Nonlinearity **10** (1997) 941–948. Printed in the UK PII: S0951-7715(97)70282-2

Rational zeta functions for the Chebyshev family of maps

Spyros J Hatjispyros

School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

Received 14 December 1995, in final form 30 January 1977
Recommended by P Cvitanovic

Abstract. We provide explicit expressions for a family of rational zeta functions originating from periodic orbits of the Chebyshev family of maps, that weigh each periodic orbit with integral powers of its derivative. Our method is algebraic.

AMS classification scheme numbers: 11R21, 11R32, 58F22

1. Introduction

This paper is devoted to the study of a family of dynamical zeta functions introduced in [1]. The rationality of these functions for the quadratic family was conjectured in [1], and recently proved in [2, ch 17] using transfer-operator techniques. A purely algebraic proof has been also given in [3], where rationality is established in the more general context of polynomial mappings over fields of characteristic zero, and for a larger class of zeta functions. Here we provide a constructive method for the zeta functions of the Chebyshev family of maps (to which the result in [3] also applies) defined recursively as

$$\begin{aligned} &f_2(z) = z^2 - 2 \qquad f_3(z) = z^3 - 3z \\ &f_\tau(z) = z f_{\tau-1}(z) - f_{\tau-2}(z) \qquad \tau \geqslant 4. \end{aligned} \tag{1.1}$$

This family is known to be *semi-conjugate* to the full shift in τ symbols, and to have as a Julia set the interval $[-2, 2]$ [4, ch 1].

We define the sequence of dynamical zeta functions

$$\zeta_{(\tau,k)}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\tau, k)\right) \tag{1.2}$$

with

$$Z_n(\tau, k) = \sum_{z_i \in \mathrm{Fix}(f_\tau^n)} (t_{\tau,n,k})^k \qquad t_{\tau,n,k} = (f_\tau^n)'(z_i) \qquad k \geqslant 0 \tag{1.3}$$

where $\mathrm{Fix}(f_\tau^n)$ is the set of fixed points of f_τ^n having cardinality τ^n. Because the weights t are multiplicative [10, 11] the zeta function takes the form of an Eulerian product

$$\zeta_{(\tau,k)}(z) = \prod_p (1 - t_{\tau,p}^k z^{n_p})^{-1} \qquad \tau \geqslant 2,\ k \geqslant 1 \tag{1.4}$$

where p runs through all admissible periodic strings of minimal length $|p| = n_p$ and $t_{\tau,p} = t_{\tau,n_p,p}$. Let $z_1, \ldots, z_{m_\tau}$, be representative points of the $m_\tau = m_\tau(n)$ orbits of f_τ of *minimal* period n. From an algebraic viewpoint, it is convenient to write the zeta function (1.4) as follows

$$\zeta_{(\tau,k)}(z)^{-1} = \prod_{n=1}^{\infty} z^{nm_\tau(n)} \delta_{\tau,n}^{(k)}(z^{-n}) \tag{1.5}$$

where the *multiplier* polynomials $\delta_{\tau,n}^{(k)}$ are defined by the product (see [1, p 322, 5, p 973])

$$\delta_{\tau,n}^{(k)}(z) = \prod_{i=1}^{m_\tau(n)} (z - (t_{\tau,n,i})^k). \tag{1.6}$$

We shall prove the following.

Theorem 1. *Let f_τ be defined by (1.1). Then for the dynamical zeta functions defined by (1.2) and (1.3) we have that*

$$\begin{aligned}
&(i)\ \tau\ \textit{even} \qquad \zeta_{(\tau,k)} = \begin{cases} \left(\dfrac{1-\tau^{2k}z}{1-\tau^k z}\right)(1-\tau^{k+1}) & k\ \textit{even} \\ \dfrac{1-\tau^{2k}z}{1-\tau^k z} & k\ \textit{odd} \end{cases} \\
&(ii)\ \tau\ \textit{odd} \qquad \zeta_{(\tau,k)} = \begin{cases} \left(\dfrac{1-\tau^{2k}z}{1-\tau^k z}\right)^2(1-\tau^{k+1}) & k\ \textit{even} \\ \dfrac{(1-\tau^{2k}z)^2}{1-\tau^k z} & k\ \textit{odd.} \end{cases}
\end{aligned} \tag{1.7}$$

In the next section we express periodic points as roots of certain polynomials whose irreducible factors are investigated. In section 3 we study in detail a family of periodic orbits which play a central role in the proof of theorem 1. The latter is completed in section 4.

2. Factorization of the primitive orbit polynomial

The fixed points of f_τ^n (also called n cycles) are roots of the polynomial

$$P_{\tau,n}(z) = f_\tau^n(z) - z \qquad n = 1, 2, \ldots \tag{2.1}$$

where f_τ^n denotes the nth iterate of f_τ (with $f_\tau = f_\tau^1$). Let $H_{\tau,n}(z)$ be the polynomial whose roots are the points of minimal period n. Then its degree deg $H_{\tau,n}$ will be $nm_\tau(n)$. We first note that [5, 7, 8]

$$P_{\tau,n}(z) = \prod_{d|n} H_{\tau,d}(z) \qquad \tau^n = \sum_{d|n} dm_\tau(d) \tag{2.2}$$

where the product is taken over all the positive divisors d of n. Then the primitive orbit polynomial $H_{\tau,n}(z)$ and its degree are extracted from (2.2) via Möbius inversion, yielding

$$H_{\tau,n}(z) = \prod_{d|n} P_{\tau,d}(z)^{\mu(n/d)} \qquad nm_\tau(n) = \sum_{d|n} \tau^d \mu\left(\frac{n}{d}\right) \qquad n = 1, 2, \ldots \tag{2.3}$$

where μ is the Möbius function [9, ch 2]. Because f_τ is monic so are $P_{\tau,n}(z)$ and $H_{\tau,n}(z)$, and the periodic points of f_τ are algebraic integers. Moreover

$$H_{\tau,n}(z) = \prod_{d|n} \left(\prod_{d_1|\tau^d-1} \Psi_{d_1}(z) \prod_{d_2|\tau^d+1} \Psi_{d_2}(z) \right)^{\mu(n/d)} \tag{2.4}$$

where

$$\Psi_n(z+z^{-1}) = z^{-\phi(n)/2}F_n(z) = \prod_{\substack{k=1\\(k,n)=1}}^{\lfloor n/2\rfloor}\left(z+z^{-1}-2\cos\left(\frac{2\pi k}{n}\right)\right) \qquad n \geqslant 1 \tag{2.5}$$

$F_n(z)$ is the nth *cyclotomic* polynomial (its roots are the primitive nth roots of unity), ϕ is the Euler totient function [12, ch 3], and $\lfloor n/2\rfloor$ is the largest integer smaller than or equal to $n/2$. In [1, pp 971–2] relation (2.4) is only proved for $\tau = 2$, the case of general τ, however, being unproblematic. It can be shown [6] that for $n \geqslant 3$, $\Psi_n(z)$ is a monic irreducible polynomial over the integers of degree $\phi(n)/2$. For $n = 1, 2$ one calculates using definition (2.5)

$$\Psi_1(z)^2 = z - 2 \qquad \Psi_2(z)^2 = z + 2.$$

Nevertheless, only the squares of $\Psi_1(z)$ and $\Psi_2(z)$ appear in equation (2.4). Indeed $H_{\tau,1}$ has the factor Ψ_1^2 for even τ and $\Psi_1^2\Psi_2^2$ for odd τ. Expanding the products on the right-hand side of relation (2.4), $H_{\tau,n}$ becomes

$$H_{\tau,n} = \Phi_{\tau,n}\underbrace{(\Psi_{d_1^-}\ldots\Psi_{d_{i_n}^-})}_{H_{\tau,n}^-}\underbrace{(\Psi_{d_1^+}\ldots\Psi_{d_{j_n}^+})}_{H_{\tau,n}^+}. \tag{2.6}$$

The subscripts d_i^- and d_j^+ in the latter relation are divisors of $\tau^n - 1$ and $\tau^n + 1$ respectively, that have a special property. Namely they are not factors of any $\tau^d - 1$ and $\tau^d + 1$ with $d|n$ and $d < n$ (as if they were $H_{\tau,n}$ would have as roots non-minimal n cycles). The polynomial $\Phi_{\tau,1}$ is equal to Ψ_1^2 and $\Psi_1^2\Psi_2^2$ for even and odd τ respectively, while $\Phi_{\tau,n} = 1$, for all $n > 1$.

3. Self-conjugate orbits and then degrees in the factorization

The semi-conjugacy between the Chebyshev polynomials and the Bernoulli shifts is given by the function $h(z) = z + z^{-1}$. Let b be the symbolic representation of an n cycle of the full shift map in τ symbols

$$b = \{b_1,\ldots,b_n,b_1,\ldots\} = \overline{b_1\ldots b_n} \qquad b_i \in \{0,1,\ldots,\tau-1\}$$

which is the τ-ary expansion of the associated periodic point of the map $z \mapsto \tau z \pmod 1$ of the interval $[0,1]$ to itself. Then the periodic point of the corresponding n cycle of f_τ is given by

$$h(b) = 2\cos\left(\frac{2\pi\lambda(b)}{\tau^n - 1}\right) \qquad \lambda(b) = \sum_{r=1}^{n}\tau^{n-r}b_r. \tag{3.1}$$

Let σ denote the shift map $(\sigma(b))_i = b_{i+1}$. Then the points of the n cycle with initial condition $h(b)$ are given by $h(\sigma^t(b))$, for $t = 0,\ldots,n-1$.

We define $C(b_i) = \tau - 1 - b_i$ and $C(b_1\ldots b_n) = C(b_1)\ldots C(b_n)$. Then $\lambda(C(b)) = \tau^n - 1 - \lambda(b)$ and $h(C(b)) = h(b)$. The latter means that the two complementary cycles b and $C(b)$ of the full shift in τ symbols are mapped under h to the same n cycle of f_τ. Of interest are the periodic strings of period $2n$ satisfying the relation

$$b' = \overline{b_1\ldots b_n C(b_1)\ldots C(b_n)}. \tag{3.2}$$

Then because $\sigma^n(b') = C(b')$ we have $h(\sigma^n(b')) = h(b')$. Thus b' halves its period under h. These are fixed points of f_τ^n with $\lambda(b') = (\tau^n - 1)(1 + \lambda(b))$ so that

$$h(b') = 2\cos\left(\frac{2\pi(1+\lambda(b))}{\tau^n + 1}\right). \tag{3.3}$$

The quantities $h(b)$ and $h(b')$, defined by equations (3.1) and (3.3), are respectively roots of the polynomials $H^-_{\tau,n}(z)$ and $H^+_{\tau,n}(z)$. More specifically the ratios $\lambda(b)/(\tau^n - 1)$ and $(1+\lambda(b))/(\tau^n + 1)$ can be written in lower terms as k^-/d^-_i and k^+/d^+_j, respectively, so that $\Psi_{d^-_i}(h(b)) = \Psi_{d^+_j}(h(b')) = 0$. From now on we shall refer to periodic strings of the form (3.2) as *self-conjugate*. Let $m'_\tau(n)$ be the number of self-conjugate prime τ-ary strings of length n, and let $\bar{m}_\tau(n)$ and $\bar{m}'_\tau(n)$ be the number of prime and self-conjugate prime cycles, respectively, of the Bernoulli shift. When τ is even and $n \geqslant 1$ we have $\bar{m}'_\tau(n)$ prime n cycles halving their period under $h(\bar{m}'_\tau(n) = 0$ for odd n). The remaining $\bar{m}_\tau(n) - \bar{m}'_\tau(n)$ n cycles of the shift map will contribute to $(\bar{m}_\tau(n) - \bar{m}'_\tau(n))/2$ prime n cycles of f_τ as complementary cycles of the shift map are mapped under h to the same n cycles of f_τ. The latter number of cycles should be increased by $\bar{m}'_\tau(2n)$—not by $\bar{m}'_\tau(2n)/2$ because b' and its complement $C(b')$ belong to the same cyclic permutation class (see [10, p 329]). Then one obtains

$$m_\tau(n) = \begin{cases} (\bar{m}_\tau(n) - \bar{m}'_\tau(n))/2 + \bar{m}'_\tau(2n) & n \text{ even} \\ \bar{m}_\tau(n)/2 + \bar{m}'_\tau(2n) & n \text{ odd} \end{cases} \qquad n \geqslant 1, \text{ even } \tau.$$

Because both maps have complete symbolic dynamics over a real phase space, then $\bar{m}_\tau = m_\tau$ and $\bar{m}'_\tau = m'_\tau$, from which we obtain

$$m'_\tau(2n) = \begin{cases} (m_\tau(n) + m'_\tau(n))/2 & n \text{ even} \\ m_\tau(n)/2 & n \text{ odd} \end{cases} \qquad n \geqslant 1, \text{ even } \tau. \tag{3.4}$$

For odd τ let b denote the 1 cycle $\overline{(\tau-1)/2}$. Then $C(b) = b$, and only $(\tau-1)/2$ cycles of minimal length 2 halve their period under h. Thus the recurrence relation defining m'_τ is given by

$$m'_\tau(2n) = \begin{cases} (m_\tau(n) + m'_\tau(n))/2 & n \text{ even} \\ m_\tau(n)/2 & n \text{ odd} \end{cases} \qquad n > 1, \; m'_\tau(2) = (\tau-1)/2, \text{ odd } \tau. \tag{3.5}$$

From our previous discussion and for all $\tau \geqslant 2$ and $n > 1$ we have deg $H^+_{\tau,n} = nm'_\tau(2n)$ and deg $H^+_{\tau,n} = n(m_\tau(n) - m'_\tau(2n))$ so that

$$\deg H^\pm_{\tau,n} = \begin{cases} n(m_\tau(n) \pm m'_\tau(n))/2 & n \text{ even} \\ nm_\tau(n)/2 & n \text{ odd} \end{cases} \qquad \tau \geqslant 2, \; n > 1. \tag{3.6}$$

For $n = 1$ we have

$$\deg H^+_{\tau,1} = \begin{cases} \tau/2 & \tau \text{ even} \\ (\tau-1)/2 & \tau \text{ odd} \end{cases} \qquad \deg H^-_{\tau,1} = \begin{cases} (\tau-2)/2 & \tau \text{ even} \\ (\tau-3)/2 & \tau \text{ odd.} \end{cases} \tag{3.7}$$

Proposition 1. *The derivatives along cycles of minimal length n are τ^n or $-\tau^n$ exactly when the initial conditions are roots of the polynomials $H^-_{\tau,n}$ and $H^+_{\tau,n}$, respectively.*

Proof. From the fact that

$$f_\tau(2\cos(2\pi\theta)) = 2\cos(2\pi\tau\theta) \qquad \tau \geqslant 2 \tag{3.8}$$

one obtains—using induction—that

$$f^n_\tau(z + z^{-1}) = z^{\tau^n} + z^{-\tau^n}$$

so that

$$(f^n_\tau)'(z + z^{-1}) = \frac{\tau^n(z^{\tau^n+1} - z^{-(\tau^n-1)})}{z^2 - 1} \qquad z \in S^1 \setminus \{\pm 1\}. \tag{3.9}$$

Let α be a periodic point of minimal length n with $H^{+}_{\tau,n}(\alpha) = 0$. Then there exist relatively prime integers d_j^+ and k_j^+ with d_j^+ dividing $\tau^n + 1$ (say $d_j^+\rho_j = \tau^n + 1$), $\Psi_{d_j^+}(\alpha) = 0$, with $\alpha = \omega + \omega^{-1}$ and $\omega = \exp(2\pi k^+/d_j^+)$. The situation is similar for $H^{-}_{\tau,n}(\alpha) = 0$ then $\Psi_{d_i^-}(\alpha) = 0$, and $d_i^-\rho_i = \tau^n - 1$ with $\omega = \exp(2\pi k^-/d_i^-)$. Then one obtains from relation (3.9)

$$\tau^{-n}(\omega^2 - 1)(f_\tau^n)'(\alpha) = \begin{cases} (\omega^{d_j^+})^{\rho_j} - \omega^2(\omega^{d_j^+})^{-\rho_j} & H^{+}_{\tau,n}(\alpha) = 0 \\ \omega^2(\omega^{d_i^-})^{\rho_i} - (\omega^{d_i^-})^{-\rho_i} & H^{-}_{\tau,n}(\alpha) = 0 \end{cases}$$

which gives the desired result. □

From relation (2.6) it follows that for $n = 1$ we have the period-1 point $\bar{z} = 2$ for even τ ($\Phi_{\tau,1} = \Psi_1^2$) and the period-1 points $\bar{z} = \pm 2$ for odd τ ($\Phi_{\tau,1} = \Psi_1^2\Psi_2^2$). The latter 1-cycles give rise to the exceptional eigenvalue τ^2 (does not follow the rule of proposition 1) calculated via (3.9) at $z = \pm 1$:

$$f_\tau'(2) = \tau^2 \qquad \tau \geqslant 2$$
$$f_\tau'(-2) = \tau^2 \qquad \tau > 2 \text{ and } \tau \text{ odd.}$$

Then from definition (1.6) we have

$$\delta^{(k)}_{\tau,1}(z) = \begin{cases} (z - \tau^{2k})(z - \tau^k)^{(\tau-2)/2}(z - (-\tau)^k)^{\tau/2} & \tau \text{ even} \\ (z - \tau^{2k})^2(z - \tau^k)^{(\tau-3)/2}(z - (-\tau)^k)^{(\tau-1)/2} & \tau \text{ odd} \end{cases} \qquad n = 1. \tag{3.10}$$

When $n > 1$ one obtains

$$\delta^{(k)}_{\tau,n}(z) = \begin{cases} (z - \tau^{nk})^{(m_\tau(n) - m'_\tau(n))/2}(z - (-\tau^n)^k)^{(m_\tau(n) + m'_\tau(n))/2} & n \text{ even} \\ (z - \tau^{nk})^{m_\tau(n)/2}(z - (-\tau^n)^k)^{m_\tau(n)/2} & n \text{ odd} \end{cases} \qquad n > 1. \tag{3.11}$$

4. Proof of theorem 1

Theorem 1 follows from three lemmas.

Lemma 1. *Let n be a positive odd integer. Then*

$$\sum_{d|n} d m_\tau(2^l d) = 2^{-l}(\tau^{2^l n} - \tau^{2^{l-1} n}) \qquad l \geqslant 1. \tag{4.1}$$

Proof. We prove (4.1) by induction on l. Using the second equation of relation (2.3) together with the fact that n is odd yields

$$\tau^{2n} = \sum_{d|2n} d m_\tau(d) = \sum_{d|n} d m_\tau(d) + \sum_{d|n} 2d m_\tau(2d) = \tau^n + 2\sum_{d|n} d m_\tau(2d)$$

which establishes relation (4.1) for $l = 1$. We observe that

$$\tau^{2^l n} = \sum_{d|2^l n} d m_\tau(d) = \sum_{0 \leqslant r \leqslant l} \sum_{d|n} 2^r d m_\tau(2^r d)$$
$$= \tau^{2n} + \sum_{1<r<l} 2^r \left(\sum_{d|n} d m_\tau(2^r d) \right) + 2^l \sum_{d|n} d m_\tau(2^l d) \tag{4.2}$$

where we have used our result for $l = 1$ and the second equation of relation (2.3). Assuming now (4.1) to hold up to $l - 1$, equation (4.2) yields

$$\tau^{2^l n} = \tau^{2n} + \sum_{1<r<l} (\tau^{2^r n} - \tau^{2^{r-1} n}) + 2^l \sum_{d|n} d m_\tau(2^l d) = \tau^{2^{l-1} n} + 2^l \sum_{d|n} d m_\tau(2^l d)$$

which gives the desired result. □

Lemma 2. *Let*

$$S_\tau(n) = \sum_{\substack{d|n \\ d \text{ odd}}} \frac{1}{d} m'_\tau \left(\frac{2n}{d}\right) \tag{4.3}$$

then, for all positive integers n *we have*

$$S_\tau(n) = \begin{cases} \tau^n/2n & \tau \text{ even} \\ (\tau^n - 1)/2n & \tau \text{ odd.} \end{cases} \tag{4.4}$$

Proof. We write $n = 2^l n'$ where n' is an odd integer and $l \geqslant 0$, and we proceed by induction on l. Let $l = 0$. Then for even τ we obtain from (3.4) and the second equation of (2.3) that

$$n' S_\tau(n') = \sum_{d|n'} \frac{n'}{d} m'_\tau \left(\frac{2n'}{d}\right) = \sum_{d|n'} d m'_\tau(2d) = \frac{1}{2} \sum_{d|n'} d m_\tau(d) = \frac{\tau^{n'}}{2}.$$

For odd τ we obtain from (3.5)

$$\begin{aligned} n' S_\tau(n') &= \sum_{\substack{d|n' \\ d<n'}} \frac{n'}{d} m'_\tau \left(\frac{2n'}{d}\right) + m'_\tau(2) \\ &= \sum_{\substack{d|n' \\ d>1}} d m'_\tau(2d) + m'_\tau(2) = \frac{1}{2} \sum_{\substack{d|n' \\ d>1}} d m_\tau(d) + m'_\tau(2) \\ &= \frac{1}{2}(\tau^{n'} - \tau) + \frac{1}{2}(\tau - 1) = \frac{\tau^{n'} - 1}{2}. \end{aligned}$$

For all τ we obtain using relations (3.4), (3.5) and lemma 1 that

$$\begin{aligned} S_\tau(2^l n') &= \sum_{d|n'} \frac{1}{d} m'_\tau \left(2\left(\frac{2^l n'}{d}\right)\right) = \frac{1}{2} \sum_{d|n'} \frac{1}{d} m_\tau \left(\frac{2^l n'}{d}\right) + \frac{1}{2} \sum_{d|n'} \frac{1}{d} m'_\tau \left(\frac{2(2^{l-1} n')}{d}\right) \\ &= \frac{1}{2n'} \sum_{d|n'} d m_\tau(2^l d) + \frac{1}{2} \sum_{\substack{d|2^{l-1}n' \\ d \text{ odd}}} \frac{1}{d} m'_\tau \left(\frac{2(2^{l-1} n')}{d}\right) \end{aligned}$$

so that

$$S_\tau(2^l n') = \frac{1}{2}\left(\frac{\tau^{2^l n'} - \tau^{2^{l-1} n'}}{2^l n'} + S_\tau(2^{l-1} n')\right).$$

Assuming that relation (4.4) holds true up to $l - 1$ the above equation gives

$$S_\tau(2^l n') = \frac{1}{2} \begin{cases} \dfrac{\tau^{2^l n'} - \tau^{2^{l-1} n'}}{2^l n'} + \dfrac{\tau^{2^{l-1} n'}}{n' 2^l} & \tau \text{ even} \\[2ex] \dfrac{\tau^{2^l n'} - \tau^{2^{l-1} n'}}{2^l n'} + \dfrac{\tau^{2^{l-1} n'} - 1}{2^l n'} & \tau \text{ odd} \end{cases}$$

which completes the proof. □

Lemma 3. *Let*

$$Q_\tau(z) = \prod_{n=1}^{\infty} \left(\frac{1 + z^n}{1 - z^n}\right)^{m'_\tau(2n)} \qquad \tau \geqslant 2 \tag{4.5}$$

then, for $|z| < \tau^{-1}$ *the product converges to*

$$Q_\tau(z) = \begin{cases} 1/(1-\tau z) & \tau \text{ even} \\ (1-z)/(1-\tau z) & \tau \text{ odd.} \end{cases} \tag{4.6}$$

Proof. By taking the logarithms and expanding $\ln(1 \pm z^n)$, (4.5) becomes

$$\ln Q_\tau(z) = 2\sum_{n=1}^{\infty} m'_\tau(2n) \sum_{r=0}^{\infty} \frac{z^{n(2r+1)}}{2r+1} = 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} m'_\tau(2n)\Theta_r r^{-1} z^{rn} \tag{4.7}$$

where Θ_r is zero for even r and one otherwise. Using the fact that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c(i,j) z^{ij} = \sum_{n=1}^{\infty} z^n \sum_{d|n} c\left(\frac{n}{d}, d\right)$$

relation (4.7), letting $c(i,j) = m'_\tau(2i)\Theta_j j^{-1}$ and definition (4.3) we obtain

$$\ln Q_\tau(z) = 2\sum_{n=1}^{\infty} z^n \sum_{d|r} \frac{1}{d} m'_\tau\left(\frac{2n}{d}\right)\Theta_d = 2\sum_{n=1}^{\infty} S_\tau(n) z^n.$$

Our result now follows from lemma 2. □

We now prove theorem 1. Let τ be odd. Then from (1.5), (3.10) and (3.11) we have

$$\zeta_{(\tau,k)}(z\tau^{-k})^{-2} = (1-\tau^k z)^4 (1-z)^{\tau-3} (1-(-1)^k z)^{\tau-1} \prod_{\substack{n\geqslant 3 \\ n \text{ odd}}} (1-z^n)^{m_\tau(n)} (1-(-1)^k z^n)^{m_\tau(n)}$$

$$\times \prod_{\substack{n\geqslant 2 \\ n \text{ even}}} (1-z^n)^{m_\tau(n)-m'_\tau(n)} (1-(-1)^k z^n)^{m_\tau(n)+m'_\tau(n)}. \tag{4.8}$$

Letting

$$G_1(z;\tau,k) = \frac{(1-\tau^k z)^4}{(1-z)^3(1-(-1)^k z)}$$
$$F(z;\tau,k) = \prod_{n=1}^{\infty}(1-z^n)^{m_\tau(n)} \prod_{n=1}^{\infty}(1-(-1)^k z^n)^{m_\tau(n)} \prod_{n=1}^{\infty}\left(\frac{1-(-1)^k z^{2n}}{1-z^{2n}}\right)^{m'_\tau(2n)} \tag{4.9}$$

one verifies that

$$\zeta_{(\tau,k)}(z\tau^{-k})^{-2} = G_1(z;\tau,k) F(z;\tau,k). \tag{4.10}$$

Let us define

$$A_\tau(z) = \prod_{n=1}^{\infty}(1-z^n)^{m_\tau(n)}.$$

Then (4.9) can be written as

$$F(z;\tau,k) = \begin{cases} A_\tau(z)^2 & k \text{ even} \\ A_\tau(z^2) Q_\tau(z^2) & k \text{ odd.} \end{cases}$$

Because

$$A_\tau(z) = \prod_{n=1}^{\infty}\prod_{|p|=n}(1-z^n) = \prod_p (1-z^{n_p}) = \exp\left(-\sum_{n=1}^{\infty}\frac{(\tau z)^n}{n}\right) = 1-\tau z \tag{4.11}$$

the former relation together with lemma 3 gives

$$F(z;\tau,k) = \begin{cases} (1-\tau z)^2 & k \text{ even} \\ 1-z^2 & k \text{ odd} \end{cases}$$

from which our theorem follows for odd τ.

For even τ we define

$$G_2(z;\tau,k)=\left(\frac{1-\tau^k z}{1-z}\right)^2.$$

Then it can be verified that

$$\zeta_{(\tau,k)}(z\tau^{-k})^{-2}=G_2(z;\tau,k)F(z;\tau,k)$$

with

$$F(z;\tau,k)=\begin{cases}(1-\tau z)^2 & k \text{ even}\\ 1 & k \text{ odd}\end{cases}$$

which finally establishes formula (1.7) for even τ.

Acknowledgments

The author is grateful to F Vivaldi for his valuable remarks. This research was supported by a European Commission HCM contract ERBXCT 940460.

References

[1] Hatjispyros S and Vivaldi F 1995 A family of rational zeta functions for the quadratic map *Nonlinearity* **8** 321–32
[2] Cvitanović P, Hansen K and Vattay G Periodic orbit theory http:\\alf.nbi.dk\~predrag\QCcource\
[3] Anderson G 1995 Notes on dynamical zeta functions, private communication
[4] Beardon A F 1991 *Iteration of Rational Functions* (New York: Springer)
[5] Vivaldi F and Hatjispyros S 1992 Galois theory of periodic orbits of polynomial maps *Nonlinearity* **5** 961–78
[6] Hatjispyros S J 1993 Arithmetical properties of rational iterations *PhD Thesis* University of London
[7] Morton P and Patel P 1994 The Galois theory of periodic points of polynomial maps *Proc. London Math. Soc.* **68** 225–63
[8] Bousch T 1992 Sur quelques problèmes de la dynamique holomorphe *PhD Thesis* Université de Paris-Sud, Centre d'Orsay
[9] Apostol T A 1984 *Introduction to Analytic Number Theory* (New York: Springer)
[10] Artuso R, Aurell E and Cvitanović P 1990 Recycling strange sets: I. Cycle expansions *Nonlinearity* **3** 325–59
[11] Artuso R, Aurell E and Cvitanović P 1990 Recycling strange sets: II. Applications *Nonlinearity* **3** 361–86
[12] Niven I 1956 *Irrational Numbers* (Washington, DC: Mathematical Association of America)