A generalization of the semi discrete scheme applied on the CIR process

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Abstract

In this paper we generalize an explicit numerical scheme for the CIR process that we have proposed before. The advantage of this scheme is that preserves positivity as well and is well posed for a broader set of parameters than the existing schemes. The order of convergence is logarithmic.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)\) be a complete probability space with a filtration and let a Wiener process \((W_t)_{t \geq 0}\) defined on this space. We consider here the CIR process,

\[ x_t = x_0 + \int_0^t (kl - kx_s)ds + \sigma \int_0^t \sqrt{x_s}dW_s, \]

(1)

where \(k, l, \sigma \geq 0\). It is well known that this sde has a unique strong solution which remain nonnegative. Let \(0 = t_0 < t_1 < \ldots < t_n = T\) and set \(\Delta = \frac{T}{n}\). We propose the following numerical scheme,

\[ y_t = \frac{\sigma}{2(1 + ka\Delta)}(W_t - W_{t_k}) + y_{t_k}(1 - \frac{k\Delta}{1 + ka\Delta}) + \frac{\Delta}{1 + ka\Delta}(kl - \frac{\sigma^2}{4(1 + ka\Delta)}) = (z_t)^2, \]

(2)

for \(t \in (t_k, t_{k+1}]\) and a parameter \(a \in [0, 1]\). Note that this scheme is the same as in [3] if we choose \(a = 0\) and it is well posed when \(a\Delta \geq \frac{\sigma^2 - 4kl}{4ka\Delta}\). This stochastic process has the differential form, for \(t \in [t_k, t_{k+1}]\),

\[ y_t = y_{t_k} + \Delta \left( kl - \frac{\sigma^2}{4(1 + ka\Delta)} - k(1 - a)y_{t_k} - kay_{t_k} \right) + \int_{t_k}^t \frac{\sigma^2}{4(1 + ka\Delta)}ds \]

\[ + \frac{\sigma}{1 + ka\Delta} \int_{t_k}^t sgn(z_s)\sqrt{y_s}dW_s, \]

(3)

At this point we would like to point out that in [3] we have claimed that the differential form (for \(a = 0\)) is

\[ y_t = y_{t_k} + \Delta(\sigma^2 - 4k) + \frac{\sigma^2}{4}ds + \sigma \int_0^t \sqrt{y_s}dW_s, \]

The difference is the appearance of \(sgn(z_s)\) but fortunately we can show that this is not a big problem and the two differential forms are close enough.

For a generalization of the semi discrete method see [4].
2 Main Results

We will use a compact form of (3), for $t \in [t_k, t_{k+1})$,$\quad y_t = x_0 + \int_0^t (k(t-k(1-a))y_s - k\sigma y_s)ds$

$\quad + \int_t^{t_k} \left( k - \frac{\sigma^2}{4(1 + ka\Delta)} - k(1-a)y_t - k\sigma y_{t+k} \right) ds + \sigma \int_0^t \text{sgn}(z_s)\sqrt{y_s}dW_s,$

where $\tilde{t} = t$, $\tilde{s} = t_k$ when $s \in (t_k, t]$.

**Assumption A** We suppose that $\mathbb{E}x_0^p < A$ for some $p \geq 2$, $d = kl - \frac{\sigma^2}{4(1 + ka\Delta)} \geq 0$ and $\Delta(1-a) \leq \frac{1}{\xi}$.

**Lemma 1 (Moment bounds)** Under Assumption A we have the moment bounds,

$\mathbb{E}y_t^p + \mathbb{E}x_t^p < C,$

for some $C > 0$.

**Proof.** Note that

$0 \leq y_t \leq x_0 + Tk + \frac{\sigma}{1 + ka\Delta} \int_0^t \text{sgn}(z_s)\sqrt{y_s}dW_s.$

Consider the stopping time $\theta_R = \inf\{t \geq 0 : v_t > R\}$. Using Ito’s formula on $v_{t,\theta_R}^p$ we obtain,

$\quad v_{t,\theta_R}^p = (x_0 + Tk)^p + \frac{p(p-1)}{2(1 + ka\Delta)^2}\sigma^2 \int_0^t v_{s,\theta_R}^{p-2}y_s \text{sgn}(z_s)\sqrt{y_s}dW_s + \frac{p\sigma}{1 + ka\Delta} \int_0^t v_{s,\theta_R}^{p-1}\text{sgn}(z_s)\sqrt{y_s}dW_s.$

Taking expectations on both sides and noting that $y_t \leq v_t$, we arrive at

$\mathbb{E}v_{t,\theta_R}^p \leq \mathbb{E}(x_0 + Tk)^p + \frac{p(p-1)}{2}\sigma^2 \int_0^t \mathbb{E}v_{s,\theta_R}^{p-1}ds + \frac{p\sigma}{1 + ka\Delta} \int_0^t (\mathbb{E}v_{s,\theta_R}^{p-1})^{\frac{p-1}{p}} ds$

Using now a Gronwall type theorem (see [7], Theorem 1, p. 360), we arrive at

$\mathbb{E}v_{t,\theta_R}^p \leq \left( \mathbb{E}(x_0 + Tk)^p \right)^{\frac{p-1}{p}} + \frac{T(p-1)\sigma^2}{2(1 + ka\Delta)^2}, \quad (4)$

But $\mathbb{E}v_{t,\theta_R}^p = \mathbb{E}(v_{t,\theta_R}^p \mathbb{1}_{(\theta_R \geq t)}) + R^p P(\theta_R < t)$. That means that $P(t \wedge \theta_R < t) = P(\theta_R < t) \to 0$ as $R \to \infty$ so $t \wedge \theta_R \to t$ in probability and noting that $\theta_R$ increases as $R$ increases we have that $t \wedge \theta_R \to t$ almost surely too, as $R \to \infty$. Going back to (4) and using Fatou’s lemma we obtain,

$\mathbb{E}v_t^p \leq \left( \mathbb{E}(x_0 + Tk)^p \right)^{\frac{p-1}{p}} + \frac{T(p-1)\sigma^2}{2(1 + ka\Delta)^2}, \quad (5)$

The same holds for $x_t$. \hfill \Box

Consider the stochastic process,

$\quad h_t = x_0 + \int_0^t (k(t-k(1-a))y_s - k\sigma y_s)ds + \frac{\sigma}{1 + ka\Delta} \int_0^t \text{sgn}(z_s)\sqrt{y_s}dW_s,$

**Lemma 2** We have the following estimates,

$\quad \mathbb{E}|h_s - y_s|^2 \leq C_1 \Delta^2$ for any $s \in [0, T]$ \\
$\quad \mathbb{E}|h_s - y_s|^2 \leq C_2 \Delta$ when $s \in [t_k, t_{k+1}]$ \\
$\quad \mathbb{E}|h_s - y_{t+k}^2| \leq C_3 \Delta$ when $s \in [t_k, t_{k+1}]$ \\
$\quad \mathbb{E}|h_s|^p < A$, for any $s \in [0, T]$ and some $p \geq 2$. 

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Proof. Using the moment bound for $y_t$ we easily obtain the fact that
\[
\mathbb{E}|h_t - y_t|^2 \leq C\Delta^2.
\]

Next, we have
\[
\mathbb{E}|h_s - y_t|^2 \leq 2\mathbb{E}|h_s - y_s|^2 + 2\mathbb{E}|y_s - y_t|^2 \leq C\Delta^2 + C\Delta \leq C\Delta.
\]

Moreover,
\[
\mathbb{E}|h_s - y_{t_k+1}|^2 \leq \mathbb{E}|h_s - y_s|^2 + 2\mathbb{E}|y_s - y_{t_k+1}|^2 \leq C\Delta^2 + C\Delta \leq C\Delta.
\]

Finally, to get the moment bound for $h_t$ we just use the fact that is close to $y_t$, i.e.
\[
\mathbb{E}h_t^2 \leq 2\mathbb{E}|h_t - y_t|^2 + 2\mathbb{E}y_t^2 \leq C.
\]

\[
\square
\]

Lemma 3 Under Assumption A we have the estimate for the following probability, when $t \in [t_k, t_{k+1}]$,
\[
\mathbb{P}\left( W_t - W_{t_k} < -\frac{2(1 + ka\Delta)}{\sigma} \sqrt{\frac{\Delta}{1 + ka\Delta}} (kl - \frac{\sigma^2}{4(1 + ka\Delta)})y_t(1 - \frac{k\Delta}{1 + ka\Delta}) \right) \leq \frac{\sigma\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{\pi^2}{4}}},
\]
with $d = \frac{kl - \frac{\sigma^2}{4(1 + ka\Delta)}}{1 + ka\Delta}$.

Proof. Since the increment $W_t - W_{t_k}$ is normally distributed with mean zero and variance $t - t_k$ we have that
\[
\mathbb{P}\left( W_t - W_{t_k} < -\frac{2(1 + ka\Delta)}{\sigma} \sqrt{\frac{\Delta}{1 + ka\Delta}} (kl - \frac{\sigma^2}{4(1 + ka\Delta)})y_t(1 - \frac{k\Delta}{1 + ka\Delta}) \right) \leq \int_{-\frac{\sigma\sqrt{\Delta}}{2\sqrt{d}}}^{\infty} e^{-\frac{u^2}{2}} du = \int_{\frac{\pi^2}{4}}^{\infty} e^{-\frac{\pi^2}{4}} du \leq \frac{\sigma\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{\pi^2}{4}}},
\]
We have used the inequality of problem 9.22, p.112 of [6] to obtain the last inequality.

\[
\square
\]

Theorem 1 If Assumption A holds then
\[
\sqrt{\mathbb{E}|x_t - y_t|^2} \leq C \frac{1}{\sqrt{n}}
\]
for any $t \in [0, T]$.

Proof. Applying Ito’s formula on $|x_t - h_t|^2$ we obtain
\[
\mathbb{E}|x_t - h_t|^2 \leq 2k(1 - a) \int_0^t \mathbb{E}|x_s - h_s||y_s - x_s| ds + 2ka \int_0^t \mathbb{E}|x_s - h_s||y_s - x_s| ds + \frac{2\sigma^2}{(1 + ka\Delta)^2} \int_0^t \mathbb{E}|x_s - y_s| ds + 2\sigma^2 \int_0^t \mathbb{E} \frac{sgn(z_s)}{1 + ka\Delta} - 1 \bigg| ds
\]
Let us estimate the above quantities.
\[
\mathbb{E}|x_s - h_s||y_s - x_s| \leq \mathbb{E}|x_s - h_s|^2 + \sqrt{\mathbb{E}|x_s - h_s|^2} \sqrt{\mathbb{E}|h_s - y_s|^2},
\]
\[
\mathbb{E}|x_s - h_s||y_s - x_s| \leq \mathbb{E}|x_s - h_s|^2 + \sqrt{\mathbb{E}|x_s - h_s|^2} \sqrt{\mathbb{E}|h_s - y_s|^2}.
\]

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Now, using Lemma 3 we get
\[ E|x_t| \frac{sgn(z)}{1 + ka\Delta} - 1|^2 \leq \frac{2}{1 + ka\Delta} E|x_t|sgn(z) - 1|^2 + \frac{2}{1 + ka\Delta} E|x_t|ka\Delta \]

But
\[ E|1 - sgn(z)|^2 x_s = E|1 - sgn(z)|^2 x_s I_{\{z_s \geq 0\}} + E|1 - sgn(z)|^2 x_s I_{\{z_s < 0\}} \]
\[ = 2E x_s I_{\{z_s < 0\}} = P(z_s < 0)E(x_s | \{z_s < 0\}) \]
\[ \leq \frac{\sigma\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{4\sigma^2}{\Delta}}} E(x_s | \{z_s < 0\}). \]

Then
\[ E|1 - sgn(z)|^2 x_s \leq \frac{\sigma\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{4\sigma^2}{\Delta}}} E(E(x_s | \{z_s < 0\})) \leq C \frac{\sqrt{\Delta}}{e^{\frac{4\sigma^2}{\Delta}}}. \]

Summing up we arrive at
\[ E|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \int_0^t E|x_s - h_s|^2 ds + \int_0^t E|x_s - h_s| ds. \] (6)

Therefore, we have to estimate \( E|x_t - h_t| \). Let the non increasing sequence \( \{e_m\}_{m \in \mathbb{N}} \) with \( e_m = e^{-m(m+1)/2} \) and \( e_0 = 1 \). We introduce the following sequence of smooth approximations of \( |x| \), (method of Yamada and Watanabe, [8])

\[ \phi_m(x) = \int_0^{|x|} dy \int_0^y \psi_m(u) du, \]

where the existence of the continuous function \( \psi_m(u) \) with \( 0 \leq \psi_m(u) \leq 2/(mu) \) and support in \( (e_m, e_{m-1}) \) is justified by \( \int_{e_{m-1}}^{e_m}(du/u) = m \). The following relations hold for \( \phi_m \in C^2(\mathbb{R}, \mathbb{R}) \) with \( \phi_m(0) = 0, \)

\[ |x| - e_{m-1} \leq \phi_m(x) \leq |x|, \quad |\phi'_m(x)| \leq 1, \quad x \in \mathbb{R}, \]

\[ |\phi''_m(x)| \leq \frac{2}{m|x|}, \quad \text{when} \ e_m < |x| < e_{m-1} \text{ and } |\phi''_m(x)| = 0 \text{ otherwise.} \]

Applying Ito’s formula on \( \phi_m(x_t - h_t) \) we obtain
\[ E\phi_m(x_t - h_t) = \int_0^t E\phi'_m(x_s - h_s)(k(1-a)(y_s - x_s) + ka(y_s - x_s))ds \]
\[ + \int_0^t \frac{1}{2} E\phi''_m(x_s - h_s) \left( \frac{\sigma}{1 + ka\Delta} sgn(z_s) \sqrt{y_s} - \sigma \sqrt{x_s} \right)^2 dW_s. \]

We continue by estimating
\[ E\phi'_m(x_s - h_s) (k(1-a)(y_s - x_s) + ka(y_s - x_s)) \]
\[ \leq kE|x_s - h_s| + k(1-a)E|h_s - y_s| + kaE|h_s - y_s| \]
\[ \leq kE|x_s - h_s| + C\sqrt{\Delta}. \]

Next,
\[ E\phi''_m(x_s - h_s) \left( \frac{\sigma}{1 + ka\Delta} sgn(z_s) \sqrt{y_s} - \sigma \sqrt{x_s} \right)^2 \leq \frac{4\sigma^2}{m(1 + ka\Delta)^2} \frac{4\sigma^2}{m(1 + ka\Delta)^2} E|\frac{h_s - y_s}{e_m}| \]
\[ + \frac{4\sigma^2}{m} E|x_s| \left| \frac{sgn(z_s)}{1 + ka\Delta} - 1 \right|^2. \]
Working as before and using Lemma 2 we get
\[ \mathbb{E}\phi''_m(x_s - h_s) \left( \frac{\sigma}{1 + ka\Delta} \text{sgn}(z_s) \sqrt{s} - \sigma \sqrt{x_s} \right)^2 \leq \frac{4\sigma^2}{m} + C \frac{\sqrt{\Delta}}{m \epsilon_m} + C \frac{\Delta}{m}. \]

Therefore,
\[ \mathbb{E}|x_t - h_t| \leq \epsilon_{m-1} + \frac{4\sigma^2}{m} + C \frac{\sqrt{\Delta}}{m \epsilon_m} + C \frac{\Delta}{m} + k \int_0^t \mathbb{E}|x_s - h_s| ds. \]

Use now Gronwall’s inequality and substitute in (6) and then again Gronwall’s inequality we arrive at
\[ \mathbb{E}|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \frac{\sqrt{\Delta}}{m \epsilon_m} + \epsilon_{m-1}. \]

Choosing \( m = \sqrt{\ln n} \) we deduce that
\[ \mathbb{E}|x_t - h_t|^2 \leq C \frac{1}{\sqrt{\ln n}}. \]

But
\[ \mathbb{E}|x_t - y_t|^2 \leq 2\mathbb{E}|x_t - h_t|^2 + 2\mathbb{E}|h_t - y_t|^2 \leq C \frac{1}{\sqrt{\ln n}}. \]

\[ \square \]

3 Conclusion

In [3], using the semi discrete method, we have proposed a numerical scheme that preserves positivity to approximate the CIR process. In this paper we generalize that result in order to be well posed for a broader set of parameters. Concerning the order of convergence we see that it is the same as the usual Euler scheme (see [2]). The difference between these methods is that ours preserve positivity. For other numerical schemes that preserves positivity see [5] and the references therein.

Below is the difference between the numerical scheme (2) for \( a = 1 \) and the scheme proposed in [1].

Figure 1: \( x_0 = 4, \Delta = 10^{-4}, k = 2, l = 1, s = 1, T = 1. \)

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References


