ |

Now we assume that $a = 5$ and $b = 0.25$. In Tables 7–9, we take $\rho = 0.1$ and $u = 9$, 100, 1000.

TABLE 7. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.1$, $u = 9$, $\psi_0(u) = 0.000355$, $\Delta(u) = 0.158478$, $D(u) = 0.78556$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.00240724 | 0 | 0.003497 |
| 0.11 | 0.00021884 | 0.0000693594 | 0.000293123 |
| 0.21 | 0.000114631 | 0.0000591667 | 0.000148644 |
| 0.31 | 0.0000776531 | 0.0000482031 | 0.0000989523 |

TABLE 8. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.1$, $u = 100$, $\psi_0(u) = 9.47 \cdot 10^{-9}$, $\Delta(u) = 0.0126497$, $D(u) = 0.828618$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $6.97409 \cdot 10^{-9}$ | $2.74757 \cdot 10^{-9}$ | $7.80178 \cdot 10^{-9}$ |
| 0.11 | $6.34008 \cdot 10^{-10}$ | $5.68588 \cdot 10^{-10}$ | $6.54614 \cdot 10^{-10}$ |
| 0.21 | $3.32099 \cdot 10^{-10}$ | $3.13013 \cdot 10^{-10}$ | $3.40291 \cdot 10^{-10}$ |
| 0.31 | $2.24971 \cdot 10^{-10}$ | $2.15898 \cdot 10^{-10}$ | $2.29859 \cdot 10^{-10}$ |

TABLE 9. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.1$, $u = 1000$, $\psi_0(u) = 1.12 \cdot 10^{-13}$, $\Delta(u) = 0.00422063$, $D(u) = 1$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $1.00376 \cdot 10^{-14}$ | $9.12512 \cdot 10^{-15}$ | $1.008 \cdot 10^{-14}$ |
| 0.11 | $9.12512 \cdot 10^{-16}$ | $9.04291 \cdot 10^{-16}$ | $9.16364 \cdot 10^{-16}$ |
| 0.21 | $4.77983 \cdot 10^{-16}$ | $4.75717 \cdot 10^{-16}$ | $4.8 \cdot 10^{-16}$ |
| 0.31 | $3.23795 \cdot 10^{-16}$ | $3.22754 \cdot 10^{-16}$ | $3.25161 \cdot 10^{-16}$ |

In Tables 10–12, we take $\rho = 0.9$ and $u = 9$, 100, 1000.

TABLE 10. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.9$, $u = 9$, $\psi_0(u) = 0.364$, $\Delta(u) = 13.6648$, $D(u) = 0.78556$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.0216652 | 0 | 0.635431 |
| 0.11 | 0.00196956 | 0 | 0.0575433 |
| 0.21 | 0.00103168 | 0 | 0.0300977 |
| 0.31 | 0.000698878 | 0 | 0.0203731 |

TABLE 11. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.9$, $u = 100$, $\psi_0(u) = 0.00013$, $\Delta(u) = 170.62$, $D(u) = 0.828618$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $6.27668 \cdot 10^{-8}$ | 0 | 0.0000129952 |
| 0.11 | $5.70607 \cdot 10^{-8}$ | 0 | $1.18089 \cdot 10^{-6}$ |
| 0.21 | $2.98889 \cdot 10^{-8}$ | 0 | $6.18538 \cdot 10^{-7}$ |
| 0.31 | $2.02474 \cdot 10^{-8}$ | 0 | $4.19004 \cdot 10^{-7}$ |

TABLE 12. Pareto case with $a = 5$, $b = 0.25$, $\rho = 0.9$, $u = 1000$, $\psi_0(u) = 1.14 \cdot 10^{-11}$, $\Delta(u) = 0.261917$, $D(u) = 1$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $9.03387 \cdot 10^{-14}$ | $8.21261 \cdot 10^{-14}$ | $1.14 \cdot 10^{-13}$ |
| 0.11 | $8.21261 \cdot 10^{-15}$ | $8.13862 \cdot 10^{-15}$ | $1.03636 \cdot 10^{-14}$ |
| 0.21 | $4.30184 \cdot 10^{-15}$ | $4.28146 \cdot 10^{-15}$ | $5.42857 \cdot 10^{-15}$ |
| 0.31 | $2.91415 \cdot 10^{-15}$ | $2.90478 \cdot 10^{-15}$ | $3.67742 \cdot 10^{-15}$ |

In the second case, we assume that the e.d.f. $F$ follows a Weibull distribution with the form (4.2). We take the parameters in (4.2) with values $\lambda = 4.52874$ and $b = 0.1$.

In Tables 13–16, we take $\rho = 0.5$ and $u = 10,\ 100,\ 1000,\ 10,000$.

TABLE 13. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.5$, $u = 10$, $\psi_0(u) = 0.00335$, $\Delta(u) = 0.00255419$, $D(u) = 0.377721$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.00631071 | 0 | 0.0158612 |
| 0.11 | 0.000573701 | 0 | 0.00103277 |
| 0.21 | 0.00030051 | 0.000040936 | 0.000464925 |
| 0.31 | 0.000203571 | 0.0000701342 | 0.000288563 |

TABLE 14. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.5$, $u = 100$, $\psi_0(u) = 0.000766$, $\Delta(u) = 0.00324607$, $D(u) = 0.431774$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.000164834 | 0 | 0.00027483 |
| 0.11 | 0.0000149849 | 0.0000120136 | 0.0000167677 |
| 0.21 | $7.84925 \cdot 10^{-6}$ | $6.98146 \cdot 10^{-6}$ | $8.39178 \cdot 10^{-6}$ |
| 0.31 | $5.31724 \cdot 10^{-6}$ | $4.9043 \cdot 10^{-6}$ | $5.58536 \cdot 10^{-6}$ |

TABLE 15. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.5$, $u = 1000$, $\psi_0(u) = 0.00012$, $\Delta(u) = 0.00803757$, $D(u) = 0.487477$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $2.90154 \cdot 10^{-6}$ | $2.33555 \cdot 10^{-6}$ | $3.22743 \cdot 10^{-6}$ |
| 0.11 | $2.63777 \cdot 10^{-7}$ | $2.56639 \cdot 10^{-7}$ | $2.7069 \cdot 10^{-7}$ |
| 0.21 | $1.38169 \cdot 10^{-7}$ | $1.3564 \cdot 10^{-7}$ | $1.4117 \cdot 10^{-7}$ |
| 0.31 | $9.35981 \cdot 10^{-8}$ | $9.21779 \cdot 10^{-8}$ | $9.5481 \cdot 10^{-8}$ |

TABLE 16. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.5$, $u = 10,000$, $\psi_0(u) = 0.0000115$, $\Delta(u) = 0.00252021$, $D(u) = 0.543501$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $3.11727 \cdot 10^{-8}$ | $3.05386 \cdot 10^{-8}$ | $3.15772 \cdot 10^{-8}$ |
| 0.11 | $2.83388 \cdot 10^{-9}$ | $2.82312 \cdot 10^{-9}$ | $2.84925 \cdot 10^{-9}$ |
| 0.21 | $1.48442 \cdot 10^{-9}$ | $1.47996 \cdot 10^{-9}$ | $1.49192 \cdot 10^{-9}$ |
| 0.31 | $1.00557 \cdot 10^{-9}$ | $1.00284 \cdot 10^{-9}$ | $1.01053 \cdot 10^{-9}$ |

In Tables 17–20, we take $\rho = 0.95$ and $u = 10,\ 100,\ 1000,\ 10,000$.

TABLE 17. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.95$, $u = 10$, $\psi_0(u) = 0.0658$, $\Delta(u) = 0.0364189$, $D(u) = 0.377721$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.0119903 | 0 | 0.0334215 |
| 0.11 | 0.00109003 | 0 | 0.00226093 |
| 0.21 | 0.000570969 | 0 | 0.0010398 |
| 0.31 | 0.000386785 | 0.0000673069 | 0.000654247 |

TABLE 18. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.95$, $u = 100$, $\psi_0(u) = 0.0156$, $\Delta(u) = 0.0753496$, $D(u) = 0.431774$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | 0.000313185 | 0 | 0.000586281 |
| 0.11 | 0.0000284714 | 0.0000195145 | 0.0000376861 |
| 0.21 | 0.0000149136 | 0.0000115302 | 0.0000189969 |
| 0.31 | 0.0000101028 | 0.00000814317 | 0.00001268 |

TABLE 19. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.95$, $u = 1000$, $\psi_0(u) = 0.00246$, $\Delta(u) = 0.0876195$, $D(u) = 0.487477$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.51293 \cdot 10^{-6}$ | $3.96143 \cdot 10^{-6}$ | $7.06109 \cdot 10^{-6}$ |
| 0.11 | $5.01175 \cdot 10^{-7}$ | $4.44331 \cdot 10^{-7}$ | $5.98762 \cdot 10^{-7}$ |
| 0.21 | $2.6252 \cdot 10^{-7}$ | $2.35044 \cdot 10^{-7}$ | $3.12459 \cdot 10^{-7}$ |
| 0.31 | $1.77836 \cdot 10^{-7}$ | $1.59779 \cdot 10^{-7}$ | $2.11381 \cdot 10^{-7}$ |

TABLE 20. Weibull case with $\lambda = 4.52874$, $b = 0.1$, $\rho = 0.95$, $u = 10,000$, $\psi_0(u) = 0.00024$, $\Delta(u) = 0.101166$, $D(u) = 0.543501$.

| $r$ | $\psi_r^*(u)$ | $\psi_r^-(u)$ | $\psi_r^+(u)$ |
|---|---|---|---|
| 0.01 | $5.92282 \cdot 10^{-8}$ | $5.31033 \cdot 10^{-8}$ | $7.07742 \cdot 10^{-8}$ |
| 0.11 | $5.38438 \cdot 10^{-9}$ | $4.91666 \cdot 10^{-9}$ | $6.39334 \cdot 10^{-9}$ |
| 0.21 | $2.82039 \cdot 10^{-9}$ | $2.57764 \cdot 10^{-9}$ | $3.34787 \cdot 10^{-9}$ |
| 0.31 | $1.91059 \cdot 10^{-9}$ | $1.74668 \cdot 10^{-9}$ | $2.26767 \cdot 10^{-9}$ |

**Acknowledgment.** We feel it is our duty to express our gratitude to the late Prof. V. V. Kalashnikov for his numerous valuable discussions on the paper.

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