Rational zeta functions for the Chebyshev family of maps

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Abstract. We provide explicit expressions for a family of rational zeta functions originating from periodic orbits of the Chebyshev family of maps, that weigh each periodic orbit with integral powers of its derivative. Our method is algebraic.

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1. Introduction

This paper is devoted to the study of a family of dynamical zeta functions introduced in [1]. The rationality of these functions for the quadratic family was conjectured in [1], and recently proved in [2, ch 17] using transfer-operator techniques. A purely algebraic proof has been also given in [3], where rationality is established in the more general context of polynomial mappings over fields of characteristic zero, and for a larger class of zeta functions. Here we provide a constructive method for the zeta functions of the Chebyshev family of maps (to which the result in [3] also applies) defined recursively as

\[ f_2(z) = z^2 - 2 \quad f_3(z) = z^3 - 3z \]
\[ f_\tau(z) = zf_{\tau-1}(z) - f_{\tau-2}(z) \quad \tau \geq 4. \]

This family is known to be semi-conjugate to the full shift in \( \tau \) symbols, and to have as a Julia set the interval \([-2, 2]\) [4, ch 1].

We define the sequence of dynamical zeta functions

\[ \zeta(\tau,k)(z) = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\tau,k) \right) \]  

(1.2)

with

\[ Z_n(\tau,k) = \sum_{z_i \in \text{Fix}(f^n_{\tau})} (t_{\tau,n,k})^{k} \quad t_{\tau,n,k} = (f^n_{\tau})'(z_i) \quad k \geq 0 \]  

(1.3)

where \( \text{Fix}(f^n_{\tau}) \) is the set of fixed points of \( f^n_{\tau} \) having cardinality \( \tau^n \). Because the weights \( t \) are multiplicative [10, 11] the zeta function takes the form of an Eulerian product

\[ \zeta(\tau,k)(z) = \prod_{p} (1 - t_{\tau,p}^{k} z^{n_p})^{-1} \quad \tau \geq 2, \ k \geq 1 \]  

(1.4)

where \( p \) runs through all admissible periodic strings of minimal length \(|p| = n_p\) and \( t_{\tau,p} = t_{\tau,n_p,p} \). Let \( z_1, \ldots, z_{m_\tau} \), be representative points of the \( m_\tau = m_\tau(n) \) orbits of \( f_\tau \) of
minimal period \( n \). From an algebraic viewpoint, it is convenient to write the zeta function (1.4) as follows

\[
\zeta_{\tau,k}(z)^{-1} = \prod_{n=1}^{\infty} z^{nm_{\tau,n}(n)} \delta_{\tau,n}^{(k)}(z^{-n})
\]  

(1.5)

where the multiplier polynomials \( \delta_{\tau,n}^{(k)} \) are defined by the product (see [1, p 322, 5, p 973])

\[
\delta_{\tau,n}^{(k)}(z) = \prod_{i=1}^{m_{\tau,n}(n)} (z - (t_{\tau,n,i})^k).
\]  

(1.6)

We shall prove the following.

**Theorem 1.** Let \( f_{\tau} \) be defined by (1.1). Then for the dynamical zeta functions defined by (1.2) and (1.3) we have that

\[
(i) \, \tau \text{ even} \quad \zeta_{\tau,k} = \left\{ \begin{array}{ll} 
\frac{1 - \tau^{2k}z}{1 - \tau^kz} & k \text{ even} \\
\frac{1 - \tau^{2k}z}{1 - \tau^kz} & k \text{ odd} 
\end{array} \right.
\]  

(1.7)

\[
(ii) \, \tau \text{ odd} \quad \zeta_{\tau,k} = \left\{ \begin{array}{ll} 
\frac{(1 - \tau^{2k}z)^2}{1 - \tau^kz} & k \text{ even} \\
\frac{(1 - \tau^{2k}z)^2}{1 - \tau^kz} & k \text{ odd} 
\end{array} \right.
\]

In the next section we express periodic points as roots of certain polynomials whose irreducible factors are investigated. In section 3 we study in detail a family of periodic orbits which play a central role in the proof of theorem 1. The latter is completed in section 4.

2. Factorization of the primitive orbit polynomial

The fixed points of \( f_{\tau}^n \) (also called \( n \) cycles) are roots of the polynomial

\[
P_{\tau,n}(z) = f_{\tau}^n(z) - z \quad n = 1, 2, \ldots
\]  

(2.1)

where \( f_{\tau}^n \) denotes the \( n \)th iterate of \( f_{\tau} \) (with \( f_{\tau}^1 = f_{\tau} \)). Let \( H_{\tau,n}(z) \) be the polynomial whose roots are the points of minimal period \( n \). Then its degree \( \deg H_{\tau,n} \) will be \( nm_{\tau}(n) \).

We first note that [5, 7, 8]

\[
P_{\tau,n}(z) = \prod_{d|n} H_{\tau,d}(z) \quad \tau^n = \sum_{d|n} dm_{\tau}(d)
\]  

(2.2)

where the product is taken over all the positive divisors \( d \) of \( n \). Then the primitive orbit polynomial \( H_{\tau,n}(z) \) and its degree are extracted from (2.2) via Möbius inversion, yielding

\[
H_{\tau,n}(z) = \prod_{d|n} P_{\tau,d}(z)^{\mu(n/d)} \quad nm_{\tau}(n) = \sum_{d|n} \tau^d \mu \left( \frac{n}{d} \right) \quad n = 1, 2, \ldots
\]  

(2.3)

where \( \mu \) is the Möbius function [9, ch 2]. Because \( f_{\tau} \) is monic so are \( P_{\tau,n}(z) \) and \( H_{\tau,n}(z) \), and the periodic points of \( f_{\tau} \) are algebraic integers. Moreover

\[
H_{\tau,n}(z) = \prod_{d|n} \left( \prod_{d_1 | \tau^s - 1} \Psi_{d_1}(z) \prod_{d_2 | \tau^s + 1} \Psi_{d_2}(z) \right)^{\mu(n/d)}
\]  

(2.4)
where
\[
\Psi_n(z + z^{-1}) = z^{-\phi(n)/2} F_n(z) = \prod_{k=1}^{\lceil n/2 \rceil} \left( z + z^{-1} - 2 \cos \left( \frac{2\pi k}{n} \right) \right) \quad n \geq 1
\]  

(2.5)

\( F_n(z) \) is the \( n \)th cyclic polynomial (its roots are the primitive \( n \)th roots of unity), \( \phi \) is the Euler totient function [12, ch 3], and \( \lceil n/2 \rceil \) is the largest integer smaller than or equal to \( n/2 \). In [1, pp 971–2] relation (2.4) is only proved for \( \tau = 2 \), the case of general \( \tau \), however, being unproblematic. It can be shown [6] that for \( n \geq 3 \), \( \Psi_n(z) \) is a monic irreducible polynomial over the integers of degree \( \phi(n)/2 \). For \( n = 1, 2 \) one calculates using definition (2.5)
\[
\Psi_1(z)^2 = z - 2 \quad \Psi_2(z)^2 = z + 2.
\]

Nevertheless, only the squares of \( \Psi_1(z) \) and \( \Psi_2(z) \) appear in equation (2.4). Indeed \( H_{\tau,1} \) has the factor \( \Psi_1^2 \) for even \( \tau \) and \( \Psi_2^2 \) for odd \( \tau \). Expanding the products on the right-hand side of relation (2.4), \( H_{\tau,n} \) becomes
\[
H_{\tau,n} = \Phi_{\tau,n} \left( \Psi_{d_1^+} \ldots \Psi_{d_m^+} \right) \left( \Psi_{d_1^-} \ldots \Psi_{d_m^-} \right).
\]

(2.6)

The subscripts \( d_1^+ \) and \( d_1^- \) in the latter relation are divisors of \( \tau^n - 1 \) and \( \tau^n + 1 \) respectively, that have a special property. Namely they are not factors of any \( \tau^d - 1 \) and \( \tau^d + 1 \) with \( d \mid n \) and \( d < n \) (as if they were \( H_{\tau,n} \) would have as roots non-minimal \( n \) cycles). The polynomial \( \Phi_{\tau,1} \) is equal to \( \Psi_1^2 \) and \( \Psi_2^2 \) for even and odd \( \tau \) respectively, while \( \Phi_{\tau,n} = 1 \), for all \( n > 1 \).

3. Self-conjugate orbits and then degrees in the factorization

The semi-conjugacy between the Chebyshev polynomials and the Bernoulli shifts is given by the function \( h(z) = z + z^{-1} \). Let \( b \) be the symbolic representation of an \( n \) cycle of the full shift map in \( \tau \) symbols
\[
b = \{b_1, \ldots, b_n, b_1, \ldots \} = \overline{b_1 \ldots b_n} \quad b_i \in \{0, 1, \ldots, \tau - 1 \}
\]

which is the \( \tau \)-ary expansion of the associated periodic point of the map \( z \mapsto z \tau \mod 1 \) of the interval \([0, 1]\) to itself. Then the periodic point of the corresponding \( n \) cycle of \( f_\tau \) is given by
\[
h(b) = 2 \cos \left( \frac{2\pi \lambda(b)}{\tau^n - 1} \right) \quad \lambda(b) = \sum_{r=1}^{n} \tau^{n-r} b_r.
\]

(3.1)

Let \( \sigma \) denote the shift map \( (\sigma(b))_i = b_{i+1} \). Then the points of the \( n \) cycle with initial condition \( h(b) \) are given by \( h(\sigma^t(b)) \), for \( t = 0, \ldots, n - 1 \).

We define \( C(b_i) = \tau - 1 - b_i \) and \( C(b_1) \ldots C(b_n) = C(b_1) \ldots C(b_n) \). Then \( \lambda(C(b_i)) = \tau^n - 1 - \lambda(b) \) and \( h(C(b)) = h(b) \). The latter means that the two complementary cycles \( b \) and \( C(b) \) of the full shift in \( \tau \) symbols are mapped under \( h \) to the same \( n \) cycle of \( f_\tau \). Of interest are the periodic strings of period \( 2n \) satisfying the relation
\[
b' = b_1 \ldots b_n C(b_1) \ldots C(b_n).
\]

(3.2)

Then because \( \sigma^n(b') = C(b') \) we have \( h(\sigma^n(b')) = h(b') \). Thus \( b' \) halves its period under \( h \). These are fixed points of \( f_\tau \) with \( \lambda(b') = (\tau^n - 1)(1 + \lambda(b)) \) so that
\[
h(b') = 2 \cos \left( \frac{2\pi (1 + \lambda(b))}{\tau^n + 1} \right).
\]

(3.3)
The quantities \( h(b) \) and \( h(b') \), defined by equations (3.1) and (3.3), are respectively roots of the polynomials \( H_{r,n}(z) \) and \( H_{r,n}^+(z) \). More specifically the ratios \( \lambda(b)/(\tau^n - 1) \) and \( (1 + \lambda(b))/(\tau^n + 1) \) can be written in lower terms as \( k^-/d^- \) and \( k^+/d^+ \), respectively, so that \( \Psi_{d^-}(h(b)) = \Psi_{d^+}(h(b')) = 0 \). From now on we shall refer to periodic strings of the form (3.2) as \textit{self-conjugate}. Let \( m_\tau'(n) \) be the number of self-conjugate prime \( \tau \)-ary strings of length \( n \), and let \( \bar{m}_\tau(n) \) and \( \bar{m}_\tau'(n) \) be the number of prime and self-conjugate prime cycles, respectively, of the Bernoulli shift. When \( \tau \) is even and \( n \geq 1 \) we have \( \bar{m}_\tau'(n) \) prime \( n \) cycles halving their period under \( h(\bar{m}_\tau'(n)) = 0 \) for odd \( n \). The remaining \( m_\tau(n) - \bar{m}_\tau'(n) \) \( n \) cycles of the shift map will contribute to \( (\bar{m}_\tau(n) - \bar{m}_\tau'(n))/2 \) prime \( n \) cycles of \( f_\tau \) as complementary cycles of the shift map are mapped under \( h \) to the same \( n \) cycles of \( f_\tau \). The latter number of cycles should be increased by \( \bar{m}_\tau'(2n) \) —not by \( \bar{m}_\tau'(2n)/2 \) because \( b' \) and its complement \( C(b') \) belong to the same cyclic permutation class (see [10, p 329]). Then one obtains

\[
m_\tau'(n) = \begin{cases} 
(\bar{m}_\tau(n) - \bar{m}_\tau'(n))/2 + \bar{m}_\tau'(2n) & n \text{ even} \\
\bar{m}_\tau(n)/2 + \bar{m}_\tau'(2n) & n \text{ odd} 
\end{cases} \quad n \geq 1, \text{ even } \tau.
\]

Because both maps have complete symbolic dynamics over a real phase space, then \( \bar{m}_\tau = m_\tau \) and \( \bar{m}_\tau' = m_\tau' \), from which we obtain

\[
m_\tau'(2n) = \begin{cases} 
(m_\tau(n) + m_\tau'(n))/2 & n \text{ even} \\
m_\tau(n)/2 & n \text{ odd} 
\end{cases} \quad n \geq 1, \text{ even } \tau. \tag{3.4}
\]

For odd \( \tau \) let \( b \) denote the 1 cycle \( (\tau - 1)/2 \). Then \( C(b) = b \), and only \( (\tau - 1)/2 \) cycles of minimal length 2 halve their period under \( h \). Thus the recurrence relation defining \( m_\tau' \) is given by

\[
m_\tau'(2n) = \begin{cases} 
(\bar{m}_\tau(n) + \bar{m}_\tau'(n))/2 & n \text{ even} \\
\bar{m}_\tau(n)/2 & n \text{ odd} 
\end{cases} \quad n > 1, \text{ odd } \tau. \tag{3.5}
\]

From our previous discussion and for all \( \tau \geq 2 \) and \( n > 1 \) we have \( \operatorname{deg} H_{r,n}^+ = nm_\tau'(2n) \) and \( \operatorname{deg} H_{r,n}^- = n(m_\tau(n) - m_\tau'(2n)) \) so that

\[
\begin{align*}
\operatorname{deg} H_{r,n}^+ &= \begin{cases} 
\tau/2 & \tau \text{ even} \\
(\tau - 1)/2 & \tau \text{ odd} 
\end{cases} \\
\operatorname{deg} H_{r,1}^- &= \begin{cases} 
(\tau - 2)/2 & \tau \text{ even} \\
(\tau - 3)/2 & \tau \text{ odd.} 
\end{cases}
\end{align*}
\]

\[
\textbf{Proposition 1.} \text{ The derivatives along cycles of minimal length } n \text{ are } \tau^n \text{ or } -\tau^n \text{ exactly when the initial conditions are roots of the polynomials } H_{r,n}^+ \text{ and } H_{r,n}^-, \text{ respectively.}
\]

\textbf{Proof.} From the fact that

\[
f_\tau(2 \cos(2\pi \theta)) = 2 \cos(2\pi \tau \theta) \quad \tau \geq 2
\]

one obtains—using induction—that

\[
f_\tau^n(z + z^{-1}) = z^{\tau^n} + z^{-\tau^n}
\]

so that

\[
(f_\tau^n)'(z + z^{-1}) = \frac{\tau^n(z^{\tau^n+1} - z^{-(\tau^n+1)})}{z^2 - 1} \quad z \in S^1 \setminus \{\pm 1\}. \tag{3.9}
\]
Let \( \alpha \) be a periodic point of minimal length \( n \) with \( H_{\tau,n}(\alpha) = 0 \). Then there exist relatively prime integers \( d_{\tau}^+ \) and \( k_{\tau}^+ \) with \( d_{\tau}^+ \) dividing \( \tau^{n} + 1 \) (say \( d_{\tau}^+ \rho_j = \tau^n + 1 \)), \( \Psi_{d_{\tau}^+}(\alpha) = 0 \), with \( \omega = \omega + \omega^{-1} \) and \( \omega = \exp(2\pi k^+/d_{\tau}^+) \). The situation is similar for \( H_{\tau,n}(\alpha) = 0 \) then \( \Psi_{d_{\tau}^-}(\alpha) = 0 \), and \( d_{\tau}^- \rho_i = \tau^n - 1 \) with \( \omega = \exp(2\pi k^-/d_{\tau}^-) \). Then one obtains from relation (3.9)

\[
\tau^{-n}(\omega^2 - 1)(f_\tau^n)'(\alpha) = \begin{cases} 
(\omega^{d_{\tau}^+})^{\rho_i} - \omega^2(\omega^{d_{\tau}^+})^{-\rho_i} & H_{\tau,n}^+(\alpha) = 0 \\
(\omega^{d_{\tau}^-})^{\rho_i} - (\omega^{d_{\tau}^-})^{-\rho_i} & H_{\tau,n}^-(\alpha) = 0
\end{cases}
\]

which gives the desired result. \( \square \)

From relation (2.6) it follows that for \( n = 1 \) we have the period-1 point \( \bar{z} = 2 \) for even \( \tau \) (\( \Phi_{r,1} = \Phi_1^2 \)) and the period-1 points \( \bar{z} = \pm 2 \) for odd \( \tau \) (\( \Phi_{r,1} = \Phi_1^2 \Phi_2^2 \)). The latter 1-cycles give rise to the exceptional eigenvalue \( \tau^2 \) (does not follow the rule of proposition 1) calculated via (3.9) at \( z = \pm 1 \):

\[
f_\tau^2(2) = \tau^2 \quad \tau \geq 2 \\
f_\tau^2(-2) = \tau^2 \quad \tau > 2 \text{ and } \tau \text{ odd.}
\]

Then from definition (1.6) we have

\[
\delta_{\tau,1}(z) = \begin{cases} 
(z - \tau^{2n})(z - \tau^{k})^{(r-2)/2}(z - (-\tau)^k)^{r/2} & \tau \text{ even } \\
(z - \tau^{2n})^2(z - \tau^{k})^{(r-3)/2}(z - (-\tau)^k)^{(r-1)/2} & \tau \text{ odd } \quad n = 1.
\end{cases}
\]

When \( n > 1 \) one obtains

\[
\delta_{\tau,n}(z) = \begin{cases} 
(z - \tau^{nk})^2z(z - (-\tau)^k)^{m_r(n)}/2(\bar{z} - (-\tau)^k)^{m_r(n)/2} & \tau \text{ even } \quad n > 1 \\
(z - \tau^{nk})^{m_r(n)/2}z(z - (-\tau)^k)^{m_r(n)/2} & \tau \text{ odd } \quad n \text{ odd}
\end{cases}
\]

(3.11)

4. Proof of theorem 1

Theorem 1 follows from three lemmas.

**Lemma 1.** Let \( n \) be a positive odd integer. Then

\[
\sum_{|d|n} dm_{\tau}(2'd) = 2^{-l}(\tau^{2n} - \tau^{2n-1}) \quad l \geq 1.
\]

**Proof.** We prove (4.1) by induction on \( l \). Using the second equation of relation (2.3) together with the fact that \( n \) is odd yields

\[
\tau^{2n} = \sum_{d_{\tau}^+} dm_{\tau}(d) = \sum_{d_{\tau}^+} dm_{\tau}(d) + \sum_{d_{\tau}^-} 2dm_{\tau}(2'd) = \tau^n + 2 \sum_{d_{\tau}^-} dm_{\tau}(2'd)
\]

which establishes relation (4.1) for \( l = 1 \). We observe that

\[
\tau^{2n} = \tau^{2n} + \sum_{l=r-l} 2l \left( \sum_{d|\tau} dm_{\tau}(2'd) \right) + 2l \sum_{d|\tau} dm_{\tau}(2'd)
\]

(4.2)

where we have used our result for \( l = 1 \) and the second equation of relation (2.3). Assuming now (4.1) to hold up to \( l - 1 \), equation (4.2) yields

\[
\tau^{2n} = \tau^{2n} + \sum_{l=r-l} (\tau^{2n} - \tau^{2n-1}) + 2l \sum_{d|\tau} dm_{\tau}(2'd) = \tau^{2n-1} + 2l \sum_{d|\tau} dm_{\tau}(2'd)
\]

which gives the desired result. \( \square \)
Lemma 2. Let
\[ S_\tau(n) = \sum_{\substack{d|n \atop d \text{ odd}}} \frac{1}{d} m'_\tau\left(\frac{2n}{d}\right) \]  \hfill (4.3)
then, for all positive integers \( n \) we have
\[ S_\tau(n) = \begin{cases} \frac{\tau^n}{2n} & \tau \text{ even} \\ \frac{(\tau^n - 1)/2n}{m'_{\tau}} & \tau \text{ odd}. \end{cases} \] \hfill (4.4)

Proof. We write \( n = 2^n n' \) where \( n' \) is an odd integer and \( l \geq 0 \), and we proceed by induction on \( l \). Let \( l = 0 \). Then for even \( \tau \) we obtain from (3.4) and the second equation of (2.3) that
\[ n' S_\tau(n') = \sum_{d|n'} \frac{n'}{d} m'_\tau\left(\frac{2n'}{d}\right) = \sum_{d|n'} dm_\tau(d) = \frac{1}{2} \sum_{d|n'} dm_\tau(d) = \frac{\tau}{2}. \]

For odd \( \tau \) we obtain from (3.5)
\[ n' S_\tau(n') = \sum_{d|n'} \frac{n'}{d} m'_\tau\left(\frac{2n'}{d}\right) + m'_\tau(2) 
= \sum_{d|n'} \frac{d}{d} m_\tau(d) + m_\tau(2) = \frac{1}{2} \sum_{d|n'} dm_\tau(d) + m'_\tau(2) 
= \frac{1}{2} (\tau^n - \tau) + \frac{1}{2} (\tau - 1) = \frac{\tau^n - \tau}{2}. \]

For all \( \tau \) we obtain using relations (3.4), (3.5) and lemma 1 that
\[ S_\tau(2^n n') = \sum_{d|n'} \frac{1}{d} m'_\tau\left(\frac{2^n n'}{d}\right) = \frac{1}{2} \sum_{d|n'} \frac{1}{d} m_\tau(d) + \frac{1}{2} \sum_{d|n'} \frac{1}{d} m_\tau\left(\frac{2^n}{d}\right) 
= \frac{1}{2n'} \sum_{d|n'} dm_\tau(2^d) + \frac{1}{2} \sum_{d|2^n - 1, d \text{ odd}} dm_\tau\left(\frac{2^d - 1}{d}\right) \]
so that
\[ S_\tau(2^n n') = \frac{1}{2} \left( \frac{\tau^{2^n} - \tau^{2^n - 1}}{2^{n'} - 1} + S_\tau(2^n - 1 n') \right) \]
Assuming that relation (4.4) holds true up to \( l - 1 \) the above equation gives
\[ S_\tau(2^n n') = \frac{1}{2} \begin{cases} \frac{\tau^{2^n} - \tau^{2^n - 1}}{2^n} + \frac{\tau^{2^n - 1} - 1}{2^n} & \tau \text{ even} \\ \frac{\tau^{2^n} - \tau^{2^n - 1}}{2^n} + \frac{\tau^{2^n - 1} - 1}{2^n} & \tau \text{ odd} \end{cases} \]
which completes the proof. \( \square \)

Lemma 3. Let
\[ Q_\tau(z) = \prod_{n=1}^{\infty} \left(\frac{1 + z^n}{1 - z^n}\right)^{m'_\tau(2n)} \quad \tau \geq 2 \] \hfill (4.5)
then, for $|z| < \tau^{-1}$ the product converges to

$$Q_\tau(z) = \begin{cases} 
1/(1 - \tau z) & \tau \text{ even} \\
(1 - z)/(1 - \tau z) & \tau \text{ odd}
\end{cases}$$

(4.6)

**Proof.** By taking the logarithms and expanding $\ln(1 \pm z^n)$, (4.5) becomes

$$\ln Q_\tau(z) = 2 \sum_{n=1}^{\infty} z^n \sum_{d|r} c(i, j) \left( \frac{2n}{d} \right) \Theta_d = 2 \sum_{n=1}^{\infty} S_\tau(n) z^n.$$  

(4.7)

Letting $G_1(z; \tau, k) = (1 - \tau z)^4 (1 - z)^3 (1 - (-1)^k z)^3 - 1 \prod_{n \geq 3} \left( 1 - z^n \right)^{m_\tau(n)} \prod_{n \geq 3} \left( 1 - (-1)^k z^n \right)^{m_\tau(n)}$ one verifies that

$$\zeta(\tau, k)(z^{\tau - k})^{-2} = G_1(z; \tau, k) F(z; \tau, k).$$

(4.10)

Letting

$$G_1(z; \tau, k) = \frac{(1 - \tau z)^d}{(1 - z)^3 (1 - (-1)^k z)}$$

$$F(z; \tau, k) = \prod_{n=1}^{\infty} (1 - z^n)^{m_\tau(n)} \prod_{n=1}^{\infty} (1 - (-1)^k z^n)^{m_\tau(n)} \prod_{n=1}^{\infty} \left( \frac{1 - (-1)^k z^n}{1 - z^n} \right)$$

(4.9)

(4.11)

(4.12)

the former relation together with lemma 3 gives

$$F(z; \tau, k) = \begin{cases} 
(1 - \tau z)^2 & \text{even} \\
1 - z^2 & \text{odd}
\end{cases}$$

(4.11)

We now prove theorem 1. Let $\tau$ be odd. Then from (1.5), (3.10) and (3.11) we have

$$\zeta(\tau, k)(z^{\tau - k})^{-2} = (1 - \tau^k z)^d (1 - z)^3 (1 - (-1)^k z)^3 - 1 \prod_{n \geq 3} \left( 1 - z^n \right)^{m_\tau(n)} \prod_{n \geq 3} \left( 1 - (-1)^k z^n \right)^{m_\tau(n)}$$

$$\times \prod_{n \geq 2} (1 - z^n)^{m_\tau(n) - m_\tau'(n)} (1 - (-1)^k z^n)^{m_\tau(n) + m_\tau'(n)}.$$  

(4.8)

Letting

$$A_\tau(z) = \prod_{n=1}^{\infty} (1 - z^n)^{m_\tau(n)}.$$  

Then (4.9) can be written as

$$F(z; \tau, k) = \begin{cases} 
A_\tau(z)^2 & \text{even} \\
A_\tau(z^2) Q_\tau(z^2) & \text{odd}.
\end{cases}$$

Because

$$A_\tau(z) = \prod_{n=1}^{\infty} \prod_{|p| = n} (1 - z^n) = \prod_{|p| = n} (1 - z^n)^{m_\tau(n)} = \exp\left( -\sum_{n=1}^{\infty} \frac{(\tau z)^n}{n} \right) = 1 - \tau z$$

(4.11)

the former relation together with lemma 3 gives

$$F(z; \tau, k) = \begin{cases} 
(1 - \tau z)^2 & \text{even} \\
1 - z^2 & \text{odd}
\end{cases}$$

(4.11)
from which our theorem follows for odd $\tau$.

For even $\tau$ we define

$$G_2(z; \tau, k) = \left(\frac{1 - \tau^k z}{1 - z}\right)^2.$$  

Then it can be verified that

$$\zeta(\tau, k)(z^{\tau} - k)^{-2} = G_2(z; \tau, k)F(z; \tau, k)$$

with

$$F(z; \tau, k) = \begin{cases} (1 - \tau z)^2 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$$

which finally establishes formula (1.7) for even $\tau$.

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References

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