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A family of rational zeta functions for the quadratic map

Spyros Hatjispyros and Franco Vivaldi
School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

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Abstract. A family of dynamical zeta functions is introduced, that weigh each periodic orbit with integral powers of its multiplier. For the quadratic map, we prove that some of these functions are rational, and conjecture their general form.

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1. Introduction

This paper is devoted to the theoretical and numerical study of a new family of dynamical zeta-functions. Our main result—the rationality of these functions—concerns the quadratic family $f_c(z) = z^2 + c$, with complex c (and in one instance the larger family $f_{c,a} = z^a + c$, where a is an integer greater than 1). The main constructs are applicable to any polynomial mapping f over a field K of characteristic zero.

The periodic points of period n of f (also called n-cycles) are the roots of the polynomial

$$P_n(z) = f^n(z) - z \qquad n = 1, 2, \dots \tag{1.1}$$

where f^n denotes the nth iterate of f (with $f = f^1$). Möbius inversion yields the polynomials $H_n(z)$ [1, 2]

$$H_n(z) = \prod_{d|n} P_d(z)^{\mu(n/d)} \qquad n = 1, 2, \dots \tag{1.2}$$

where μ is the Möbius function ([3], chapter 2). (The inversion (1.2) is valid even if K has non-zero characteristic [2]). The roots of $H_n(x)$ are the periodic points of *essential* period n [4]. They include the orbits of *minimal* period n, and, possibly, bifurcational orbits of lower period. The degree of $H_n(z)$ is a multiple of n, and $m = \partial H_n/n$ is the number of orbits of essential period n, including multiplicities. One has $\partial H_n \sim (\partial f)^n$. If f is monic (unit leading coefficient) so are $P_n(z)$ and $H_n(z)$. For $f = f_{c,a}$, P_n and H_n are monic polynomials over $Z[c]$, and the periodic orbits of $f_{c,a}$ are integers in algebraic extensions of $Z[c]$.

Let ξ be a periodic point of minimal period n. Its *multiplier* is the derivative of f^n at ξ, which is a prominent symmetric function of the points of the orbit. For the family $f_{c,a}$, we have $f'_{c,a}(z) = az^{a-1}$, and we shall replace the multiplier by the product of the points in the orbit (which we still call multiplier)

$$w_n(\xi) = \xi \cdot f_{c,a}(\xi) \cdots f_{c,a}^{n-1}(\xi) \qquad a^n w_n(\xi)^{a-1} = (f_{c,a}^n)'(\xi). \tag{1.3}$$

There are m such multipliers, and they are roots of a monic polynomial over $\mathbb{Z}[c]$, which we call the nth *multiplier polynomial* of f, denoted by $\delta_n(z)$. Multiplier polynomials can be computed by means of Galois theory [1].

This construction can be generalized. For any integer $k > 0$, we define $\delta_n^k(z)$ to be the monic polynomial whose roots are the kth power of the roots of $\delta_n(z)$. Specifically, if $\xi_1, \ldots, \xi_m$ are representative points of the m orbits of f of essential period n, we define

$$\delta_n^k(z) = \prod_{i=1}^{m(n)} (z - w_n(\xi_i)^k). \tag{1.4}$$

Thus $\delta_n(z) = \delta_n^1(z)$, and all polynomials δ_n^k have the same degree m.

We now define a sequence of dynamical zeta functions which are naturally associated to multiplier polynomials

$$\zeta_k(z)^{-1} = \prod_{n=1}^{\infty} z^{nm(n)}\,\delta_n^k(1/z^n) \qquad k = 1, 2, \ldots. \tag{1.5}$$

The aim of this paper is to provide evidence that for $f = f_{c,a}$ these products converge for all $k > 0$ to rational functions of z and c. More precisely, we have

Proposition 1.1. *Let $f_{c,a}(z) = z^a + c$, with a an integer greater than 1, and let $\zeta_k^a(z)$ be the corresponding zeta function (1.5). Then, for all c, the product (1.5) defines a rational function of z, given by*

$$\zeta_{a-1}^a(z) = \frac{1-z}{1-az}. \tag{1.6}$$

Of note is the independence of $\zeta_{a-1}^a(z)$ from the parameter c. For fixed a, the value $k = a-1$ appears to be the only one for which ζ_k^a is independent of c. For $a = 2$ (quadratic family) the broader picture is given by the following conjecture

Conjecture. *For the quadratic family, and all integers $k > 1$ we have, (dropping the superscript a)*

$$\zeta_k(z) = \frac{1-z}{1-2z}\,\frac{G_{k-1}(z/2, c)}{G_k(z, c)} \tag{1.7}$$

where $G_k(z, c)$ is an integral polynomial in z and c of z–degree $\lfloor k/2 \rfloor$ ($\lfloor x \rfloor$ is the largest integer not exceeding x).

Assuming the validity of (1.7), we note that all functions ζ_k share the factor $(1-z)/(1-2z)$, which is equal to $\zeta_1(z)$, from proposition 1.1. Also of interest is the existence of a close relationship between the poles of $\zeta_{k-1}(z)$ and the zeros of $\zeta_k(z)$, which is consistent with a conjecture put forward by Artuso *et al* on a related family of zeta functions ([5] part I, p. 349). Finally, one sees that the infinite sequence of polynomials $G_k(x, c)$ provide an alternative encoding of the information stored in the multipliers of the system.

The plan of this paper is as follows. In the next section, after reviewing some ideas concerning zeta functions, we consider the convergence of the products (1.5), and then relate the coefficients of their logarithmic derivative to multiplier polynomials. In section 4 we prove proposition 1.1, and in the following section we provide numerical and theoretical support for the conjecture. In appendix 1 we display the polynomials $G_k(z, c)$ for $k = 1, \ldots, 12$, while the multiplier polynomials $\delta_n(z)$ are tabulated in appendix 2 for $n = 1, \ldots, 6$.

2. Dynamical zeta functions

For a background reference for this section, see [5] and references therein.

Let $\mathrm{Fix}(f^n)$ be the set of periodic points of period n of f, and let M_n be its cardinality. We construct a sequence Z_n of complex numbers, defined as follows

$$Z_n = \sum_{i=1}^{M_n} t_{i,n} \tag{2.1}$$

where $t_{i,n}$ is a *weight* associated to the ith n–cycle. The corresponding *dynamical zeta function* $\zeta(z)$ is defined formally as

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{Z_n z^n}{n}\right). \tag{2.2}$$

The zeta function is related to the generating function of the sequence Z_n via the logarithmic derivative

$$\sum_{n=1}^{\infty} Z_n z^n = z\frac{\mathrm{d}}{\mathrm{d}z}\ln(\zeta(z)). \tag{2.3}$$

We say that the weights $t_{i,n}$ are *multiplicative* when they assume the same value at all points of a periodic orbit, and when the weight at a point of an orbit repeating r times a minimal orbit is the rth power of the weight of the minimal orbit itself. When this is the case, let us denote by p the symbolic representation of a minimal orbit of length $|p| = n_p$, with associated weight $t_p = t_{p,n_p}$. Then the zeta function (2.2) can be written in the form of an eulerian product

$$\zeta(z) = \prod_p (1 - t_p z^{n_p})^{-1} \tag{2.4}$$

where p runs over all admissible periodic strings of minimal length n_p, for all $n_p \geqslant 1$. The expressions (2.3) and (2.4) are valid wherever the series in (2.2) converges unconditionally.

For example, taking the weights $t_{i,n}$ to be

$$t_{i,n} = \left|(f^n)'(x_i)\right|^{\beta} \qquad -\infty < \beta < \infty \tag{2.5}$$

thermodynamical quantities such as generalized dimensions, entropies and Lyapunov exponents can be extracted from the knowledge of the leading singularities of the series (2.2), as functions of the real parameter β [6–9].

Algebraically speaking, the multipliers themselves are more natural than their absolute value (2.5), and in what follows we shall consider the multiplicative weights $t_{i,n} = w_n(\xi_i)^k$, where w_n is given by (1.3), k is a positive integer, and ξ_i is a point in the ith n-cycle. The quantities Z_n defined in (2.1) are now sums over multipliers

$$Z_n^k = \sum_{\xi \in \mathrm{Fix}(f^n)} w_n(\xi)^k \tag{2.6}$$

and the zeta function becomes

$$\zeta_k(z) = \exp\left(\sum_{n=1}^{\infty} \frac{Z_n^k z^n}{n}\right) = \prod_p (1 - w_p^k z^{n_p})^{-1}. \tag{2.7}$$

The equality between these functions and those defined in (1.5) will be proven in section 3. In reference [5] it was conjectured, based on a transfer operator argument, that the zeros of the zeta function (2.4) should be given by the poles of the zeta function

$$\zeta^*(z) = \prod_p \left(1 - t_p^* z^{n_p}\right)^{-1} \quad \text{with} \quad t_p^* = t_p / \left(f^{n_p}\right)'(z_p).$$

Formula (1.7) is a refinement of this conjecture for the zeta function (2.7). The appearance of the argument $z/2$ results from having chosen the product of points in an orbit rather than the muliplier (cf. (1.3)).

The convergence of the series in (2.7) derives from the boundness of the Julia set of a polynomial (see [10], prop. 4.2 and 4.10). Let B be an upper bound for the modulus of all periodic points of f. Then

$$|Z_n^k| \leqslant \sum_{\xi \in \mathrm{Fix}(f^n)} B^{nk} = M_n B^{nk}$$

and $\zeta_k(z)$ converges and does not vanish in the disc $|z| < (aB^k)^{-1}$, where a is the degree of f. We compute B as the radius of the smallest circle centered at the origin encircling the Julia set of f, which is given by

$$B = \inf\left\{r > 0 \mid |z| \leqslant r \Rightarrow |f^{-1}(z)| \leqslant r\right\}.$$

For the quadratic family, one verifies that $B = B(c) = 1/2 + \sqrt{1/4 + |c|}$. (The case in which the w_p^k are multipliers—rather than products of points in the orbits—can be treated similarly.)

In the remainder of this section we establish a relation between the coefficients of the multiplier polynomials, and those of the elements of the sequence (2.6). We first recall a standard result on symmetric functions. Let $\xi_1, \ldots, \xi_n$ be elements of a commutative ring. Their kth *power sum* is the expression $S_k = \sum_{i=1}^n \xi_i^k$. If $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric functions of the ξ_i, then the power sums—themselves symmetric functions—can be expressed in terms of them. This is achieved via the so-called *Newton's relations*

Lemma 2.1. *Let ξ_k, S_k and σ_k be as above. Then*

$$\begin{aligned} &\sum_{i=0}^{k-1} (-1)^i \sigma_i S_{k-i} + (-1)^k k \sigma_k = 0 \quad 1 \leqslant k \leqslant n \\ &\sum_{i=0}^{n} (-1)^i \sigma_i S_{k-i} = 0 \quad k \geqslant n. \end{aligned} \qquad \sigma_0 = 1 \tag{2.8}$$

For a proof, see [11], page 51.

We denote by T_n^k the kth power sum of the roots of $\delta_n(z)$, that is, the trace of $\delta_n^k(z)$. Since the w–weights are multiplicative, we obtain

$$Z_n^k = \sum_{d|n} d\, T_d^{kn/d}. \tag{2.9}$$

To express T in terms of Z we cannot perform the usual Möbius inversion, as the summands on the right hand side of equation (2.9) depend on d as well as n/d. Nonetheless an inversion is possible, which, together with the Newton's relations (2.8), allows one to express the coefficients of the multiplier polynomials in terms of the coefficients of the series (2.3).

Proposition 2.2. [12] *Let Z_n^k and T_n^k be as above. Then the following holds*

$$T_n^k = \frac{1}{n}\sum_{d|n}\mu(d)Z_{n/d}^{kd}. \tag{2.10}$$

Proof. We define the sequence

$$\Phi_k(n) = \frac{1}{n}\sum_{d|n}\mu(d)\sum_{r|\frac{n}{d}} rT_r^{kn/r}.$$

It suffices to show that for all $n \geqslant 1$, $\Phi_k(n) = T_n^k$. Let t be the number of distinct prime factors of n. We write the prime decomposition of n in the form $n = n_t = n_{t-1}p_t^{a_t}$, with $n_0 = 1$. Then

$$\begin{aligned}
\Phi_k(n) &= \frac{1}{n_{t-1}p_t^{a_t}}\sum_{d|n_{t-1}p_t^{a_t}}\mu(d)\sum_{r|n_{t-1}p_t^{a_t}/d} rT_r^{kn_{t-1}p_t^{a_t}/r}\\
&= \frac{1}{n_{t-1}p_t^{a_t}}\sum_{l=0}^{a_t}\sum_{d|n_{t-1}}\mu(p_t^l d)\sum_{r|n_{t-1}p_t^{a_t}/p_t^l d} rT_r^{kn_{t-1}p_t^{a_t}/r}\\
&= \frac{1}{n_{t-1}p_t^{a_t}}\sum_{d|n_{t-1}}\mu(d)\left(\sum_{r|\frac{n_{t-1}}{d}p_t^{a_t}} rT_r^{kn_{t-1}p_t^{a_t}/r} - \sum_{r|\frac{n_{t-1}}{d}p_t^{a_t-1}} rT_r^{kn_{t-1}p_t^{a_t}/r}\right)\\
&= \frac{1}{n_{t-1}p_t^{a_t}}\sum_{d|n_{t-1}}\mu(d)\left(\sum_{l=0}^{a_t}\sum_{r|\frac{n_{t-1}}{d}} p_t^l rT_{p_t^l r}^{kn_{t-1}p_t^{a_t-l}/r} - \sum_{l=0}^{a_t-1}\sum_{r|\frac{n_{t-1}}{d}} p_t^l rT_{p_t^l r}^{kn_{t-1}p_t^{a_t-l}/r}\right)\\
&= \frac{1}{n_{t-1}}\sum_{d|n_{t-1}}\mu(d)\sum_{r|\frac{n_{t-1}}{d}} rT_{rp_t^{a_t}}^{kn_{t-1}/r}.
\end{aligned}$$

By performing the same algebraic manipulations on the last expression one obtains

$$\Phi_k(n) = \frac{1}{n_{t-2}}\sum_{d|n_{t-2}}\mu(d)\sum_{r|\frac{n_{t-2}}{d}} rT_\beta^\alpha$$

where $\alpha = kn_{t-2}/r$ and $\beta = rp_{t-1}^{a_{t-1}}p_t^{a_t}$. Thus in a finite number of steps we obtain

$$\Phi_k(n) = \frac{1}{n_0}\sum_{d|n_0}\sum_{r|\frac{n_0}{d}} rT_{rp_1^{a_1}\cdots p_t^{a_t}}^{kn_0/r} = T_{p_1^{a_1}\cdots p_t^{a_t}}^k = T_n^k$$

which completes the proof. □

3. Proof of proposition 1.1

Proposition 1.1 is a corollary of the following result

Lemma 3.1. *Let $f(x) = f_{c,a}(z) = z^a + c$, with $a \geqslant 2$, and let Z_n^1 be as in (2.6). Then, for all complex c*

$$\exp\left(\sum_{n=1}^{\infty} \frac{Z_n^1 z^n}{n}\right) = \frac{1-az}{1-a^2 z}. \tag{3.1}$$

Proof. The polynomials $P_n(z)$ for $f_{c,a}$ satisfy the recursion relation

$$P_{n+1}(z) = f\left(f^n(z)\right) - z = P_n(z)^a + \sum_{r=1}^{a-1} \binom{a}{r} P_n(z)^{a-r} z^r + P_1(z) \quad n \geqslant 1$$

where $P_1(z) = z^a - z + c$. It can be verified that for all $a > 1$ and $n \geqslant 1$ one has

$$P_n(z) = Q_n(z) - z + (-1)^{a^n} f^n(0)$$

where

$$Q_n(z) = \sum_{j=0}^{a^{n-1}-1} (-1)^{ja} \sigma_{ja} z^{a^n - ja}; \quad \sigma_0 = 1$$

and the σ_i's are the elementary symmetric functions of the roots ξ_i of $P_n(z)$. Then

$$\begin{aligned} a^n Z_n^1 &= \sum_{P_n(\xi)=0} Q_n'(\xi) = \sum_{i=1}^{a^n} \sum_{j=0}^{a^{n-1}-1} (-1)^{ja} \left(a^n - ja\right) \sigma_{ja} \xi_i^{a^n - ja - 1} \\ &= \sum_{j=0}^{a^{n-1}-1} (-1)^{ja} (a^n - ja) \sigma_{ja} S_{a^n - ja - 1}. \end{aligned} \tag{3.2}$$

From the fact that

$$\sigma_k = \begin{cases} 0 & k \not\equiv 0 \pmod{a} \quad k \neq a^n - 1 \\ (-1)^{a^n} & k = a^n - 1 \end{cases}$$

and the first of the Newton's relations (2.8), we obtain an $a^{n-1} \times a^{n-1}$ linear system in the unknowns $S_{a^n - ja - 1}$, for $j = 0, 1, \ldots, a^{n-1} - 1$, namely

$$\begin{aligned} & S_{a-1} = 0 \\ & S_{2a-1} + (-1)^a \sigma_a S_{a-1} = 0 \\ & S_{3a-1} + (-1)^a \sigma_a S_{2a-1} + (-1)^{2a} \sigma_{2a} S_{a-1} = 0 \\ & \quad \vdots \\ & S_{a^n - a - 1} + (-1)^a \sigma_a S_{a^n - 2a - 1} + \cdots + (-1)^{a^n - 2a} \sigma_{a^n - 2a} S_{a-1} = 0 \\ & S_{a^n - 1} + (-1)^a \sigma_a S_{a^n - a - 1} + \cdots + (-1)^{a^n - a} \sigma_{a^n - a} S_{a-1} = (-1)^{a^n} (a^n - 1) \sigma_{a^n - 1}. \end{aligned}$$

This gives

$$S_{a^n - ka - 1} = \begin{cases} 0 & 1 \leqslant k \leqslant a^{n-1} - 1 \\ a^n - 1 & k = 0. \end{cases}$$

Substituting the latter into (3.2), we have $Z_n^1 = a^n - 1$, for all values of the parameter c. Using (2.2) we finally get

$$
\begin{aligned}
\zeta(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{(az)^n}{n} - \sum_{n=1}^{\infty}\frac{z^n}{n}\right)\\
&= \exp\left(\ln\left(\frac{1}{1-az}\right) - \ln\left(\frac{1}{1-z}\right)\right)\\
&= \frac{1-z}{1-az}.
\end{aligned}
$$

□

Proof of proposition 1.1. Let $f = f_{c,a}$ be as in lemma 3.1, let $\delta_n(z) = \delta_n^1(z)$ denote the nth multiplier polynomial (1.4), and let m be the number of orbits of minimal period n. All that remains to be proved is the equality of the products (1.5) and (2.7). One the one hand, we have

$$\prod_p \left(1 - w_p z^{n_p}\right) = \prod_{n=1}^{\infty}\left(\prod_{|p|=n}\left(1 - w_p z^n\right)\right).$$

On the other hand

$$
\begin{aligned}
\prod_{|p|=n}\left(1 - w_p z^n\right) &= 1 - \left(\sum_i^m w_i\right) z^n + \left(\sum_{i<j} w_i w_j\right) z^{2n} + \cdots\\
&\quad + (-1)^m \left(w_1 \cdots w_m\right) z^{mn}\\
&= z^{mn}\delta_n\left(z^{-n}\right).
\end{aligned}
$$

This equation, together with lemma 3.1 and the identity

$$\prod_p \left(1 - w_p z^{n_p}\right) = \zeta_1(z)^{-1}$$

establishes our result. □

4. Support for the conjecture

In this section we provide theoretical and experimental support for the validity of formulae (1.7).

We begin to justify the presence of the factor $\zeta_1(z) = (1-z)/(1-2z)$ in each $\zeta_k(z)$. For each n and k, the quantity T_n^k—the trace of $\delta_n^k(z)$—is an integral polynomial in c. This is because $\delta_n^k(z)$ is the minimal polynomial of a matrix whose entries are integral polynomials in c (see [1], section 4). Then Z_n^k has the same property, from equation (2.9). We write

$$Z_n^k(c) = Z_n^k(0) + \bar{Z}_n^k(c) \tag{4.1}$$

where $\bar{Z}_n^k(c)$ is also integral in c. For $c = 0$, the non-zero fixed points ξ of f_c^n are the $(2^n - 1)$-th roots of unity, with multiplier $w_n(\xi) = 1$. Then $Z_n^k(0) = 2^n - 1$, and equation (4.1) gives

$$\zeta_k(z)^{-1} = \frac{1-2z}{1-z} \exp\left(-\sum_{n=1}^{\infty} \frac{\bar{Z}_n^k z^n}{n}\right). \tag{4.2}$$

In order to deal with the exponential term in (4.2), we expand ζ_k^{-1} as follows

$$\zeta_k(z)^{-1} = \prod_{n=1}^{\infty} z^{nm(n)} \delta_n^k(1/z^n) = 1 - \sum_{r=1}^{\infty} g_r^k z^r. \tag{4.3}$$

Next we express the coefficients g_r^k in terms of the trace T_n^k and the norm N_n^k of the polynomial $\delta_n^k(z)$. The first six coefficients are given by

$$\begin{aligned}
g_1^k &= T_1^k \\
g_2^k &= T_2^k - N_1^k \\
g_3^k &= T_3^k - g_1^k T_2^k \\
g_4^k &= T_4^k - g_1^k T_3^k + T_2^k N_1^k \\
g_5^k &= T_5^k - g_1^k T_4^k - g_2^k T_3^k \\
g_6^k &= T_6^k - g_1^k T_5^k - g_2^k T_4^k + g_1^k T_2^k T_3^k - N_3^k
\end{aligned}$$

From the knowledge of the first six multiplier polynomials (see appendix 2), and making use of the Newton's relations (2.8), we calculate the first six coefficients g_r^k, for $k = 2, 3, 4$. We obtain

$$\begin{aligned}
g_1^2 &= 1 - 2c \\
g_r^2 &= 1 + 2c \quad r = 2, \ldots, 6 \\[1ex]
g_1^3 &= 1 - 3c \\
g_r^3 &= g_{r-1}^3 - 3(c+2)(-c)^{r-1} \quad r = 2, \ldots, 6 \\[1ex]
g_1^4 &= 1 - 4c + 2c^2 \\
g_2^4 &= 1 + 4c + 6c^2 + 4c^3 \\
g_r^4 &= g_{r-1}^4 - 2^r(c+1)(c+2)(-c)^{r-1} \quad r = 3, \ldots, 6
\end{aligned}$$

There are two values of c for which these recursion relations for g_r^k are true for all r (and k), namely $c = 0$, and, less trivially, $c = -2$, the Ulam mapping [12, 13]. This suggests that they may be valid for *all* values c. Assuming this, equations (4.2) and (4.3) give

$$\begin{aligned}
\zeta_2(z) &= \zeta_1(z) \frac{1}{1+2cz} \\
\zeta_3(z) &= \zeta_1(z) \frac{1+cz}{1+4cz} \\
\zeta_4(z) &= \zeta_1(z) \frac{1+2cz}{1-2c(c-3)z-8c^3z^2}
\end{aligned} \tag{4.4}$$

where $\zeta_1(z) = (1-z)/(1-2z)$. In order to cast (4.4) in the form (1.7), it is sufficient to define

$$\begin{aligned} G_1(z,c) &= 1 & G_3(z,c) &= 1+4cz \\ G_2(z,c) &= 1+2cz & G_4(z,c) &= 1-2c(c-3)z-8c^3z^2. \end{aligned}$$

Assuming (1.7), we can compute $G_k(z,c)$. We illustrate the computation in the case $k=5$. We first calculate the coefficients g_r^5 for $r=1,\ldots,6$, and equate the coefficients up to degree six in the equation

$$1-g_1^5z-\cdots-g_6^5z^6-\cdots = \frac{1-2z}{1-z}\frac{G_5(z,c)}{G_4(z/2,c)}.$$

Letting $G_5(z,c) = 1+\eta_1 z+\cdots+\eta_6 z^6$, we then solve the corresponding 6×6 linear system in the unknowns $\eta_1,\ldots,\eta_6$, to obtain

$$\begin{aligned} \eta_1 &= 8c-6c^2 \\ \eta_2 &= -32c^3 \\ \eta_r &= 0 \quad r=3,\ldots,6 \end{aligned}$$

which gives $G_5(z,c) = 1-2c(3c-4)z-32c^3z^2$. With an analogous procedure, we have obtained the polynomials $G_k(z,c)$ for $k\leqslant 12$, which are displayed in appendix 1.

Finally, we have verified the validity of (1.7) numerically, for the function $\zeta_{12}(z)$, at the parameter values $c=1$ and $c=10^{-3}(1+i)$. Specifically, the terms up to degree 12 in z of the expansion of the reciprocal of the right hand side of (1.7) were found to agree with those of the expansion (4.3), where the coefficients g_r^{12} for $r=1,\ldots,12$ were computed using the numerical values of the periodic points of period up to 12.

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Appendix 1

The polynomials $G_k(z,c)$ appearing in (1.7), for $k\leqslant 12$

$$\begin{aligned} G_1(z,c) &= 1 \\ G_2(z,c) &= 1+2cz \\ G_3(z,c) &= 1+4cz \end{aligned}$$

$$
\begin{aligned}
G_4(z,c) &= 1 - 2c(c-3)z - 8c^3z^2\\
G_5(z,c) &= 1 - 2c(3c-4)z - 32c^3z^2\\
G_6(z,c) &= 1 + 2c(c^2-6c+5)z - 4c^3(3c^2-5c+20)z^2 - 64c^6z^3\\
G_7(z,c) &= 1 + 4c(2c^2-5c+3)z - 16c^3(5c^2-6c+10)z^2 - 512c^6z^3\\
G_8(z,c) &= 1 - 2c(c^3-10c^2+15c-7)z - 8c^3(2c^4-7c^3+41c^2-35c+35)z^2\\
&\quad + 32c^6(5c^3-7c^2+16c-70)z^3 + 1024c^{10}z^4\\
G_9(z,c) &= 1 - 2c(5c^3-20c^2+21c-8)z - 8c^3(20c^4-49c^3+125c^2-80c+56)z^2\\
&\quad + 64c^6(35c^3-40c^2+64c-112)z^3 + 16384c^{10}z^4\\
G_{10}(z,c) &= 1 + 2c(c^4-15c^3+35c^2-28c+9)z - 4c^3(5c^6-30c^5+238c^4\\
&\quad - 401c^3+625c^2-315c+168)z^2 - 16c^6(20c^6-61c^5+215c^4\\
&\quad - 1115c^3+1029c^2-1152c+1176)z^3 + 128c^{10}(35c^4-45c^3\\
&\quad + 88c^2-224c+1008)z^4 + 32768c^{15}z^5\\
G_{11}(z,c) &= 1 + 2c(6c^4-35c^3+56c^2-36c+10)z - 8c^3(35c^6-144c^5+500c^4\\
&\quad - 618c^3+679c^2-280c+120)z^2 - 32c^6(224c^6-549c^5+1363c^4\\
&\quad - 3072c^3+2320c^2-1920c+1344)z^3 + 2048c^{10}(63c^4-70c^3\\
&\quad + 110c^2-192c+336)z^4 + 1048576c^{15}z^5\\
G_{12}(z,c) &= 1 - 2c(c^5-21c^4+70c^3-84c^2+45c-11)z - 4c^3(6c^8-55c^7\\
&\quad + 573c^6-1557c^5+3308c^4-3171c^3+2660c^2-924c+330)z^2\\
&\quad + 16c^6(35c^9-186c^8+919c^7-5877c^6+10837c^5-18744c^4\\
&\quad + 26121c^3-16401c^2+10560c-5544)z^3 + 64c^{10}(224c^8\\
&\quad - 645c^7+1961c^6-6399c^5+31267c^4-29952c^3+36592c^2\\
&\quad - 44352c+44352)z^4 - 4096c^{15}(63c^5-77c^4+140c^3-306c^2\\
&\quad + 816c-3696)z^5 - 2097152c^{21}z^6.
\end{aligned}
$$

Appendix 2

The first six multipliers polynomials for the quadratic family [1, 12]

$$
\begin{aligned}
\delta_1(x) &= x^2 - x + c.\\
\delta_2(x) &= x - (c+1).
\end{aligned}
$$

$$\delta_3(x) = x^2 - (c+2)x + (c^3 + 2c^2 + c + 1).$$

$$\delta_4(x) = x^3 + (c^2 - 3)x^2 + (-c^4 - c^3 + c^2 + 3)x - \\ - (c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1)$$

$$\delta_5(x) = x^6 + k_5x^5 + k_4x^4 + k_3x^3 + k_2x^2 + k_1x + k_0.$$

$$k_5 = c^2 - c - 6$$

$$k_4 = 3c^5 + 3c^4 - 6c^3 - 2c^2 + 5c + 15$$

$$k_3 = 2c^7 + 9c^6 + 17c^5 + 21c^4 + 13c^3 - 2c^2 - 10c - 20$$

$$k_2 = 3c^{10} + 11c^9 + 6c^8 - 20c^7 - 42c^6 - 53c^5 - 37c^4 - 3c^3 + 8c^2 + 10c + 15$$

$$k_1 = c^{12} + 7c^{11} + 20c^{10} + 33c^9 + 40c^8 + 37c^7 + 21c^6 + 7c^5 - c^4 - 9c^3 - 7c^2 \\ - 5c - 6$$

$$k_0 = c^{15} + 8c^{14} + 28c^{13} + 60c^{12} + 94c^{11} + 116c^{10} \\ + 114c^9 + 94c^8 + 69c^7 + 44c^6 + 26c^5 + 14c^4 + 5c^3 + 2c^2 + c + 1$$

$$\delta_6(x) = x^9 + k_8x^8 + k_7x^7 + k_6x^6 + k_5x^5 + k_4x^4 + k_3x^3 + k_2x^2 + k_1x + k_0$$

$$k_8 = -c^3 + c^2 + c - 9$$

$$k_7 = -4c^6 - 2c^5 + 8c^4 - 4c^3 - 6c^2 - 8c + 36$$

$$k_6 = 4c^9 + 2c^8 - 34c^7 - 63c^6 - 54c^5 - 6c^4 + 42c^3 + 14c^2 + 28c - 84$$

$$k_5 = 6c^{12} + 14c^{11} - 11c^{10} - 8c^9 + 101c^8 + 192c^7 + 231c^6 \\ + 142c^5 - 83c^4 - 112c^3 - 14c^2 - 56c + 126$$

$$k_4 = -6c^{15} - 20c^{14} + 15c^{13} + 124c^{12} + 189c^{11} + 186c^{10} \\ + 49c^9 - 212c^8 - 208c^7 - 109c^6 - 5c^5 + 225c^4 + 140c^3 + 70c - 126$$

$$k_3 = -4c^{18} - 22c^{17} - 22c^{16} + 112c^{15} + 441c^{14} + 853c^{13} \\ + 1157c^{12} + 1143c^{11} + 774c^{10} + 322c^9 - 39c^8 - 308c^7 - 334c^6 \\ - 250c^5 - 230c^4 - 84c^3 + 14c^2 - 56c + 84$$

$$
\begin{aligned}
k_2 = & 4c^{21} + 30c^{20} + 80c^{19} + 57c^{18} - 195c^{17} - 734c^{16} - 1441c^{15} \\
& - 1941c^{14} - 1881c^{13} - 1344c^{12} - 570c^{11} + 126c^{10} + 467c^{9} + 593c^{8} \\
& + 566c^{7} + 375c^{6} + 212c^{5} + 92c^{4} + 14c^{3} - 14c^{2} + 28c - 36 \\
k_1 = & c^{24} + 10c^{23} + 41c^{22} + 84c^{21} + 60c^{20} - 155c^{19} \\
& - 642c^{18} - 1347c^{17} - 2036c^{16} - 2455c^{15} - 2522c^{14} \\
& - 2272c^{13} - 1842c^{12} - 1385c^{11} - 923c^{10} - 536c^{9} - 290c^{8} \\
& - 132c^{7} - 61c^{6} - 26c^{5} + c^{4} + 8c^{3} + 6c^{2} - 8c + 9 \\
k_0 = & -(c^{27} + 13c^{26} + 78c^{25} + 293c^{24} + 792c^{23} + 1672c^{22} \\
& + 2892c^{21} + 4219c^{20} + 5313c^{19} + 5892c^{18} + 5843c^{17} \\
& + 5258c^{16} + 4346^{15} + 3310c^{14} + 2331c^{13} + 1525c^{12} \\
& + 927c^{11} + 536c^{10} + 298c^{9} + 155c^{8} + 76c^{7} + 35c^{6} + 17c^{5} \\
& + 7c^{4} + 3c^{3} + c^{2} - c + 1).
\end{aligned}
$$

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