On the order of convergence for the semi discrete method applied on the CIR process

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Abstract

In this paper we show that the order of convergence of the semi discrete numerical scheme that we have proposed before to approximate the CIR process is tending to be one as \( \Delta \to 0 \).

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1 Introduction

Let \( (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t) \) be a complete probability space with a filtration and let a Wiener process \((W_t)_{t \geq 0}\) defined on this space. We consider here the CIR process,

\[
x_t = x_0 + \int_0^t (k l - k x_s) ds + \sigma \int_0^t \sqrt{x_s} dW_s.
\]

It is well known that this sde has a unique strong solution which remain nonnegative. The aim of this paper is to study a previously proposed numerical scheme that preserves positivity. Let \( 0 = t_0 < t_1 < ... < t_n = T \) and set \( \Delta = \frac{T}{n} \). In [2] we have proposed the following numerical scheme,

\[
y_t = \left( \frac{\alpha}{2} (W_t - W_{t_k}) + \sqrt{y_{t_k} + \Delta (k l - \frac{\sigma^2}{4} - k y_{t_k})} \right)^2 = (z_t)^2,
\]

for \( t \in (t_k, t_{k+1}] \). This stochastic process has the differential form,

\[
y_t = y_{t_k} + \Delta (k l - \frac{\sigma^2}{4} - k y_{t_k}) + \int_{t_k}^t \frac{\sigma^2}{4} ds + \sigma \int_{t_k}^t \text{sgn}(z_s) \sqrt{y_s} dW_s,
\]
and not
\[ y_t = y_{t_k} + \Delta (k l - \frac{\sigma^2}{4} - k y_{t_k}) + \int_{t_k}^t \frac{\sigma^2}{4} ds + \sigma \int_{t_k}^t \sqrt{y_s} dW_s, \]
as we have claimed in [2]. However, these two forms are close enough as we will see later.

For a generalization of the semi discrete method see [3].

## 2 Main Results

We will use also a compact form of (2),
\[ y_t = x_0 + \int_0^t (k l - k y_s)ds + \int_t^t (k l - \frac{\sigma^2}{4} - k y_s)ds + \sigma \int_0^t sgn(z_s)\sqrt{y_s}dW_s, \]
where \( \hat{t} = t_{k+1} \) when \( t \in (t_k, t_{k+1}] \).

Consider the following stochastic process,
\[ p_t = \int_0^t k(x_s - y_s)ds + \Delta kl + \sigma \int_0^t (sgn(z_s)\sqrt{y_s} - \sqrt{x_s})dW_s, \]
\[ q_t = \int_0^t k(x_s - y_s)ds - \Delta (kl + \frac{\sigma^2}{4} + k y_s) + \sigma \int_0^t (sgn(z_s)\sqrt{y_s} - \sqrt{x_s})dW_s. \]

Then, it is clear that \( q_t \leq e_t \leq p_t \) and that \( |e_t|^r \leq u_t = \max\{|p_t|^r, |q_t|^r\} \).

**Assumption A** We suppose that \( E x_0^r < A \) for any \( r \geq 2 \), \( d = kl - \frac{\sigma^2}{4} \geq 0 \) and \( 1 - k \Delta \geq 0 \).

**Lemma 1 (Moment bounds)** Under Assumption A we have the moment bounds,
\[ E y_t^r + E p_t^r + E q_t^r < A(r). \]

**Proof.** Note that
\[ 0 \leq y_t \leq v_t = x_0 + Tkl + \int_0^t \sigma sgn(z_s)\sqrt{y_s}dW_s. \]

Consider the stopping time \( \theta_R = \inf\{t \geq 0 : y_t > R\} \). Using Ito’s formula on \( v_t^r \) we obtain,
\[ v_{t \wedge \theta_R}^r = (x_0 + Tkl)^r + \frac{r(r - 1)}{2} \sigma^2 \int_0^t v_{s \wedge \theta_R}^{r-2} y_{s \wedge \theta_R} ds + r\sigma \int_0^t v_{s \wedge \theta_R}^{r-1} sgn(z_s)\sqrt{y_{s \wedge \theta_R}} dW_s. \]

Taking expectations on both sides and noting that \( y_t \leq v_t \), we arrive at
\[
E v_{t \wedge \theta_R}^r \leq E(x_0 + Tkl)^r + \frac{r(r - 1)}{2} \sigma^2 \int_0^t E v_{s \wedge \theta_R}^{r-1} ds \\
\leq E(x_0 + Tkl)^r + \frac{r(r - 1)}{2} \sigma^2 \int_0^t (E v_{s \wedge \theta_R}^{r-1})^{\frac{1}{r-1}} ds
\]
Using now a Gronwall type theorem (see [6], Theorem 1, p. 360), we arrive at

\[ \mathbb{E} v^r_t \leq \left( \mathbb{E}|x_0| + Tk + \frac{T(r-1)\sigma^2}{2} \right)^r, \]  

(3)

But \( \mathbb{E} v^r_t = \mathbb{E}(v^r_t \mathbb{1}_{\{\theta_R \geq t\}}) + R^r P(\theta_R < t). \) That means that \( P(t \land \theta_R < t) = P(\theta_R < t) \to 0 \) as \( R \to \infty \) so \( t \land \theta_R \to t \) in probability and noting that \( \theta_R \) increases as \( R \) increases we have that \( t \land \theta_R \to t \) almost surely too, as \( R \to \infty \). Going back to (3) and using Fatou’s lemma we obtain,

\[ \mathbb{E} v^r_t \leq \left( \mathbb{E}|x_0| + Tk + \frac{T(r-1)\sigma^2}{2} \right)^r, \]

for any \( r \geq 2 \). The same holds for \( x_t \).

Let us estimate \( p_t \). We consider the stopping time \( \theta_R = \inf\{t \geq 0 : p_t > R\} \). Again, using Ito’s formula on \( p_t \) we obtain

\[ \mathbb{E}|p_t \land \theta_R|^r \leq \Delta^r k^r T + rk \int_0^t \mathbb{E}|p_{s \land \theta_R}|^{r-1}|x_s - y_s| + \frac{r(r-1)\sigma^2}{2} \mathbb{E}|p_{s \land \theta_R}|^{r-2}sgn(z_s)\sqrt{y_s} - \sqrt{x_s}^2 ds. \]

Now we have, using Young’s inequality and the moment bounds for \( x_t, y_t \),

\[ \mathbb{E}|p_{s \land \theta_R}|^{r-1}|x_s - y_s| \leq \frac{r-1}{r} \mathbb{E}|p_{s \land \theta_R}|^r + \frac{1}{r} \mathbb{E}|x_s - y_s|^r \leq \frac{r-1}{r} \mathbb{E}|p_{s \land \theta_R}|^r + C(r). \]

With similar arguments we have

\[ \mathbb{E}|p_{s \land \theta_R}|^{r-2}sgn(z_s)\sqrt{y_s} - \sqrt{x_s}^2 \leq \frac{r^2}{r} \mathbb{E}|p_{s \land \theta_R}|^r + C(r). \]

Therefore, using Gronwall inequality we arrive at the desired result.

The same holds for \( q_t \). \( \square \)

Lemma 2 Under Assumption A we have the estimate for the following probability, when \( t \in [t_k, t_{k+1}] \),

\[ \mathbb{P}\left(W_t - W_{t_k} < -\frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}} + y_{tk} (1 - k\Delta)\right) \leq \frac{\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{2\Delta}{\sigma^2}}} \]

Proof. Since the increment \( W_t - W_{t_k} \) is normally distributed with mean zero and variance \( t - t_k \) we have have that

\[ \mathbb{P}\left(W_t - W_{t_k} < -\frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}} + y_{tk} (1 - k\Delta)\right) \]

\[ \leq \mathbb{P}\left(W_t - W_{t_k} < -\frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}}\right) \]

\[ = \mathbb{P}\left(W_t - W_{t_k} > \frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}}\right) \]

\[ = \int_{\frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}}}^{\infty} e^{-\frac{u^2}{2(t - t_k)}} du = \int_{\frac{2}{\sigma} \sqrt{\Delta kl - \frac{\sigma^2}{4}}}^{\infty} e^{-u^2/2} du \leq \frac{\sqrt{\Delta}}{2\sqrt{d}} \frac{1}{e^{\frac{2\Delta}{\sigma^2}}} \]

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where \( d = kl - \frac{\sigma^2}{4} \). We have used the inequality of problem 9.22, p.112 of [5] to obtain the last inequality.

\[ \square \]

**Lemma 3** Under Assumption A, for any \( t \in [t_k, t_{k+1}] \), we have
\[
\mathbb{E}|y_t - y_{t_k}|^r \leq C(r) \Delta.
\]

**Proof.** Using Ito’s formula and the moment bounds we obtain,
\[
\mathbb{E}|y_t - y_{t_k}|^r \leq \Delta^r \mathbb{E}|kl - \frac{\sigma^2}{4} - ky_t|^r + \int_{t_k}^t \mathbb{E}|y_s - y_{t_k}|^{r-1} \frac{\sigma^2}{4} ds + \frac{\sigma^2 r(r-1)}{2} \mathbb{E}|y_s - y_{t_k}|^{r-2} ds \leq C(r) \Delta
\]

\[ \square \]

**Theorem 1** Fix some \( r \geq 2 \). If Assumption A holds then there exists a \( C(r) \) such that
\[
\sqrt{\mathbb{E}|x_t - y_t|^2} \leq C(r) \Delta^{1-\frac{1}{r}}
\]
for any \( t \in [0, t] \).

**Proof.**

Let \( e_t \) the difference between the true and the approximate solution, i.e.
\[
e_t = y_t - x_t = \int_0^t k(y_s - x_s) ds + \int_0^t (kl - \frac{\sigma^2}{4} - ky_s) ds + \int_0^t \sigma (sgn(z_s) \sqrt{y_s} - \sqrt{x_s}) dW_s.
\]

Using Ito’s formula for \( t \in [0, t_1] \),
\[
|p_t|^r = \Delta^r k^r \tau^r + rk \int_0^t |p_s|^{r-2} p_s(x_s - y_s) + \frac{r(r-1)\sigma^2}{2} |p_s|^{r-3} p_s(sgn(z_s) \sqrt{y_s} - \sqrt{x_s})^2 ds
\]
\[
+ r \sigma \int_0^t |p_s|^{r-2} p_s(sgn(z_s) \sqrt{y_s} - \sqrt{x_s}) dW_s
\]
\[
\leq \Delta^r k^r \tau^r + rk \int_0^t u_s ds + rk \int_0^t |p_s|^{r-1} y_s - y_s| ds + \frac{r(r-1)\sigma^2}{2} \int_0^t 2u_s \tau^{-1}
\]
\[
+ 2(1 - sgn(z_s)) x_s |p_s|^{r-2} ds + r \sigma \int_0^t |p_s|^{r-2} p_s(sgn(z_s) \sqrt{y_s} - \sqrt{x_s}) dW_s
\]
\[
\leq \Delta^r k^r \tau^r + rk \int_0^t u_s ds + rk \int_0^t |p_s|^{r-1} y_s - y_s| ds + \frac{r(r-1)\sigma^2}{2} \int_0^t 2u_s \tau^{-1}
\]
\[
+ 2(1 - sgn(z_s)) x_s |p_s|^{r-2} ds + r \sigma \int_0^t |p_s|^{r-2} p_s(sgn(z_s) \sqrt{y_s} - \sqrt{x_s}) dW_s
\]

Note that
\[
|p_s|^{r-1} |x_s - y_s| = (|p_s|^r)^{\frac{r-1}{r}} (|x_s - y_s|^r)^{\frac{1}{r}} \leq u_s,
\]
Therefore we have,

\[ |q_t| \leq \Delta^r (kl + \frac{\sigma^2}{4} + ky_t)^r + rk \int_0^t u_s ds + rk \int_0^t |q_s|^{r-1}|y_s - y_t| ds + \]

\[ \frac{r(r-1)\sigma^2}{2} \int_0^t 2u_s^{\frac{r-1}{r}} + 2(1 - sgn(z_s))x_s |q_s|^{r-2} ds + r\sigma \int_0^t |q_s|^{r-2} q_s(sgn(z_s)\sqrt{y_s} - \sqrt{x_s}) dW_s. \]

Therefore we obtain,

\[ u_t \leq \Delta^r (kl + \frac{\sigma^2}{4} + ky_t)^r + rk \int_0^t u_s ds + \max \left\{ rk \int_0^t |p_s|^{r-1}|y_s - y_t| ds, rk \int_0^t |q_s|^{r-1}|y_s - y_t| ds \right\}

\[ + \frac{r(r-1)\sigma^2}{2} \int_0^t 2u_s^{\frac{r-1}{r}} + 2(1 - sgn(z_s))x_s u_s^{\frac{r-2}{r}} ds \]

\[ + \max \left\{ 2\sigma \int_0^t |p_s|^{r-2} p_s(sgn(z_s)\sqrt{y_s} - \sqrt{x_s}) dW_s, 2\sigma \int_0^t |q_s|^{r-2} q_s(sgn(z_s)\sqrt{y_s} - \sqrt{x_s}) dW_s \right\}. \]

Thus,

\[ \mathbb{E} u_t \leq \Delta^r \mathbb{E} (kl + \frac{\sigma^2}{4} + ky_t)^r + rk \int_0^t \mathbb{E} u_s ds + \int_0^t \mathbb{E} u_s^{\frac{r-1}{r}} |y_s - y_t| ds + \sigma^2 \int_0^t (\mathbb{E} u_s)^{\frac{r-1}{r}} ds \]

\[ + C(r) \frac{\sigma \sqrt{\Delta}}{2\sqrt{d} e^{\frac{2r}{2\Delta}}} + 2\sigma \int_0^t |p_s|^{r-2} p_s(sgn(z_s)\sqrt{y_s} - \sqrt{x_s}) dW_s \]

\[ + 2\sigma \int_0^t |q_s|^{r-2} q_s(sgn(z_s)\sqrt{y_s} - \sqrt{x_s}) dW_s. \]

But

\[ \mathbb{E} u_s^{\frac{r-1}{r}} |y_s - y_t| \leq (\mathbb{E} u_s)^{\frac{r-1}{r}} (\mathbb{E} |y_s - y_t|^r)^{\frac{1}{r}} \leq C(r) \Delta^{\frac{1}{r}} (\mathbb{E} u_s)^{\frac{r-1}{r}}. \]

We have used Lemma 2 to prove that

\[ \mathbb{E} (1 - sgn(z_s))x_s u_s^{\frac{r-2}{r}} = \mathbb{E} (1 - sgn(z_s))x_s u_s^{\frac{r-2}{r}} \mathbb{I}_{\{z_s \geq 0\}} + \mathbb{E} (1 - sgn(z_s))x_s u_s^{\frac{r-2}{r}} \mathbb{I}_{\{z_s < 0\}} \]

\[ = 2\mathbb{E} x_s u_s^{\frac{r-2}{r}} \mathbb{I}_{\{z_s < 0\}} = \mathbb{P}(z_s < 0) \mathbb{E}(x_s u_s^{r-2} | \{z_s < 0\}) \]

\[ \leq \frac{\sigma \sqrt{\Delta}}{2\sqrt{d} e^{\frac{2r}{2\Delta}}} \mathbb{E} \left( x_s u_s^{r-2} | \{z_s < 0\} \right). \]

Therefore we have

\[ \mathbb{E} (1 - sgn(z_s))x_s u_s^{\frac{r-2}{r}} \leq \frac{\sigma \sqrt{\Delta}}{2\sqrt{d} e^{\frac{2r}{2\Delta}}} \mathbb{E} \left( x_s u_s^{r-2} | \{z_s < 0\} \right) \leq C(r) \frac{\sqrt{\Delta}}{e^{\frac{2r}{2\Delta}}}. \]

If \( h_t = \mathbb{E} u_t \) we arrive at,

\[ h_t \leq C(r) \Delta^r + C(r) \frac{\sqrt{\Delta}}{e^{\frac{2r}{2\Delta}}} + rk \int_0^t h_s ds + (C(r) \Delta^{\frac{1}{r}} + \sigma^2) \int_0^t h_s^{\frac{r-1}{r}} ds \]

\[ \leq C(r) \Delta^r + rk \int_0^t h_s ds + (C(r) \Delta^{\frac{1}{r}} + \sigma^2) \int_0^t h_s^{\frac{r-1}{r}} ds. \]
Using a Gronwall type inequality (see [6], Theorem 1, p. 360) we obtain,

\[ h_t \leq C(r)\Delta^r \text{ for } t \in [0,t_1] \]

We continue by estimating for \( t \in [t_1,t_2] \) using the estimate that we have obtained already and thus, step by step, we are able to show that

\[ h_t \leq C(r)\Delta^{r-1} \]

for any \( t \in [0,T] \). That means,

\[ \sqrt{\mathbb{E}|x_t - y_t|^2} \leq (\mathbb{E}|x_t - y_t|^r)^{\frac{1}{r}} \leq C(r)\Delta^{1-\frac{1}{r}}. \]

\[ \Box \]

## 3 Conclusion

In [2], using the semi discrete method, we have proposed a numerical scheme that preserves positivity to approximate the CIR process. In this paper we have studied the order of convergence and find that the order is tending to one as \( \Delta \to 0 \). For other numerical schemes that preserves positivity see [4] and the references therein.

Below, is the difference between our method and the method that first proposed in [1].

Figure 1: \( x_0 = 4, \Delta = 10^{-4}, k = 2, l = 1, s = 1 \ T = 1. \)

Another issue is to construct numerical methods to approximate the CIR process which can be applied for a broader class of parameters. One such a scheme can be constructed by the semi discrete idea and is the following,

\[ y_t = \left( \frac{\sigma(W_t - W_{t_k})}{2(1+k\Delta)} + \sqrt{\frac{y_{t_k}}{1+k\Delta} + \frac{\Delta}{1+k\Delta}(kl - \frac{\sigma^2}{4(1+k\Delta)})} \right)^2. \tag{4} \]

This scheme is well posed when \( \sigma^2 \leq 4(1+k\Delta)kl \). Below is the difference between the numerical scheme (4) and the scheme proposed in [1].

As one can see we can produce several schemes, using the semi discrete method, for this and for other sdes but the difficulty is to prove that converges in some sense and that is we are going to do in the future.
Figure 2: \( x_0 = 4, \Delta = 10^{-4}, k = 2, l = 1, s = 1 \) \( T = 1 \).

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**References**


