On the strong convergence for discontinuous SDEs

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We consider the following stochastic differential equation.

$$X_t = X_0 + \int_0^t f(X_s)ds + AW_t. \tag{1}$$

We do not assume that f is continuous but is monotonically increasing. We would like to give some remarks on [1].

Assumption A Suppose that there exists an upper solution U and a lower solution L such that

$$\mathbb{E}U_t^2 < A, \quad \mathbb{E}L_t^2 < A$$

for some $A > 0$ with $L_0 = X_0 = U_0$.

We construct the following discrete stochastic processes.

$$\begin{aligned} L_{n+1} &= \min\{L_{t_{n+1}}, L_n + f(L_n)\Delta + A\Delta W_n\}, \\ U_{n+1} &= \max\{U_{t_{n+1}}, U_n + f(U_n)\Delta + A\Delta W_n\}. \end{aligned}$$

It is clear that $L_n \leq L_{t_n}$ and $U_n \geq U_{t_n}$ and we have proved in [1] that $L_n \leq X_n^\Delta \leq U_n$. However, it is not true (under these assumptions) that the Euler scheme converges (even in probability). We have to impose further conditions.

Assumption B Suppose that there exists square integrable adapted processes $\tilde{L}_t, \tilde{U}_t$ such that $\tilde{L}_{t_n} \leq L_n$ and $\tilde{U}_{t_n} \geq U_n$. Suppose further that $X^\Delta \to X$ in probability where X is a solution to our problem.

Corollary 1 *Under Assumptions* **A,B** *we have that* $X^\Delta \to X$ *in* L^2.

1 Example

We will apply our ideas to the following example,

$$X_t = X_0 + \int_0^t H(X_s - 1)ds + AW_t, \tag{2}$$

where $H(x)$ is the Heaviside function.

We have shown in [1] that the upper and lower solutions are

$$\begin{aligned} L_t &= X_0 + AW_t \\ U_t &= X_0 + t + AW_t \end{aligned}$$

Therefore, we will show by induction, that $L_n = L_{t_n}, U_n = U_{t_n}$ thus $\tilde{L} = L, \tilde{U} = U$. For $n = 1$ we have and using the fact that $H(x) \geq 0$,

$$L_1 = \min\{X_0 + W_{t_1}, X_0 + H(X_0 - 1)\Delta + AW_{t_1}\} = L_{t_1}$$

Suppose now that $L_n = L_{t_n}$ for $n = 1, 2, ..., k$ and will show that $L_{k+1} = L_{t_{k+1}}$.

$$L_{k+1} = \min\{X_0 + W_{t_{k+1}}, L_{t_k} + H(L_{t_k} - 1)\Delta + AW_{t_{k+1}} - AW_{t_k}\}$$
$$= \min\{X_0 + W_{t_{k+1}}, X_0 + AW_{t_k} + H(L_{t_k} - 1)\Delta + AW_{t_{k+1}} - AW_{t_k}\} = L_{t_{k+1}}.$$

The same arguments holds for the equality $U_n = U_{t_n}$.

In order to prove now that the Euler scheme converges in probability we will use Corollary 2.6 of [2]. Set $D = (1, +\infty)$, $D_k = (1, k)$, $V(t, x) = x$ and $X_0 > 1$ a.s. Then we have that the Euler scheme converges in probability to the unique solution of problem (2). Therefore, the convergence is also in L^2.

References

[1] N. Halidias and P. Kloeden *A note on the EulerMaruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient*, BIT Numerical Mathematics (2008) 48: 5159

[2] I. Gyongy and N. Krylov *Existence of strong solutions for Ito's stochastic equations via approximations*, Probab. Theory Relat. Fields 105, 143-158 (1996)