Στα επόμενα θεωρούμε ότι όλα συμβαίνουν σε ένα χώρο πιθανότητας $\left(\Omega, \mathcal{F}, P\right)$.

Modes of convergence: Οι τρόποι σύγκλισης μιας ακολουθίας τ.μ. $\left\{X_n\right\}_{n\geq 1}$ σε μια τ.μ. X είναι οι εξής:

1. Ισχυρή σύγκλιση – strong convergence
   $X_n \xrightarrow{a.s.} X \Leftrightarrow P\left\{\lim_{n\to\infty} X_n = X\right\} = 1$.

2. Σύγκλιση ως προς πιθανότητα – convergence in probability
   $X_n \xrightarrow{P} X \Leftrightarrow \lim_{n\to\infty} P\left\{\left|X_n - X\right| \geq \varepsilon\right\} = 0 \Leftrightarrow \lim_{n\to\infty} P\left\{\left|X_n - X\right| < \varepsilon\right\} = 1,\ \forall \varepsilon > 0$.

3. $L^p -$Σύγκλιση – $L^p -$ convergence
   $X_n \xrightarrow{L^p} X \Leftrightarrow \lim_{n\to\infty} \mathbb{E}\left\{\left|X_n - X\right|^p\right\} = 0,\ 0 < p < \infty$.

4. Σύγκλιση ως προς νόμο (ή κατά κατανομή) – convergence in probability law (in distribution)
   $X_n \xrightarrow{d} X \Leftrightarrow \lim_{n\to\infty} F_{X_n}\left(x\right) = F_X\left(x\right),\ x \in \mathbb{R}$.

<u>Πρόταση</u>:

$$X_n \xrightarrow{a.s.} X$$
$$\Downarrow$$
$$X_n \xrightarrow{\mathcal{L}^2} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$$

Θα δείξουμε ότι

1. $X_n \xrightarrow{L^p} X \Rightarrow X_n \xrightarrow{P} X$
2. $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$
3. $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$
4.

1. $\lim_{n\to\infty} P\left\{\left|X_n - X\right| \geq \varepsilon\right\} \leq \frac{1}{\varepsilon^p} \lim_{n\to\infty} \mathbb{E}\left\{\left|X_n - X\right|^p\right\} = 0$

2. Έχουμε ότι $X_n \xrightarrow{a.s.} X \Leftrightarrow P\left\{\left|X_n - X\right| \geq \varepsilon,\ i.o.\right\} = 0, \forall \varepsilon > 0$.

Θέτουμε $B_n = \left\{\left|X_n - X\right| \geq \varepsilon\right\}$, τότε

$$\lim_{n\to\infty} P\left(B_n\right) \leq \lim_{n\to\infty} P\left(J_n\left(\mathbb{B}\right)\right) = P\left(\lim_{n\to\infty} J_n\left(\mathbb{B}\right)\right) = P\left(\inf\left\{J_n\left(\mathbb{B}\right) : n \geq 1\right\}\right)$$

$$= P\left(\limsup_{n\to\infty} B_n\right) = P\left\{\left|X_n - X\right| \geq \varepsilon,\ i.o.\right\} = 0 \Rightarrow \lim_{n\to\infty} P\left\{\left|X_n - X\right| \geq \varepsilon\right\} = 0,\ \forall\, \varepsilon > 0$$

3. Ισχύει ότι $\left\{X_n \leq x\right\} \subset \left\{X_n + \varepsilon < X \leq x + \varepsilon\right\}$ το τελευταίο ενδεχόμενο, είναι η τομή των ενδεχομένων $\left\{X \leq x + \varepsilon\right\}$ και $\left\{X - X_n > \varepsilon\right\}$, και έτσι έχουμε
$\left\{X_n \leq x\right\} \subset \left\{X \leq x + \varepsilon\right\} \cup \left\{X - X_n > \varepsilon\right\}$ που δίνει

$$P\left\{X_n \leq x\right\} \leq P\left\{X \leq x + \varepsilon\right\} + P\left\{X - X_n > \varepsilon\right\} \leq P\left\{X \leq x + \varepsilon\right\} + P\left\{\left|X - X_n\right| > \varepsilon\right\},$$

ή ότι

$$F_{X_n}\left(x\right) \leq F_X\left(x + \varepsilon\right) + P\left\{\left|X - X_n\right| > \varepsilon\right\}.$$

Παίρνοντας το $\limsup_{n\to\infty}$ της προηγούμενης σχέσης, και επειδή
$\limsup_{n\to\infty} P\left\{\left|X - X_n\right| > \varepsilon\right\} = \lim_{n\to\infty} P\left\{\left|X - X_n\right| > \varepsilon\right\} = 0$, έχουμε

$$\limsup_{n\to\infty} F_{X_n}\left(x\right) \leq F_X\left(x + \varepsilon\right)$$

Ισχύει ότι $\left\{X \leq x - \varepsilon\right\} \subset \left\{X + \varepsilon < X_n \leq x\right\}$ το τελευταίο ενδεχόμενο, είναι η τομή των ενδεχομένων $\left\{X_n - x > \varepsilon\right\}$ και $\left\{X_n \leq x\right\}$, και έτσι έχουμε
$\left\{X \leq x - \varepsilon\right\} \subset \left\{X_n \leq x\right\} \cup \left\{X_n - X > \varepsilon\right\}$ που δίνει

$$P\left\{X \leq x - \varepsilon\right\} \leq P\left\{X_n \leq x\right\} + P\left\{X_n - X > \varepsilon\right\} \leq P\left\{X_n \leq x\right\} + P\left\{\left|X - X_n\right| > \varepsilon\right\}.$$

Παίρνοντας το $\liminf_{n\to\infty}$ της προηγούμενης σχέσης, και επειδή
$\liminf_{n\to\infty} P\left\{\left|X - X_n\right| > \varepsilon\right\} = \lim_{n\to\infty} P\left\{\left|X - X_n\right| > \varepsilon\right\} = 0$, έχουμε

$$F_X\left(x - \varepsilon\right) \leq \liminf_{n\to\infty} F_{X_n}\left(x\right).$$

Συνολικά λοιπόν

$$F_X\left(x - \varepsilon\right) \leq \liminf_{n\to\infty} F_{X_n}\left(x\right) \leq \limsup_{n\to\infty} F_{X_n}\left(x\right) \leq F_X\left(x + \varepsilon\right),$$

και αν το x είναι σημείο συνέχειας της F_X, θα έχουμε

$$F_X\left(x\right) \leq \liminf_{n\to\infty} F_{X_n}\left(x\right) \leq \limsup_{n\to\infty} F_{X_n}\left(x\right) \leq F_X\left(x\right).$$

Δηλαδή $\liminf_{n\to\infty} F_{X_n}(x) = F_X(x) = \limsup_{n\to\infty} F_{X_n}(x)$ ή ότι $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$.

Lemma: $X_n \xrightarrow{P} X$ as $n \to \infty$ implies $X_n - X_m \xrightarrow{P} 0$ as $n, m \to \infty$, or equivalently $\lim_{n\to\infty} P\{|X - X_n| > \varepsilon\} = 0,\ \forall \varepsilon > 0 \Rightarrow \lim_{\substack{n\to\infty \\ m\to\infty}} P\{|X_m - X_n| > \varepsilon\} = 0,\ \forall \varepsilon > 0$.

$$\{|X_m - X_n| > \varepsilon\} = \{|X_m - X| + |X_n - X| \geq |X_m - X_n| > \varepsilon\} \subseteq \{|X_m - X| + |X_n - X| > \varepsilon\}$$

$$\subseteq \left\{|X_m - X| > \frac{\varepsilon}{2}\right\} \cup \left\{|X_n - X| > \frac{\varepsilon}{2}\right\},\ \forall \varepsilon > 0.$$

Taking probabilities $P\{|X_m - X_n| > \varepsilon\} \leq P\left\{|X_m - X| > \frac{\varepsilon}{2}\right\} + P\left\{|X_n - X| > \frac{\varepsilon}{2}\right\}$ and then the limit as $n, m \to \infty$ gives

$$\lim_{\substack{n\to\infty \\ m\to\infty}} P\{|X_m - X_n| > \varepsilon\} \leq \lim_{m\to\infty} P\left\{|X_m - X| > \frac{\varepsilon}{2}\right\} + \lim_{n\to\infty} P\left\{|X_n - X| > \frac{\varepsilon}{2}\right\}$$

$$\Rightarrow \lim_{\substack{n\to\infty \\ m\to\infty}} P\{|X_m - X_n| > \varepsilon\} = 0,\ \forall \varepsilon > 0 \Rightarrow X_n - X_m \xrightarrow{P} 0.$$

Remark: The latter lemma tells us that if a sequence of r.v.s converges in probability then it is Cauchy in probability. The converse is also true (theorems 4, 5, and, 6 Shiryaev AN p 259).

Theorem (Protter P, Jacod J Probability Essentials p 143)

$$X_n \xrightarrow{P} X \Leftrightarrow \lim_{n\to\infty} \mathbb{E}\left\{\frac{|X_n - X|}{1 + |X_n - X|}\right\} = 0.$$

W.l.o.g. it suffices to show that $X_n \xrightarrow{P} 0 \Leftrightarrow \lim_{n\to\infty} \mathbb{E}\left\{\frac{|X_n|}{1 + |X_n|}\right\} = 0$.

$$(\Rightarrow)\ \frac{|X_n|}{1+|X_n|} = \frac{|X_n|}{1+|X_n|} 1\{|X_n| > \varepsilon\} + \frac{|X_n|}{1+|X_n|} 1\{|X_n| \leq \varepsilon\}$$

When $\varepsilon \geq 1$ it is always true that $\frac{|X_n|}{1+|X_n|} \leq \varepsilon$. When $0 < \varepsilon < 1$ we have

$\frac{|X_n|}{1+|X_n|} \leq \varepsilon \Leftrightarrow |X_n| \leq \frac{\varepsilon}{1-\varepsilon}$ which is true for $|X_n| \leq \varepsilon$ as $\varepsilon \leq \frac{\varepsilon}{1-\varepsilon}$.

Therefore $\frac{|X_n|}{1+|X_n|} \leq \frac{|X_n|}{1+|X_n|} 1_{\{|X_n|>\varepsilon\}} + \varepsilon\, 1_{\{|X_n|\leq\varepsilon\}} \leq 1_{\{|X_n|>\varepsilon\}} + \varepsilon$

$$\Rightarrow \mathbb{E}\left\{\frac{|X_n|}{1+|X_n|}\right\} \leq P\{|X_n| > \varepsilon\} + \varepsilon \Rightarrow \lim_{n\to\infty} \mathbb{E}\left\{\frac{|X_n|}{1+|X_n|}\right\} \leq \lim_{n\to\infty} P\{|X_n| > \varepsilon\} + \varepsilon .$$

Because $X_n \xrightarrow{P} 0$ we have $\lim_{n\to\infty} P\{|X_n| > \varepsilon\} = 0$ and $\lim_{n\to\infty} \mathbb{E}\left\{\frac{|X_n|}{1+|X_n|}\right\} \leq \varepsilon,\ \forall \varepsilon > 0$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}\left\{\frac{|X_n|}{1+|X_n|}\right\} = 0 .$$

$(\Leftarrow)$ The function $f(x) = \frac{x}{1+x} \uparrow$ then $f(\varepsilon) 1_{\{|X_n|>\varepsilon\}} \leq f(|X_n|) 1_{\{|X_n|>\varepsilon\}} \leq f(|X_n|)$.

Taking expectations and then limits yields

$$f(\varepsilon) \lim_{n\to\infty} P\{|X_n| > \varepsilon\} \leq \lim_{n\to\infty} f(|X_n|) = 0 \Rightarrow X_n \xrightarrow{P} 0 .$$

<u>Exercise</u>: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing, bounded, and continuous with $f(0) = 0$, then $X_n \xrightarrow{P} X \Leftrightarrow \lim_{n\to\infty} \mathbb{E}\{f(|X_n - X|)\} = 0$.

W.l.o.g. it suffices to show that $X_n \xrightarrow{P} 0 \Leftrightarrow \lim_{n\to\infty} \mathbb{E}\{f(|X_n|)\} = 0$.

$(\Rightarrow)$ $0 \leq f(|X_n|) = f(|X_n|) 1_{\{|X_n|>\varepsilon\}} + f(|X_n|) 1_{\{|X_n|\leq\varepsilon\}}$

$\leq f(|X_n|) 1_{\{|X_n|>\varepsilon\}} + f(\varepsilon) 1_{\{|X_n|\leq\varepsilon\}} \leq M 1_{\{|X_n|>\varepsilon\}} + f(\varepsilon),\ f(x) \leq M < \infty,\ x \geq 0$.

Taking expectations and then limits yields

$$0 \leq \lim_{n\to\infty} \mathbb{E}\{f(|X_n|)\} \leq M \lim_{n\to\infty} P\{|X_n| > \varepsilon\} + f(\varepsilon) = f(\varepsilon) .$$

Then taking the limit as $\varepsilon \downarrow 0$ gives $0 \le \lim_{n\to\infty} \mathbb{E}\left\{ f\left(\left|X_n\right|\right)\right\} \le f\left(0+\right) = 0$.

$(\Leftarrow)$ The function $f(x)\uparrow$ then $f(\varepsilon)1_{\{|X_n|>\varepsilon\}} \le f\left(\left|X_n\right|\right)1_{\{|X_n|>\varepsilon\}} \le f\left(\left|X_n\right|\right)$.

Taking expectations and then limits yields

$f(\varepsilon)\lim_{n\to\infty} P\left\{\left|X_n\right| > \varepsilon\right\} \le \lim_{n\to\infty} f\left(\left|X_n\right|\right) = 0 \Rightarrow X_n \xrightarrow{P} 0$.

Observe that: The functions $f(x) = \dfrac{x}{1+x}$, $f(x) = x \wedge 1$ and $f(x) = \arctan(x)$ are increasing, bounded, and continuous with $f(0) = 0$.

Exercise: Show that if $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ as $n \to \infty$, then $X_n + Y_n \xrightarrow{P} X + Y$ as $n \to \infty$.

$$\left\{\left|\left(X_n + Y_n\right) - \left(X + Y\right)\right| > \varepsilon\right\} = \left\{\left|X_n - X\right| + \left|Y_n - Y\right| \ge \left|\left(X_n + Y_n\right) - \left(X + Y\right)\right| > \varepsilon\right\}$$

$$\subseteq \left\{\left|X_n - X\right| + \left|Y_n - Y\right| > \varepsilon\right\} \subseteq \left\{\left|X_n - X\right| > \frac{\varepsilon}{2}\right\} \cup \left\{\left|Y_n - Y\right| > \frac{\varepsilon}{2}\right\},\ \forall \varepsilon > 0.$$

Taking probabilities and then the limit as $n \to \infty$ gives

$$P\left\{\left|\left(X_n + Y_n\right) - \left(X + Y\right)\right| > \varepsilon\right\} \le P\left\{\left|X_n - X\right| > \frac{\varepsilon}{2}\right\} + P\left\{\left|Y_n - Y\right| > \frac{\varepsilon}{2}\right\}$$

$$\Rightarrow \lim_{n\to\infty} P\left\{\left|\left(X_n + Y_n\right) - \left(X + Y\right)\right| > \varepsilon\right\} \le \lim_{n\to\infty} P\left\{\left|X_n - X\right| > \frac{\varepsilon}{2}\right\} + \lim_{n\to\infty} P\left\{\left|Y_n - Y\right| > \frac{\varepsilon}{2}\right\}$$

$$\Rightarrow \lim_{n\to\infty} P\left\{\left|\left(X_n + Y_n\right) - \left(X + Y\right)\right| > \varepsilon\right\} = 0,\ \forall \varepsilon > 0 \Rightarrow X_n + Y_n \xrightarrow{P} X + Y.$$

Different types of convergence have different types of limits for the same sequence $(X_n)_{n\geq 1}$. Assume $(X_n)_{n\geq 1}$ converges simultaneously P-a.s., in probability, and in L^1- , L^2-norms. Denote for a definiteness by X, X^p, X', and X'' the limits respectively. A remarkable fact is given below.

Lemma 2.1.

$$P(X = X^p = X' = X'') = 1.$$

Proof. It suffices to show that $P(X \neq X^p) = 0$. Write

$$P(X \neq X^p) = P(|X - X^p| > 0)$$

and notice that $P(|X - X^p| > 0) = \lim_{\varepsilon\to 0} P(|X - X^p| > \varepsilon)$. On the other hand, since $|X_n - X| \xrightarrow[n\to\infty]{P-a.s.} 0$, also $I(|X_n - X| > \frac{\varepsilon}{2}) \xrightarrow[n\to\infty]{P-a.s.} 0$ and so, by taking the expectation we, find that $P(|X_n - X| > \frac{\varepsilon}{2}) \to 0$. The use now the triangular inequality $|X - X^p| \leq |X - X_n| + |X_n - X^p|$ for any $\varepsilon > 0$ provides

$$P(|X - X^p| > \varepsilon) \leq P\left(|X - X_n| > \frac{\varepsilon}{2}\right) + P\left(|X_n - X^p| > \frac{\varepsilon}{2}\right) \xrightarrow[n\to\infty]{} 0$$

and, thus, $\lim_{\varepsilon\to 0} P(|X - X^p| > \varepsilon) = 0$. □

3

Assume that a sequence of r.v.s $\{X_n\}_{n\geq 1}$ have simultaneously a $P-a.s.$ limit X, a limit in probability X^P, a L^1 limit $X^{(1)}$ and a L^2 limit $X^{(2)}$. Then
$X = X^P = X^{(1)} = X^{(2)}$, $P-a.s.$

We show that $P\{X = X^P\} = 1 \Leftrightarrow P\{X \neq X^P\} = 0 \Leftrightarrow P\{|X - X^P| > 0\} = 0$. We observe that $P\{|X - X^P| > 0\} = \lim_{\varepsilon\to 0+} P\{|X - X^P| > \varepsilon\}$. To see that, let $\varepsilon_n \downarrow 0$ as, $n \to \infty$ and

$$A_n = \{|X - X^P| > \varepsilon_n\} = \{|X - X^P| > \varepsilon_n \geq \varepsilon_{n-1}\} \subseteq \{|X - X^P| > \varepsilon_{n-1}\} = A_{n-1}$$

$$\Rightarrow \lim_{n\to\infty} P\{|X - X^P| > \varepsilon_n\} = P\left(\bigcap_{n=1}^{\infty}\{|X - X^P| > \varepsilon_n\}\right) = P\{|X - X^P| > 0\} = 0.$$

But $P\{|X - X^P| > \varepsilon_n\} \leq P\left\{|X^P - X_n| > \frac{\varepsilon_n}{2}\right\} + P\left\{|X - X_n| > \frac{\varepsilon_n}{2}\right\}$, then by taking the limit as $n \to \infty$ we have $P\{|X - X^P| > 0\} = 0$.