A random dynamical system model of a stylized equity market

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Abstract

We propose a random dynamical systems model for a stylized equity market. The model generalises previous deterministic models for price formation in equity markets. We provide analytic results (existence of fixed points, existence of invariant measures) as well as numerical results indicating the dynamical richness of this simple model. The model can be used to assess the effects of uncertainty on the fundamentals on stock price dynamics.

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1 Introduction

The rôle of agents on the formation of prices in the stock market is a subject that has fascinated economists for a long time. There is a vast related literature on the subject to which thinkers of the caliber of John Maynard Keynes [9] or Fischer Black [2] have contributed. In recent years models for the formation of prices using the theory of complex dynamical systems have been proposed. Richard Day [3], [4], [5] has proposed a dynamical systems model for an equity market. Eventhough the model was very simple (an one dimensional map) it was designed in a very clever way so as to reflect well established stylized facts for the stock market structure. As a result the model could reproduce very well qualitative features of stock market prices such as transition between bull and bear regimes etc. The model was purely deterministic.

The basic features of the model were the following. Each stock was supposed to somehow reflect the value of the fundamental it was representing. The stock market was assumed to consist of two types of investors, α-investors and β-investors and a mediator through which transactions were made. Each type of agent followed a different policy towards investment decisions. Type α-investors were those using sophisticated estimates of long run investment value. Such investors are those trying to buy when prices are below investment value (i.e. when chances of capital gain seem to be high) and trying to sell in the opposite case. Type β-investors cannot afford to be as sophisticated as α-investors. Such investors make investments decisions based on current investment value. In some sense β-investors are the noise traders (to follow the terminology of Fisher Black) entering the market when prices are are high and exiting the market in the opposite case. The third type of agent is the market maker (or mediator) who sets the price in response to excess demand or supply. The prices are supposed to be formed by Walrasian tattonement.

This simple dynamical model was found to present complicated behaviour giving rise to irregular oscillations of the prices (chaos). Furthermore, despite its simplicity it reproduced well qualitative features of the prices, and could be used to understand the relative importance of the various types of investors in the market. As a result it was greeted with great interest by the community (see the discussion following [5]).

The model of Day was a purely deterministic model. Eventhough there have been followups of Day's research (see e.g. [7] and references therein) all of them, at least to the best of our knowledge, have kept the deterministic nature of Day's original model. It is the aim of the present model to include randomness in Day's model. The randomness is introduced through the current investment value which is used by the β-investors in their invesment decisions. The model can then be formulated as a random dynamical system. Through the use of recent advances of the theory of random dynamical systems a qualitative and quantitative study of the evolution of prices in the stock market as a result of the interaction of different types of agents is possible. We see that the salient features and findings of the Day model (such as the existence of fixed points or the existence of invariant measures) are robust in the presence of random effects. Furthermore, the existence of unstable dynamics in the stock market, leading possibly to erratic oscillations, is shown through the calculation of Lyapunov exponents. The model allows us to monitor the effect of uncertainty of the fundamental to the stock prices and assess the effect of the underlying market dynamics on the propagation of uncertainty from the level of the fundamental to the level of the stock. The model, despite its simplicity contributes to the interesting and important field of market microstructure; an active area of financial mathematics the study of which is necessary for the understanding of the structure and function of financial markets.

The paper is organized as follows: In section 2 we present the general features of our model and provide two special forms, a piecewise linear form and a piecewise monotone form. In section 3 we present several qualitative and analytic results on the models, that is the existence of fixed points, the existence and approximation of invariant measures and a method for the calculation of Liapunov exponents, based on the existence of the invariant measure. In section 4 we present our numerical results on the proposed models, which illustrate the dynamical richness of the random dynamical system governing the market dynamics. Finally in section 5 we summarize our findings and propose topics for further research.

2 The model

2.1 General features of the model

The model is based upon the model of Day and coworkers of a stylized equity market. The model uses the following two basic concepts for the value of a share, the current fundamental value v and the investment value u. The current fundamental value is based on the most recent fundamental data (earnings, dividents etc) while the investment value is an estimate of the expected future value of v based on comprehensive information and analysis of long run considerations.

Following day we assume that the market consists of two investor types using distinct strategies. The first kind of investors (called hereafter α-investors) base their decisions on the investment value u, they buy when stock prices are below investment value and sell when stock prices are above their estimated investment value. Their strategy can be described mathematically using an excess demand function $\alpha(p;u) = \alpha(p)$ with the properties

$$a^{'}(p) < 0, \quad a(u) = 0.$$

The second kind of investors (called hereafter $\beta-$investors) are those whose strategy is based on the fundamental value. Their strategy depends on the spread $p - v$ which can be taken as a signal of future price. β-investors buy in a rising market and sell in a falling market. Their strategy can be described mathematically using an excess demand function $\beta(p;v) = \beta p$ with the following properties

$$\beta^{'}(p) > 0, \quad \beta(v) = 0.$$

In a market the β type of investors correspond to the vast majority of 'un-informed' investors, while α investors correspond to well organized, well-informed investors.

The price of the stocks is obtained through a Walrasian tattônement procedure. We assume the existence of mediator whose rôle is to supply α and β investors with their demands. The price of the stocks at time $t+1$, denoted by p_{t+1}, will be related to the price at the previous time t, denoted by p_t, by the following recursive relation

$$p_{t+1} = p_t + \lambda[\alpha(p_t;u) + \beta(p_t;v)], \tag{1}$$

where λ is a factor related to the behaviour of mediators and provides a measure of how fast the market reaches equilibrium. Equation (1) defines a discrete dynamical system the evolution of which describes the (long-time) behaviour of stock prices. Day and coworkers have studied the behaviour of this dynamical system for the case $u = v$ and have shown that even when the system is completely deterministic the system may display irregular behaviour (chaos). Using the techniques of chaotic dynamics certain typical stochastic patterns concerning stock prices have been reproduced.

In this paper we wish to extend model (1) to include the effects of randomness. We argue that randomness should be included in the model in a very natural way: The β investors base their strategy on the current fundamental value v. The current fundamental value of a stock is subject to uncertainty arising form a multitude of sources (misinformation etc). Thus in general $v = v_t$ some stochastic process. On the contrary α-investors base their strategy on anticipated investment value u. This may be obtained by the application of some expectation operator (or conditional expectation operator) on the stochastic process v_t. Therefore, the process u will be a better behaved process, which up to a point may be considered to even be some deterministic trend. Other sources of randomness may be randomness in the mediator's behaviour etc. For the sake of simplicity here (and without significant loss of generality) we will only assume that randomness is introduced in the dynamical system for the price evolution through the current fundamental value v. All other parameters will be considered to be constant.

Based on the above assumptions we will reformulate the dynamical system

$$p_{t+1} = p_t + \lambda(\alpha(p_t, u) + \beta(p_t, v_t)), \tag{2}$$

as a random dynamical system. The random dynamical system consists of two parts, a model for the noise v_t and a model for the dynamics. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let us by θ denote a dynamical system on the probability space corresponding to the random perturbation. We will now rewrite the stochastic process v_t as $v_t = v(\theta^t\omega)$ and reformulate the dynamical system (2) as follows

$$p_{t+1} = p_t + \lambda(\alpha(p_t; u) + \beta(p_t; v(\theta^t\omega)) := h(\theta^t\omega, p_t). \tag{3}$$

This is a random difference equation on some space $X \subset \mathbb{R}$ where the map $h(\omega, \cdot) := h(\omega) : X \to X$ is assumed to be measurable and measurable invertible. We now use the basic formulation of stochastic difference equations as random dynamical systems provided by Arnold and coworkers [1]. We may now define the state of the random dynamical system at time t, having started at $x_0 = x$ under the perturbation determined by ω by

$$\phi(t, \omega, x) = \begin{cases} h(\theta^{t-1}\omega) \circ \ldots \circ h(\omega)x & t \geq 1 \\ x & t = 0 \\ h(\theta^t\omega)^{-1} \circ \ldots \circ h(\omega)^{-1}x & t \leq -1 \end{cases}.$$

The above defines a two-sided dynamical system in time. That requires the invertibility of the map $h(\omega) : X \to X$. If the map is not invertible (as may be the case here) we will only define the map for $t \geq 0$.

The family of maps $\phi(t, \omega, x)$ is a random dynamical system in the sense that $\phi : \mathbb{T} \times \Omega \times X \to X$ is a mapping with the properties

$$\phi(0, \omega) = Id_X, \quad \phi(s + t, \omega) = \phi(t, \theta^s\omega) \circ \phi(s, \omega),$$

for all $s, t \in \mathbb{T}$ and $\omega \in \Omega$. This property is called the cocycle property of the model.

At this point we will introduce two different specific versions of the model.

2.2 A piecewise linear model

Following Day and coworkers [4], [5], consider $p^B < \underline{p} < u < \overline{p} < p^T$ and let the α-strategy be continuous and skew symmetric about the investment value u

$$\alpha(p) = A\mathcal{X}_{(-\infty, p^B]}(p) + a(\underline{p} - p)\mathcal{X}_{[p^B, \underline{p}]}(p) - a(p - \overline{p})\mathcal{X}_{[\overline{p}, p^T]}(p) - A\mathcal{X}_{(p^T, \infty]}(p), \tag{4}$$

where p^B and p^T are the estimations, of the α-investors, for the bottoming and topping price of the stock respectively. For prices below the bottoming price or above the topping price buys and sells are constant and equal to A. In the interval $[\underline{p}, \overline{p}]$ the α-investors hold their positions, in the interval $[p^B, \underline{p}]$ they buy, and they sell in the interval $[\overline{p}, p^T]$. Due to continuity at $p = p^B$ and $p = p^T$ we have $\underline{p} + \overline{p} = p^B + p^T$ and $A = a(\underline{p} - p^B)$. Assuming the model $\beta(p) = b(p - v)$ for the β-strategy and letting $u = v$, $p^B = 0$ and $p^T = 1$ one obtains

$$\Theta(p) = p + \lambda(\alpha(p) + \beta(p)) = \begin{cases} p(1 + \lambda b) + \lambda(a\underline{p} - bv) & p \leq 0 \\ -p(\lambda(a - b) - 1) + \lambda(a\underline{p} - bv) & 0 \leq p \leq \underline{p} \\ p(1 + \lambda b) - \lambda bv & \underline{p} \leq p \leq \overline{p} \\ -p(\lambda(a - b) - 1) + \lambda(a\overline{p} - bv) & \overline{p} \leq p \leq 1 \\ p(1 + \lambda b) - \lambda(a\underline{p} + bv) & p \geq 1 \end{cases}, \tag{5}$$

where all parameters a, b and λ are positive. It can be readily verified that the latter transformation is expansive in $X = [0,1]$ provided that

$$\lambda > \lambda_1 = \frac{2}{a-b}, \quad a > b. \tag{6}$$

Letting p^m and p^M to be the minimum and maximum of Θ in X, one has $p^m = \Theta(\underline{p})$ and $p^M = \Theta(\overline{p})$. Moreover the set $Z = [p^m, p^M]$ is a trapping set (that is $\Theta\,(X) \subseteq Z$) whenever

$$\Theta(p^m) > p^m, \quad \Theta(p^M) < p^M. \tag{7}$$

By direct substitution it can be verified that both inequalities in relation (7) are satisfied when $a - b > 1/\lambda$. The latter inequality holds trivially when the system is expansive due to relation (6). Following (Day) we decompose $Z = Z^b \cup \{v\} \cup Z^B$ to bear and bull zones $Z^b = [p^m, v)$ and $Z^B = (v, p^M]$ respectively. Asking $\Theta(Z^b) \cap Z^B \neq \emptyset$ and $\Theta(Z^B) \cap Z^b \neq \emptyset$ one obtains the condition

$$\Theta(p^m) > v > \Theta(p^M)\,, \tag{8}$$

which guaranties that the bear and the bull zones cannot themselves be attracting. After some algebraic manipulations it can be verified that the switching condition (8) is satisfied for

$$\lambda > \lambda_2 = \frac{b + \sqrt{ab}}{b(a-b)}. \tag{9}$$

The latter condition implies relation (6) because $a > b$ gives $\lambda_2 > \lambda_1$. We define the random map $h(\omega) : X \to X$

$$\phi(1, \omega, p) = h(\omega, p) = \Theta(p) + \Theta(\underline{p})\,\eta(\omega)\,\sigma = p + \lambda(\alpha(p) + \beta(\omega, p)), \tag{10}$$

where $X = [0,1]$, $0 \leq \sigma \leq 1$ and

$$\beta(\omega, p) = b\left\{p - \left(v - \frac{\sigma\Theta(\underline{p})}{\lambda\, b}\,\eta(\omega)\right)\right\}, \tag{11}$$

with $\mathrm{prob}\{\eta = 1\} = r$ and $\mathrm{prob}\{\eta = -1\} = 1 - r$. Due to the discrete nature of the stochastic perturbation and denoting by Θ^u and Θ^d the expression in relation (10), for η equal to 1 and -1 respectively, the random map attains the iterated function system (IFS) representation

$$T = \left\{\Theta^u,\, \Theta^d;\; r,\, 1 - r\right\}. \tag{12}$$

Because $\underline{p} + \overline{p} = 1$ we have $\Theta(\underline{p}) + \Theta(\overline{p}) = 1$ exactly for $v = 1/2$. In fact for $v = 1/2$ and $\underline{p} = 1/4$ we have $\Theta^d(\underline{p}) \geq 0$ and $\Theta^u(\overline{p}) \leq 1$ whenever

$$\lambda b \ \leq 1. \tag{13}$$

Then the invariant set of the IFS in relation (12) is a subset of X. For the special case of $\sigma = 1$ we have $\Theta^d(\underline{p}) = 0$ and $\Theta^u(\overline{p}) = 1$ and consequently $h(\omega)(X) = X$. The union of the trapping sets

$$Z^d = [\Theta^d(\underline{p}), \Theta^d(\overline{p})], \quad Z^u = [\Theta^u(\underline{p}), \Theta^u(\overline{p})] \tag{14}$$

for the Θ^d and Θ^u maps respectively, become the whole phase space X.

Our aim is to find the conditions on the parameters a, b and λ for which both maps Θ^d and Θ^u are ergodic over some subsets of X for all values of σ in $[0,1]$. Let us denote the minima of the maps Θ^d and Θ^u over X with $p^{m,d}$, $p^{m,u}$ respectively. We denote with $p^{M,d}$, $p^{M,u}$ the associated maxima. Then it can be verified that the following two identities hold

$$\Theta^u\left(p^{m,u}\right) - p^{m,u} = \frac{b}{4}(a-b)(\sigma+1)\left(\lambda - \frac{1}{a-b}\right)\left(\lambda - \frac{\sigma}{b(\sigma+1)}\right), \quad 0 \leq \sigma \leq 1, \tag{15}$$

$$\Theta^u\left(p^{M,u}\right) - p^{M,u} = \begin{cases} \frac{b}{4}(a-b)(\sigma-1)\left(\lambda - \frac{1}{a-b}\right)\left(\lambda - \frac{\sigma}{b(\sigma-1)}\right), & 0 \leq \sigma < 1 \\ \frac{1}{4} - \frac{1}{4}\lambda(a-b)\,, & \sigma = 1 \end{cases} \tag{16}$$

In analogy with relation (7) we impose the condition

$$\Theta^u\left(p^{m,u}\right) > p^{m,u}, \quad \Theta^u\left(p^{M,u}\right) < p^{M,u}. \tag{17}$$

Then it can be seen that for

$$\lambda > \lambda_1' = \max_{0\le\sigma\le 1}\left\{\frac{1}{a-b},\ \frac{\sigma}{b(\sigma+1)}\right\} = \max\left\{\frac{1}{a-b},\ \frac{1}{2b}\right\}, \tag{18}$$

the trapping conditions for Θ^u are satisfied. The following relations together with relations (19) and (16) assure us that the same condition (18) applies for the map Θ^d

$$\begin{aligned}\Theta^u\left(p^{m,u}\right) + \Theta^d\left(p^{M,d}\right) &= p^{m,u} + p^{M,d} = 1\\ \Theta^u\left(p^{M,u}\right) + \Theta^d\left(p^{m,d}\right) &= p^{m,d} + p^{M,u} = 1.\end{aligned} \tag{19}$$

Let us denote with p^d and p^u the fixed points of the maps Θ^d and Θ^u in $(\underline{p},\overline{p})$ which clearly they satisfy $p^d \to 1/2$ and $p^u \to 1/2$ when $\sigma \to 0$. Then the following pair of identities hold

$$\Theta^u\left(p^{m,u}\right) - p^u = \frac{b}{4}(a-b)(\sigma+1)\left(\lambda - \frac{\sigma}{b(\sigma+1)}\right)\left(\lambda - \frac{b+\sqrt{ab}}{b(a-b)}\right)\left(\lambda - \frac{b-\sqrt{ab}}{b(a-b)}\right),\ 0\le\sigma\le 1, \tag{20}$$

$$\Theta^u\left(p^{M,u}\right) - p^u = \begin{cases} \frac{b}{4}(a-b)(\sigma-1)\left(\lambda - \frac{\sigma}{b(\sigma-1)}\right)\left(\lambda - \frac{b+\sqrt{ab}}{b(a-b)}\right)\left(\lambda - \frac{b-\sqrt{ab}}{b(a-b)}\right), & 0\le\sigma<1\\ -\frac{a-b}{4}\left(\lambda - \frac{b+\sqrt{ab}}{b(a-b)}\right)\left(\lambda - \frac{b-\sqrt{ab}}{b(a-b)}\right), & \sigma = 1\end{cases} \tag{21}$$

In analogy with relation (8) we impose the condition

$$\Theta^u\left(p^{m,u}\right) > p^u > \Theta^u\left(p^{M,u}\right) \tag{22}$$

then it can be seen that the switching inequalities for the map Θ^u are satisfied for

$$\lambda > \lambda_2' = \max_{0\le\sigma\le 1}\left\{\frac{b+\sqrt{ab}}{b(a-b)},\ \frac{\sigma}{b(\sigma+1)}\right\} = \max\left\{\frac{b+\sqrt{ab}}{b(a-b)},\ \frac{1}{2b}\right\}. \tag{23}$$

Again thanks to symmetry

$$\Theta^u\left(p^{M,u}\right) + \Theta^d\left(p^{m,d}\right) = \Theta^u\left(p^{M,u}\right) + \Theta^d\left(p^{m,d}\right) = p^u + p^d, \tag{24}$$

condition (22) applies for the map Θ^d.

We observe that for $\lambda > \lambda_2'$

$$\lambda > \lambda_2' \ge \max\left\{\frac{2}{a-b},\ \frac{1}{2b}\right\} \ge \lambda_1', \tag{25}$$

meaning that condition (22) implies, in addition to switching, the trapping and expansitivity conditions for both Θ^u and Θ^d maps and for all σ in $[0,1]$. Thus we have shown the following proposition

Proposition 2.1 *The transformations $\Theta^u : Z^u \to Z^u$ and $\Theta^d : Z^d \to Z^d$ in relation (12) for $v = 1/2$ and $\underline{p} = 1/4$, are ergodic for all $a > b > 0$ and σ in $[0,1]$ for*

$$\max\left\{\frac{b+\sqrt{ab}}{b(a-b)},\ \frac{1}{2b}\right\} < \lambda \le \frac{1}{b} \tag{26}$$

Lemma 2.1 *Consider the random map $h(\omega) : X \to X$*

$$h(\omega, p) = \Theta_\lambda(p) + \sigma(1/4 - \lambda/16)\eta(\omega), \quad 0 \le \sigma \le 1, \tag{27}$$

or its equivalent representation $T_\lambda = \{\Theta_\lambda^u, \Theta_\lambda^d;\ r, 1-r\}$ with

$$\Theta_\lambda(p) = \begin{cases} (1-\lambda)p + 3\lambda/16 & 0 \le p \le 1/4\\ (1+\lambda/4)p - \lambda/8 & 1/4 \le p \le 3/4\\ (1-\lambda)p + 13\lambda/16 & 3/4 \le p \le 1\end{cases}. \tag{28}$$

The member maps $\Theta_\lambda^u : Z^u \to Z^u$ and $\Theta_\lambda^d : Z^d \to Z^d$ are ergodic for $\lambda^* = 1+\sqrt{5} < \lambda \le 4$ and for all σ in $[0,1]$.

Proof: The inequality (26) for λ in the previous proposition is satisfied for $a = 5/4$ and $b = 1/4$. $\square$

In figures 1(a)-(d), we plot for $\sigma = 1$ the member maps Θ_λ^u and Θ_λ^d of the IFS in the previous lemma for $\lambda = 1.5,\ 2.0,\ 3.0,\ 3.5$. In all cases we have $\Theta_\lambda^u(3/4) = 1$ and $\Theta_\lambda^d(1/4) = 0$ and as a consequence the interval $[0,1]$ is left invariant under the action of the random map $h(\omega)$.

2.3 A piecewise monotone model

We choose after Day and coworkers [4], [5], the α-strategy to be the function

$$\alpha(p) = A\mathcal{X}_{(-\infty,p^B]}(p) + a(u-p)f(p)\mathcal{X}_{[p^B,p^T]}(p) - A\mathcal{X}_{[p^T,\infty)}(p). \tag{29}$$

Where

$$f(p) = \frac{1}{\sqrt{(p-p^B+\epsilon)(p^T+\epsilon-p)}}, \tag{30}$$

is a function representing the choice of lost opportunity either to fail to buy when the market is low or fail to sell when the market is high. Then for positive a, b, and λ and for

$$u = \frac{p^B+p^T}{2},\quad A = \frac{a(p^T-p^B)}{\sqrt{\epsilon(\epsilon+p^T-p^B)}},\quad \epsilon = 0.01, \tag{31}$$

it can be verified that the function in relation (29) is continuous and skew symmetric about the investment value u. Setting the β-strategy to $\beta(p) = b(p-v)$ and letting $u = v$ one obtains for $p^B = 0$ and $p^T = 1$

$$\Theta(p) = p + \lambda(\alpha(p)+\beta(p)) = \begin{cases} (1+\lambda b)p + \lambda(A - bv) & p \le 0 \\ p + \lambda(a(v-p)\,f(p) + b(p-v)) & 0 \le p \le 1 \\ (1+\lambda b)p - \lambda(A+bv) & p \ge 1 \end{cases} \tag{32}$$

It can be seen that the latter transformation for

$$c := \frac{b(1+2\epsilon)}{2a} > 1, \tag{33}$$

has three fixed points in $X = [0,1]$ namely $p^* = v = 1/2$, and, p^- and p^+ which are given by

$$p^{\pm} = \frac{1}{2} \pm \frac{a}{b}\sqrt{c^2-1}. \tag{34}$$

Further the latter three fixed points are unstable, $\Theta'(v) > 1$ and $\Theta'(p^{\pm}) < -1$, exactly when

$$\lambda > \frac{2}{b(c^2-1)}. \tag{35}$$

Let us denote by $p^m = \Theta(\underline{p})$ and $p^M = \Theta(\overline{p})$ the maximum and minimum value of Θ in the intervals $Z^b = [p^m, v)$ and $Z^B = (v, p^M]$; the bear and the bull zones respectively. Then according to our previous discusion the set $Z = [p^m, p^M]$ is a trapping set for the dynamics whenever $\Theta(p^m) > p^m$ and $\Theta(p^M) < p^M$. If in addition a condition of the form $\Theta(p^m) > v > \Theta(p^M)$ is satisfied, the bear and the bull zones cannot themselves be attracting. Nevertheless it is difficult to analytically identify the regions in the parameter space for which the map in relation (32) becomes totally expansive over $X = [0,1]$. The situation becomes even less analytically tractable when one considers the random system $h(\omega) : X \to X$

$$\phi(1,\omega,p) = h(\omega,p) = p + \lambda(\alpha(p) + \beta(\omega,p)) = \Theta(p) + \lambda\, b\, \sigma\, \eta(\omega),\quad 0 \le \sigma \le \sigma^*, \tag{36}$$

with $\beta(\omega,p)) = b(p - v(\omega))$, $v(\omega) = 1/2 - \sigma\eta(\omega)$, $\text{prob}\{\eta = 1\} = r$ and $\text{prob}\{\eta = -1\} = 1 - r$. With the latter map we consider additive iid perturbations of magnitude σ around the fundamental value $v = 1/2$. We would like to restrict the dynamics of the random map in relation (36) to the interval $X = [0,1]$. Having this in mind we impose the restriction $\Theta^u(0) = 1$, equivalently $\Theta^d(1) = 0$, thus obtaining λ as a function of a and b

$$\lambda = \frac{2}{a\,f(0) - b\,(1 - 2\,\sigma)}. \tag{37}$$

We then let σ^*, to be the maximum value of σ for which, for fixed a and b, one has $h(\omega, X) \subseteq X$. Using the same notation as in the piecewise linear case we denote by Θ^u and Θ^d the corresponding deterministic maps in relation (36), for η equal to $+1$ and -1 respectively. We calculate σ^* by asking for $\sigma = \sigma^*$

$$(\Theta^u)'(\overline{p}) = 0, \quad \Theta^u(\overline{p}) = 1, \tag{38}$$

equivalently, $\left(\Theta^d\right)'(\underline{p}) = 0$, $\Theta^d(\underline{p}) = 0$. Then the dynamics of the random map $h(\omega)$ are entirely confined to X for all values of σ in the interval $[0, \sigma^*]$.

3 Analysis of the model

3.1 Fixed points of the model

One important property of the model is the existence of fixed points (equilibrium solutions). The fixed point in a random dynamical system is defined differently than in the case of deterministic dynamical systems. We have the following definition:

Definition 3.1 *A random fixed point of a random dynamical system ϕ on X is a random variable $x^* : \Omega \to X$ such that*

$$x^*(\theta\omega) = \phi(1, \omega, x^*(\omega)) := h(\omega, x^*(\omega))$$

A random fixed point is a stationary solution of the random difference equation in the sense that

$$x^*(\theta^{t+1}\omega) = h(\theta^t\omega, x^*(\theta^t\omega))$$

We now show the existence of unstable random fixed points for the piecewise linear system. We will only treat the case where the piecewise linear system has three branches which lead to three unstable fixed points. Other cases (e.g. the case of existence of a single stable fixed point) can be treated similarly.

Proposition 3.1 *Consider the piecewise linear map*

$$p_{t+1}(\omega) = h(\theta^t\omega)p_t(\omega) = \begin{cases} a_1(\theta^t\omega)p_t + b_1(\theta^t\omega) & \underline{p} \le p \le p_1 \\ a_2(\theta^t\omega)p_t + b_2(\theta^t\omega) & p_1 \le p \le p_2 \\ a_3(\theta^t\omega)p_t + b_3(\theta^t\omega) & p_2 \le p \le \bar{p} \end{cases},$$

where $a_{i,d} \le a_i(\theta^t\omega) \le a_{i,u}$, $b_{i,d} \le b_i(\theta^t\omega) \le b_{i,u}$, $i = 1, 2, 3$ and $a_{i,u} < -1$, $i = 1, 3$ and $1 < a_{3,d}$. Then this piecewise linear dynamical system has three unstable random fixed points, each situated at the intervals $I_1 = [\underline{p}, p_1]$, $I_2 = [p_1, p_2]$,$I_3 = [p_2, \bar{p}]$.

Proof: We will work with the inverse map at each of the intervals I_i. For instance let us take the middle branch of the map. The inverse map for $p \in I_2$ is of the form

$$p_t = \frac{1}{a_2(\theta^t\omega)}p_{t+1} - \frac{b_2(\theta^t\omega)}{a_2(\theta^t\omega)} \tag{39}$$

We now need to find an invariant (random) interval for the inverse map. Let us call this interval $I_2' = (p_d^{(2)}, p_u^{(2)})$ where the ends of the interval are as yet unspecified. To check the invariance of the interval under the inverse map we will have to show that this interval is mapped into itself. Indeed, let us denote by $p_u^{(2)'}$ and $p_d^{(2)'}$ the images of $p_u^{(2)}$ and $p_d^{(2)}$ under the inverse map respectively. We have to ensure that the following ordering holds

$$p_d^{(2)} \le p_d^{(2)'} \le p_u^{(2)'} \le p_u^{(2)}.$$

This will hold as long as the following two conditions are valid

$$\begin{gathered} a_2(\theta^t\omega)p_d^{(2)} \le p_d^{(2)} - b_1(\theta^t\omega) \\ p_u^{(2)} - b_2(\theta^t\omega) \le a_2(\theta^t\omega)p_u^{(2)} \end{gathered}$$

These conditions are equivalent to

$$p_d^{(2)} \leq \frac{b_2(\theta^t\omega)}{1-a_2(\theta^t\omega)}$$
$$p_u^{(2)} \geq \frac{b_2(\theta^t\omega)}{1-a_2(\theta^t\omega)}$$

Thus it is enough to choose

$$p_d^{(2)} = \min_{t,\omega} \frac{b_2(\theta^t\omega)}{1-a_2(\theta^t\omega)} = \frac{b_{2,d}}{1-a_{2,u}}$$
$$p_u^{(2)} = \max_{t,\omega} \frac{b_2(\theta^t\omega)}{1-a_2(\theta^t\omega)} = \frac{b_{2,u}}{1-a_{2,d}}$$

We will now look for a stochastic fixed point of the inverse map in this interval. The inverse map is an affine map of the form

$$p_{t+1} = \alpha(\theta^t\omega)p_t + \beta(\theta^t\omega)$$

where we have reversed time $(t+1 \to t-1)$ and

$$\alpha(\theta^t\omega) = \frac{1}{a_2(\theta^t\omega)}, \quad \beta(\theta^t\omega) = -\frac{b_2(\theta^t\omega)}{a_2(\theta^t\omega)}$$

This map is now a contraction since $\mid \alpha(\theta^t\omega) \mid < 1$. The candidate for a random fixed point of this map is the random variable

$$p^*(\omega) = \beta(\theta^{-1}\omega) + \sum_{i=1}^{\infty} \beta(\theta^{-(i+1)}\omega) \prod_{j=1}^{i} \alpha(\theta^{-j}\omega) \tag{40}$$

We will assume that

$$E[\log(\alpha)] < \infty, \quad E[\beta] < \infty$$

so that the random fixed point is well defined. We can easily check that $p^*(\omega)$ is the fixed point for the random dynamical system. One may also easily check from the above series that $p^*(\omega)$ lies in the invariant interval. Indeed we may see that since $\underline{\alpha} \leq \alpha(\theta^t\omega) \leq \bar{\alpha}$ and $\underline{\beta} \leq \beta(\theta^t\omega) \leq \bar{\beta}$ the following bounds are valid for $p^*(\omega)$:

$$\frac{\underline{\beta}}{1-\underline{\alpha}} = \underline{\beta} + \sum_{i=1}^{\infty} \underline{\beta} \prod_{j=1}^{i} \underline{\alpha} \leq x^*(\omega) \leq \bar{\beta} + \sum_{i=1}^{\infty} \bar{\beta} \prod_{j=1}^{i} \bar{\alpha} = \frac{\bar{\beta}}{1-\bar{\alpha}}$$

The left and the right estimates are easily seen to be identical to those for the ends of the invariant interval.

The temperedness of the random variable $p^*(\omega)$, i.e. the property that $\lim_{t\to\infty} e^{-\delta t} \mid\mid p^*(\theta^t\omega) \mid\mid = 0$ for all $\delta > 0$, can be shown in a manner similar to that followed in [12].

We may finally show that the fixed point $p^*(\omega)$ is an unstable fixed point. For the inverse map we may study the evolution of an initial condition near the fixed point, $p_0 = p^*(\omega) + \bar{x}_0$. Under the inverse map we may find that the evolution of the perturbation $\bar{x}_0$ will be given by the formula

$$\bar{x}_t = \prod_{j=0}^{t-1} \alpha(\theta^j\omega)\bar{x}_0$$

Since $\mid \alpha \mid < 1$ we see that under the inverse map $p_t \to p^*(\theta^t\omega)$ and thus the fixed point is stable under the inverse map. We may now define by ψ the dynamical system defined by the inverse map $\psi(t,\omega) = \phi^{-1}(t,\omega)$. Since the fixed point is stable for the dynamical system ψ is is unstable for the original dynamical system ϕ.

We may work similarly for the other branches. The only minor modification is the definition of the invariant interval and the conditions for invariance. In this case the candidate for the invariant interval is $[p_d^{(1)}, p_u^{(1)}]$ and let $p_d^{(1)'}$ and $p_u^{(1)'}$ be the images of the left and right end respectively under the inverse map. Then we will need the following ordering

$$p_d^{(1)} \leq p_u^{(1)'} \leq p_d^{(1)'} \leq p_u^{(1)},$$

which will be used to obtain estimates for the invariant interval. The rest follows. This concludes the proof. □

The existence of fixed points for the piecewise monotone model can be shown with the use of random generalizations of Banach's fixed point theorem proposed in [13].

3.2 Existence of invariant measures

Consider the random dynamical system $T : X \times W \to X$ such that

$$p_{t+1} = T(p_t, \xi_t), \quad t \geq 0, \tag{41}$$

that maps the point $(p_t, \xi_t) \in X \times W$ to some point p_{t+1} of X, where X and W are closed and Borel measurable subsets of $\mathbb{R}$ respectively. The tranformation F is such that for fixed $y \in W$ the function $T(x, y)$ is continuous in x and for every fixed $x \in X$ it is measurable in y. The random variables ξ_t are independent and identically distributed (iid) for all $t \geq 0$. Using the notation introduced in the beginning of section 3 we rewrite the random dynamical system in relation (41) as

$$p_{t+1}(\omega) = h(\theta^t\omega)\, p_t(\omega) = h(\theta^t\omega, p_t(\omega)) = \phi(t+1, \omega, p). \tag{42}$$

Let $\{\mu_t\}$ denote the sequence of distributions statistically describing the latter dynamical system in the sense that $\mu_t(A) = \text{prob}(p_t \in A)$ for all Borel sets A of X, and ν the distribution of each element in the sequence of the independent random variables $\{\xi_t\}$. Does an invariant distribution μ_* for dynamical system (41) exists in the sense that $\mu_*(A) = \text{prob}(p_t \in A)$ for all positive t? Let us fix $p_t = p \in X$. Then the probability that in the the next step $h(\omega)p$ is in the Borel set A is given by

$$\text{prob}(h(\omega)p \in A) = \int_{h(\omega)^{-1}A} \text{prob}(d\omega) = \int_{\Omega} \mathcal{X}_A(h(\omega)p)\, \text{prob}(d\omega) = \int_W \mathcal{X}_A(T(p,y))\, \nu(dy). \tag{43}$$

In other words we consider p_t to be a Markov process with

$$\mathbb{P}(p, A) = \text{prob}\{p_{t+1} \in A \,|\, p_t = p\} = \text{prob}(h(\omega)p \in A)$$

a time-independent 1-step transition probability. Rewriting the transition probability $\mathbb{P}(p, A)$ as a conditional expectation

$$\mathbb{P}(p, A) = \mathbb{E}_\xi \left\{\mathcal{X}_A(p_{t+1}) \,|\, p_t = p\right\}, \tag{44}$$

where $\mathbb{E}_\xi$ is the expectation operator over all possible values of the noise, and averaging the random variable $\mathbb{E}_\xi \left\{\mathcal{X}_A(p_{t+1}) | p_t\right\}$ yields

$$\mu_{t+1}(A) = \mathbb{E}_p \left\{\mathcal{X}_A(p_{t+1})\right\} = \mathbb{E}_p \left\{\mathbb{E}_\xi \left\{\mathcal{X}_A(p_{t+1}) \,|\, p_t\right\}\right\}. \tag{45}$$

In words the probability the next period the system is in A is the sum of the probability of p moving to A in one step over all possible values of p, weighted by the probability $\text{prob}(p_t \leq p \leq p_t + dp_t) = \mu_t(dp)$. Then equation (45) gives a recurrence relation for the measures μ_t

$$\mu_{t+1}(A) = \int_{p\in X} \left\{\int_{y \in W} \mathcal{X}_A(T(p,y))\nu(dy)\right\} \mu_t(dp), \ \forall t \geq 0, \ \forall A \in \mathcal{B}(X). \tag{46}$$

We naturally associate the process of mapping measure μ_t to μ_{t+1} with the action of an operator P_T. It is not difficult to see that P_T is a Markov operator on measures (for a definition see [11]), thus mapping the set of probability measures supported over X to itself via the relation

$$\mu_{t+1} = P_T \mu_t = \cdots = P_T^{t+1} \mu_0,$$

where μ_0 is an initial distribution to the system. The measure μ_* is an invariant measure for the random dynamical system in relation (41) when it is invariant in the average due to relation (44), that is

$$\mu_*(A) = \int_{p\in X} \mathbb{P}(p, A)\mu_*(dp) = (P_T \mu_*)(A), \ \forall A \in \mathcal{B}(X), \tag{47}$$

or equivalently when μ_* is a fixed point of P_T i.e. $P_T \mu_* = \mu_*$.

Because the set X in relation (41) is compact there is a stationary measure for the random dynamical system T (theorem 12.5.1 in [11]).

For the special case when ν is discrete T attains the IFS representation

$$T = \{\Theta_1, \ldots, \Theta_m; p_1, \ldots, p_m\}, \tag{48}$$

with $\Theta_i : X \to X$ and $p_i = \text{prob}\{T(p) = \Theta_i(p)\}$ for $1 \leq i \leq m$ for all $t \geq 0$. Then the 1-step transition probability $\mathbb{P}(p, A)$ becomes

$$\mathbb{P}(p, A) = \sum_{i=1}^{m} p_i \, \mathcal{X}_A \left(\Theta_i(p)\right).$$

In view of the previous equation, relation (46) reads

$$P_T \mu_t(A) = \sum_{i=1}^{m} p_i \, \mathbb{E}_p \left\{\mathcal{X}_A \left(\Theta_i(p_t)\right)\right\} = \sum_{i=1}^{m} p_i \, \mu_t \left(\Theta_i^{-1}(A)\right). \tag{49}$$

Let μ_t have density f_t with respect to the Lebesque measure and $\hat{P}_{\Theta_i}$ and $\hat{P}_T$ be the associated Frobenious-Perron operators, acting on densities, for the transformations Θ_i and T respectively. Because

$$\mu_t \left(\Theta_i^{-1}(A)\right) = \int_{\Theta_i^{-1}(A)} f_t(x)\, dx = \int_A \hat{P}_{\Theta_i} f_t(x)\, dx, \tag{50}$$

relation (49) gives

$$\hat{P}_T f_t = \sum_{i=1}^{m} p_i \, \hat{P}_{\Theta_i} f_t. \tag{51}$$

In words the transition operator for the random map T is the averaged Frobenius-Perron operator for the maps Θ_i. The invariant density f_* is a solution of the eigenvalue equation

$$f_* = \sum_{i=1}^{m} p_i \, \hat{P}_{\Theta_i} f_*. \tag{52}$$

3.3 Approximations to the fixed point of the operator $\hat{P}_T$

Let the state space X be a compact subset of the real line. In what follows we approximate the action of $\hat{P}_T$ on a n-partition of equal lengths $X^{(n)} = \{X_i^{(n)}\}_{i=1}^n$. We effectively project the action of $\hat{P}_T$ on infinite dimensional space $L^1(X, \lambda)$, where λ denotes the Lebesgue measure, to a n-dimensional subspace $L^{(n)}\left(X^{(n)}, \lambda\right)$ spanned by simple functions defined over $X^{(n)}$. Then the approximations to the stationary density f_* are simple functions $f^{(n)}$

$$f^{(n)} = \sum_{k=1}^{n} f_k^{(n)} \, \mathcal{X}_{X_i^{(n)}}, \quad f_k^{(n)} \geq 0, \tag{53}$$

which under the normalization condition $\int_X f^{(n)} dx = 1$ and $X = [0, 1]$ they satisfy $\sum_{k=1}^n f_k^{(n)} = n$. The sequence of densities $f^{(n)} = (f_1^{(n)}, \ldots, f_n^{(n)})'$ (( )$'$ denotes transposition) are the dominant eigenvectors (those corresponding to eigenvalue 1) of the $n \times n$ stochastic matrix $\mathcal{K}_T^{(n)}$ with entries

$$\left(\mathcal{K}_T^{(n)}\right)_{ij} = \text{prob}\left(p_{t+1} \in X_i^{(n)} \mid p_t \in X_j^{(n)}\right). \tag{54}$$

It can be verified that the matrix in the previous relation plays indeed the rôle of a n-dimensional approximation to the operator $\hat{P}_T$ in relation (51). To see this observe that the event $\{p_{t+1} \in X_i^{(n)}\}$ is the union of the mutually disjoint events $\{p_t \in \Theta_k^{-1}(X_i)$ and $\xi_t = k\}$ for $1 \leq k \leq m$. The latter together with the fact that p_t is dependent only on $\xi_1, \ldots, \xi_{t-1}$ gives

$$\mathcal{K}_T^{(n)} = \sum_{k=1}^{m} p_k \, \mathcal{K}_{\Theta_k}^{(n)}, \quad \left(\mathcal{K}_{\Theta_k}^{(n)}\right)_{ij} = \text{prob}\left(\Theta_k(p_t) \in X_i^{(n)} \mid p_t \in X_j^{(n)}\right) \tag{55}$$

where $\mathcal{K}_{\Theta_k}^{(n)}$ is the corresponding finite dimensional approximation to the $\hat{P}_{\Theta k}$ operator. Then the finite dimensional approximation to relation (52) is

$$f^{(n)} = \left(\sum_{k=1}^{m} p_k \, \mathcal{K}_{\Theta_k}^{(n)}\right) f^{(n)}, \tag{56}$$

3.4 Liapunov exponents

For the estimation of the Liapunov exponents we need a more general setting, which we explain briefly. For a more detailed approach see [8], [12] and references therein.

Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\omega \in \Omega$ being infinite sequences $\omega = (\omega_0, \omega_1, \ldots)$ and $\omega_i \in \{1, \ldots, m\}$ the indices in relation (48). Consider the dynamical system $\tilde{T} : \Omega \times X \to \Omega \times X$ with $\tilde{T}(\omega, p) = (\sigma(\omega), \Theta_{\omega_0}(p))$ and $\sigma : \Omega \to \Omega$ the left shift map $(\sigma(\omega))_i = \omega_{i+1}$, describing the noise trajectory, being the θ dynamical system over Ω of relation (3). Then the random orbit $T^t(p)$ of the random dynamical system in relation (48) is the projection to X of $\tilde{T}^t(\omega, p)$

$$\tilde{T}^t(\omega, p) = (\sigma^t(\omega), T^t(p)), \quad T^t(p) = \Theta_{\omega_{t-1}} \circ \cdots \circ \Theta_{\omega_0}(p). \tag{57}$$

It is clear that starting at p, the probability of observing the sequence of compositions $\Theta_{\omega_{t-1}} \circ \cdots \circ \Theta_{\omega_0}$, is $\text{prob}\{\omega_{t-1}, \cdots, \omega_0\} = p_{t-1} \cdots p_0$. Then the probability measure P can be obtained by extending the latter product to an infinite sequence of compositions. It can be shown that P is an invariant measure of σ (see [10]) i.e. $P = P \circ \sigma^{-1}$ and that $\sigma : \Omega \to \Omega$ is an ergodic transformation i.e. the only invariant sets are sets of full or zero measure. Let $\tilde{\mu}$ be an invariant measure of $\tilde{T} : \Omega \times X \to \Omega \times X$ i.e. $\tilde{\mu} = \tilde{\mu} \circ \tilde{T}^{-1}$ then by construction the measures μ_* and P are marginals of the measure $\tilde{\mu}$

$$\mu_*(A) = \tilde{\mu}(\Omega \times A), \quad P(B) = \tilde{\mu}(B \times X), \tag{58}$$

for all measurable subsets A and B of the sets X and Ω respectively. Therefore an ergodicity of μ_* would imply

$$\lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} f \circ \tilde{T}^k(\omega, p) = \int_{\Omega \times X} f(\omega, p)\, \tilde{\mu}(d\omega, dp), \tag{59}$$

for all continuous functions $f : \Omega \times X \to \Omega \times X$ and for $P \times \lambda$ almost all (ω, p). Letting

$$f(\omega, p) = \log \left| \frac{\partial \phi(1, \omega, p)}{\partial p} \right| = \log \left| T'_{\omega_0}(p) \right|, \tag{60}$$

and using the fact that

$$\frac{\partial \phi(t+1, \omega, p)}{\partial p} = \frac{\partial h(\sigma^t(\omega), h(\sigma^{t-1}(\omega), h(\ldots)))}{\partial h(\sigma^{t-1}(\omega), h(\ldots))} \cdots \frac{\partial h(\omega, p)}{\partial p} = \prod_{k=0}^{t-1} \frac{\partial \phi(k+1, \omega, p)}{\partial \phi(k, \omega, p)}, \tag{61}$$

one obtains using relation (59)

$$\begin{aligned} \lambda(\tilde{\mu}) &= \lim_{t \to \infty} \frac{1}{t} \log \left| \frac{\partial \phi(t+1, \omega, p)}{\partial p} \right| = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \log \left| \frac{\partial \phi(k+1, \omega, p)}{\partial \phi(k, \omega, p)} \right| \\ &= \int_{\Omega \times X} \log \left| \frac{\partial \phi(1, \omega, p)}{\partial p} \right| \tilde{\mu}(d\omega, dp). \end{aligned} \tag{62}$$

Whenever the integral in the previous relation converges, the real number $\lambda(\tilde{\mu})$ measures the (exponential) rate of contraction or divergence of nearby orbits. Because the noise is iid the events $\{\Theta_{\omega_{t-1}} \circ \cdots \circ \Theta_{\omega_0}(p) \in A\}$ and $\{\omega_t, \omega_{t+1}, \ldots) \in B$ are independent and $\tilde{\mu}(B \times A) = P(B)\, \mu_*(A)$ one obtains the decomposition $\tilde{\mu}(d\omega, dp) = P(d\omega)\, \mu_*(dp)$. Then using the IFS representation in relation (48), relation (62) gives

$$\lambda(\tilde{\mu}) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \log \left| \Theta'_{\omega_k} \left(\Theta_{\omega_{k-1}} \circ \cdots \circ \Theta_{\omega_0}(p) \right) \right| = \sum_{k=1}^{m} p_k \int_X \log \left| \Theta'_k(p) \right| \mu_*(dp). \tag{63}$$

Using the approximation in relation (53) of the invariant density, under the additional assumption that the derivatives Θ'_k are approximately constant over the interval $X_i^{(n)}$, for large n, one obtains from the previous relation

$$\lambda(\tilde{\mu}) \approx \sum_{k=1}^{m} p_k \sum_{i=1}^{n} f_i^{(n)} \log \left| \Theta'_k(p_{i,n}) \right|, \tag{64}$$

where $p_{i,n}$ is the center of the interval $X_i^{(n)}$.

4 Numerical results

We present in this section some results on the dynamics of our market model obtained by numerical simulation.

4.1 The piecewise linear case

Here all numerical simulations have been carried out for the random system h in lemma 2.1 for $\sigma = 1$.

The case where the switching conditions are not satisfied: For $\lambda \leq \lambda^*$ the switching conditions are not satisfied. The bull and bear zones of Θ^d are

$$Z^{b,d} = \left(0, \frac{4+\lambda}{4\lambda}\right), \quad Z^{B,d} = \left(\frac{4+\lambda}{4\lambda}, \frac{2+\lambda}{8}\right), \tag{65}$$

respectively. Both intervals $Z^{b,d}$ and $Z^{b,d}$, are invariant sets of the tranformation Θ^d. Approximations to the invariant densities of the maps $\Theta^d : Z^{b,d} \to Z^{b,d}$ and $\Theta^d : Z^{B,d} \to Z^{B,d}$ are given in figures 2(a) and (b) respectively, for $\lambda = \lambda^*$. The bull and bear zones for Θ^u are

$$Z^{b,u} = \left(\frac{4-\lambda}{8\lambda}, \frac{3\lambda-4}{4\lambda}\right), \quad Z^{B,u} = \left(\frac{3\lambda-4}{4\lambda}, 1\right), \tag{66}$$

respectively, and are both invariant under Θ^u. In figures 2(c) and (d) we give approximations to the invariant densities of the map Θ^u restricted to the intervals $Z^{b,u}$ and $Z^{B,u}$ respectively, for $\lambda = \lambda^*$. The situation changes when one considers the orbits of the random map $T_\lambda = \{\Theta^u_\lambda, \Theta^d_\lambda;\ 1/2, 1/2\}$. The interval $[0, 1]$ is the unique invariant set of positive measure of the random map T_λ. In figure 2(e) we give an approximation to its invariant density for $\lambda = \lambda^*$.

The case where the switching conditions are satisfied: For $\lambda > \lambda^*$ the switching conditions are met. In Figures 3(a) and (b) we plot approximations to the invariant densities of the deterministic maps Θ^d and Θ^u for $\lambda = \lambda^* + 0.1$. For this value of λ, the intervals Z^d and Z^u in relation (14) are the only invariant sets of positive measure for the maps Θ^d and Θ^u respectively. In figure 3(c) we plot an approximation to the invariant density of the corresponding random system h in relation (27) for prob$\{\eta = 1\} = 0.5$ and $\lambda = \lambda^* + 0.1$.

Approximating the Liapunov exponent when the switching conditions are satisfied : We consider $\lambda = \lambda^* + 0.6$ and we approximate the invariant density of the random system h by equi-partitioning its invariant set $X = [0, 1]$ initially into 4 sets. Namely $X_i^{(4)} = [(i-1)/4, i/4]$, $1 \leq i \leq 4$. Then the stochastic matrix $\mathcal{K}_T^{(4)}$ in relation (54) reads

$$\mathcal{K}_T^{(4)} = \begin{pmatrix} 0.38032 & 0.46486 & 0 & 0 \\ 0.36144 & 0.52486 & 0.01030 & 0.25799 \\ 0.25824 & 0.01028 & 0.52485 & 0.36176 \\ 0 & 0 & 0.46485 & 0.38024 \end{pmatrix}. \tag{67}$$

The latter stochastic matrix defines a 4-state Markov chain with states $\{X_i^{(4)}\}$, and, stationary density

$$f^{(4)} = \begin{pmatrix} 0.85687 & 1.14225 & 1.14333 & 0.85755 \end{pmatrix}', \tag{68}$$

which is unique, and satisfies $\int_0^1 \sum_{i=1}^4 f_i^{(4)} \mathcal{X}_{X_i^{(4)}}(x)\, dx = 1$. Now letting $X_1 = [0, 1/4] \cup [3/4, 1]$ and $X_2 = [1/4, 3/4]$, and, considering an equipartition $\{X_i^{(n)}\}$ of X having the property that its elements are proper subsets of X_1 or X_2, relation (63) yields

$$\begin{aligned} \lambda(\tilde{\mu}) \approx \lambda^{(n)} &= r \int_0^1 \log |(\Theta^u)'(x)| \mu_*^{(n)}(dx) + (1-r) \int_0^1 \log |(\Theta^d)'(x)| \mu_*^{(n)}(dx) \\ &= \tfrac{1}{n} \left\{ \log(\lambda - 1) \textstyle\sum_{X_i^{(n)} \subset X_1} f_i^{(n)} + \log(\lambda/4 + 1) \sum_{X_i^{(n)} \subset X_2} f_i^{(n)} \right\} \\ &= \log(\lambda - 1)\mu_*^{(n)}(X_1) + \log(\lambda/4 + 1)(1 - \mu_*^{(n)}(X_1)), \end{aligned} \tag{69}$$

for all $\lambda^* < \lambda \leq 4$ and $0 \leq \sigma \leq 1$. The latter formula still depends on the choice of the probability r as $f^{(n)}$ is the eigenvector that corresponds to eigenvalue 1 of the stochastic matrix in relation (56), for $m = 2$, $p_1 = r$ and $p_2 = 1 - r$. In Table 4.1 we give approximations to the Liapunov exponent for $n = 2^k$, $2 \leq k \leq 6$. Eventhough the resolution of the density function improves as n increases – see figures 4(c) and (d) – the measure of the sets X_1 and X_2 remains approximately the same, and the value of the Liapunov exponent in a 3 digits accuracy is $\lambda(\tilde{\mu}) = 0.831$.

4.2 The piecewise monotone case

Here all numerical simulations have been carried out for the random system h in relation (36).

Noise induced instability: For $a = 0.20$, $b = 0.75$ and λ given by relation (37), using the system of equations in relation (38) we calculate in a four digits accuracy, $\sigma^* = 0.0192$ and $\overline{p} = 0.8846$, figure 5(a). Trapping conditions, like those in relation (17) for tranformation Θ^u, are satisfied for both members Θ^u and Θ^d of the random map h in relation (36). Likewise for $\sigma = \sigma^*$ the union of the two trapping sets $Z^d = [0, 0.9545]$ and $Z^u = [0.0455, 1]$ is the interval $[0, 1]$ which remains invariant under the action of the random map h. Nevertheless it can be seen that the associated switching conditions are not met. Letting p^u and p^d, with $0 < p^u \leq 1/2 \leq p^d < 1$, be the central fixed points of the tranformations Θ^u and Θ^d respectively, one has due to symmetry

$$\Theta^d(p^{m,d}) - p^d = p^u - \Theta^u(p^{M,u}), \quad \Theta^d(p^{M,d}) - p^d = p^u - \Theta^u(p^{m,u}), \quad p^u + p^d = 1. \tag{70}$$

Then we need to check only the validity of the switching inequalities for Θ^u in relation (22). For $\sigma = \sigma^*$ we find that both quantities $\Theta^u(p^{M,u}) - p^u$ and $\Theta^u(p^{m,u}) - p^u$ are negative. Using relation (70) one obtains the results on Table 4.2. In fact for these parameter values the associated deterministic dynamical systems, Θ^u and Θ^d from $[0, 1]$ to itself, are stable as there exist 4-cycles, namely the iterates of $p^* = 0.1112$ under Θ^u situated in $Z^{b,u}$, and the iterates of $1 - p^*$ under Θ^d situated in $Z^{B,d}$ such that

$$\rho^* = \left((\Theta^u)^4\right)'(p^*) = \left((\Theta^d)^4\right)'(1 - p^*) = -0.7306. \tag{71}$$

In figures 5(b),(c) the invariant measure of Θ^u and Θ^d is the sum of four δ-functions. An approximation to the invariant measure of the random map h for prob$\{\eta = 1\} = r = 0.5$ is given in figure 5(d). This is a case of noise induced switching. The Liapunov exponent λ_μ, as a function of r, is given in figure 6. The $\lambda_\mu(r)$ function is symmetric with respect to the value $r = 0.5$ due to the symmetric dynamics of the two constituent deterministic maps. It takes on its maximum value $\lambda_\mu = 0.6623$ for $r = 0.5$. Its minimum value is the Liapunov exponent of the deterministic maps Θ^u and Θ^d, at $\lambda_\mu = -0.0785$, for $r = 0$, 1 which naturally satisfies, $\lambda_\mu = \ln|\rho^*|/4$.

Noise induced intermittent behaviour: Again for $a = 0.20$, $b = 0.75$ but for smaller values of σ we have a different situation. Here it can be verified that the bear and bull zones of Θ^u and Θ^d are all attractive. If we let $Z^{ud} = (p^u, p^d)$, by our previous setting it is to be understood that

$$Z^{b,u} \subseteq Z^{b,d}, \quad Z^{B,d} \subseteq Z^{B,u}, \quad Z^{b,d} \cap Z^{B,u} = Z^{ud}. \tag{72}$$

If we iterate the random map h with an initial condition $p \in Z^{B,u} \backslash Z^{ud}$, then a necessary condition for switching to the set $Z^{b,d} \backslash Z^{ud}$, is $h(\sigma^t(\omega), p) \in Z^{ud}$. Nevertheless such a situation is rare as the length of the interval Z^{ud} is small when σ is small, and, p^u and p^d are unstable. The open interval Z^{ud} serves as a bridge between the two attracting zones $Z^{b,d}$ and $Z^{B,u}$. For small σ, the bridge is rarely visited. As a result, for small values

n	$\lambda^{(n)}$	$\mu_*^{(n)}(X_1)$
4	0.83102	0.42861
8	0.83088	0.42820
16	0.83162	0.43024
32	0.83124	0.42920
64	0.83068	0.42770

Table 1: Approximations to the Liapunov exponent of the random map h and the measure of the set X_1 in relation (69)

Zone	Attracting
$Z^{b,d}$	No
$Z^{B,d}$	Yes
$Z^{b,u}$	Yes
$Z^{B,u}$	No

Table 2: The bear and bull zones of Θ^d and Θ^u for $a = 0.20$, $b = 0.75$ and $\sigma = \sigma^*$

of σ the invariant measure will have a minimum situated at $p = 0.5$, the center of the open interval Z^{ud}. In such cases convergence to the invariant measure using a test orbit, is extremely slow. Such a case arrise for $\sigma = 0.00770$. For these parameter values most realizations of h of length less than 10^8, starting in $Z^{B,d} \backslash Z^{ud}$ cannot switch to $Z^{b,u}$. For values of σ slightly larger, a slow switching manifests itself. In figures 7(a),(b) we give two realizations of h of length 5×10^5 for $\sigma = 0.00780$ and $\sigma = 0.00785$ respectively. As σ gets larger switching becomes faster. In figure 7(c) we give an approximation to the invariant measure of h for $\sigma = 0.00785$. Noise induced intermittent behaviour is a quite common feature. For $a = 0.20$ and $b = 0.45$ (then $\sigma^* = 0.3657$) the dynamics of the deterministic pair of maps are again symmetric. As σ increases from 0 to σ^* we have observed a complete period doubling cascade of bifurcations. Specificaly for $\sigma = 0.0375$ both deterministic maps exhibit stable 1-cycles. In figures 8(a)-(d), the behaviour of the random map h is intermittent, and, the invariant measure is bimodal. The piecewise linear model shares this feature when $\lambda < \lambda^*$. Then by lemma 2.1 switching conditions are not satisfied. In figure 9(a)-(d) we plot a bimodal and a multimodal case of slow switching.

Noise induced stabilization: We start by considering the case $a = 0.20$ and $b = 0.45$. We are interested on the effect produced by increasing the probability r of choosing map Θ^u. Here the dynamics of the deterministic pair of maps are symmetric implying that the Liapunov exponent function satisfies $\lambda_\mu(\sigma, r) = \lambda_\mu(\sigma, 1-r)$. Both maps exhibit, as σ increases from $\sigma_0 = 0.0142$ to σ^* a period doubling bifurcation cascade; Θ^u in its bull zone and Θ^d in its bear zone, which are both attracting. For $0 \leq \sigma \leq \sigma_0$ the attracting regions are reversed. The Liapunov exponent curve $\lambda_\mu(\sigma, 0)$ for $r = 0$ is typical for systems with a quadratic minimum or maximum; Θ^d and Θ^u restricted on their bear and bull zones respectively are such systems. In figures 10(a) and (b) we plot σ - cross sections of the Liapunov exponent $\lambda_\mu(\sigma, r)$, and, the $\mathrm{Var}_\mu(p)$ variance of the price (with respect to the invariant measure) functions. As undecidability on the value of the fundamental enters i.e. there is a non zero probability r of choosing Θ^u, the finite variability of the Liapunov function decreases. Schematically, as r tends to 0.5, the curves approach the $\lambda_\mu = 0$ line. In figures 10(c) and (d) we plot r - cross sections of the $\lambda_\mu(\sigma, r)$, and, $\mathrm{Var}_\mu(p)$ functions. We observe that for curves that correpond to a σ with a stable underlying dynamics, an increase in undecidability, makes the system less stable with a Liapunov exponent attaining its maximum value at $r = 0.5$. Nevertheless when the underlying dynamics are unstable, in the sense that they possess positive Liapunov exponents, the effect of increasing r to 0.5, leads to a stabilization of the random dynamical system, with a Liapunov exponent attaining its minimum value at $r = 0.5$. An alternative way of investigating noise dependence is by considering the variance of the price as a function of the variance of the fundamental. From the definition of the random map h in relation (36), one obtains $\mathrm{Var}(v) = 4\sigma^2 r(1-r)$. Fixing the σ parameter and letting $\mathrm{Var}(v) = \xi(\sigma) \in [0, 4\sigma^2 r(1-r)]$ one calculates $r_{\xi(\sigma)}$, which in turn is used for the approximation of the invariant density of h, and, finally the computation of the variance $\mathrm{Var}_\mu(p)$. In figure 11 we plot six such curves. Low period stable dynamics of the underlying deterministic maps correspond to an abrupt monotone amplification of the variance of the price under small increments of the variance of the fundamental. Nevertheless for more complicated underlying dynamics (large stable periodic or erratic motion) the picture changes dramatically. The associatted σ - curves possess minima that corresponds to a noise induced stabilization of the dynamics.

Monotone switching satisfied: We finally investigate the case where $a = 0.1295$, $b = 0.5$. Then $\sigma^* = 0.004301$, and, the switching inequalities in relation (22) are satisfied for $0 \leq \sigma \leq \sigma^*$. The underlying deterministic maps are ergodic in their trapping regions. In figure 12 we give finite dimensional approximations to the invariant measure of the random map h.

5 Discussion and conclusions

One of the aims of the present paper was to examine the robustness of the qualitative features of the original model of Day and coworkers [3],[4],[5] under the effect of uncertainty on the value of the fundamental. Assuming that the fundamental follows a stochastic process Day's original model takes the form of a random dynamical system with additive or multiplicative noise. Through the study of this random dynamical system we found the following persistent qualitative features between the deterministic and the random model:

1. Existence of random fixed points that are random processes invariant under the dynamics.
2. Existence of an invariant measure.
3. Existence of positive Liapunov exponents

We provide an approximation scheme for the density of the invariant measure using finite dimension approximations of the Frobenius Perron operator as well as a scheme for the calculation of dynamical quantities of interest such as Liapunov exponents or the dispersion of the stock prices. Other examples can be estimates of quantities related to the viability of the market such us the maximum transaction fees $H_{\max} := \mathbb{E}_p\,(\alpha(p) + \beta(p))$

Two specific models are treated in detail; a piecewise linear model and a piecewise monotone case. Both models are stochastic generalizations of their corresponding deterministic counterparts. The morphology of the invariant measure allows us to visualize features of the dynamics that exhibit a very rich behaviour. For example we see that for certain parameter values the invariant measure presents a single mode, or it becomes bimodal or even multimodal. We observe features that are generated by noise and thus are absent in the original deterministic model, such as for instance the intermitent behaviour of the stock prices between bull and bear zones which are caused by the uncertainty of the fundamental. Another interesting feature of the model is that it allows us to assess the effect of market dynamics on the transmition of uncertainty from the level of the fundamentals to the level of the stock prices. One can imagine that the market mechanism (the tattônement process) acts as nonlinear filter on the uncertainty at the level of the fundamental values. We can quantify the above argument through the examination of the Liapunov exponent and the variance of the prices as a function of the variance of the fundamental In particular we find that regular market dynamics tend to amplify the effects of the uncertainty of the fundamentals on the stock prices whereas on the contrary chaotic (unstable) market dynamics tend to supress it.

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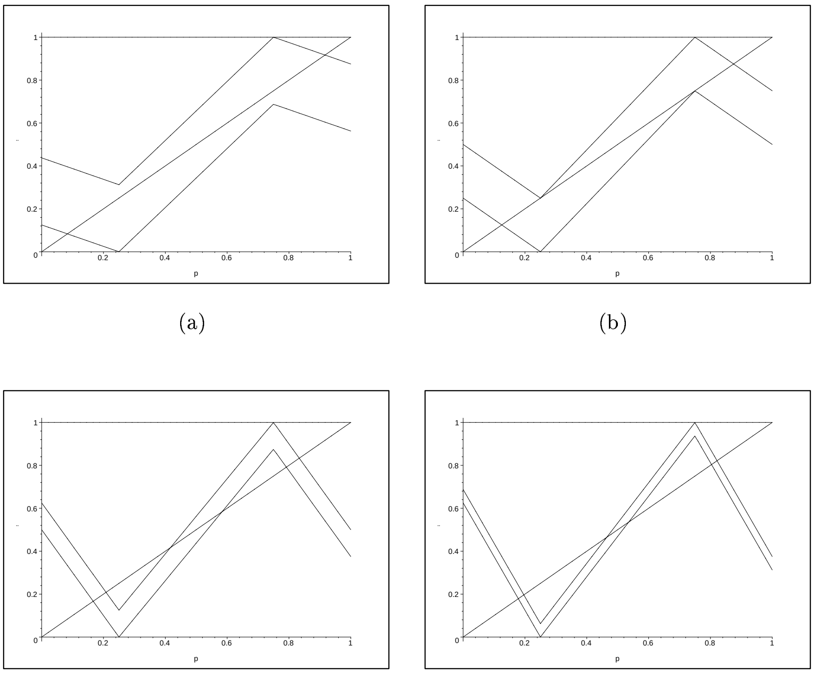


(a) (b)

(c) (d)

Figure 1: The member maps of the IFS for $\lambda = 1.5,\ 2.0,\ 3.0,\ 3.5$

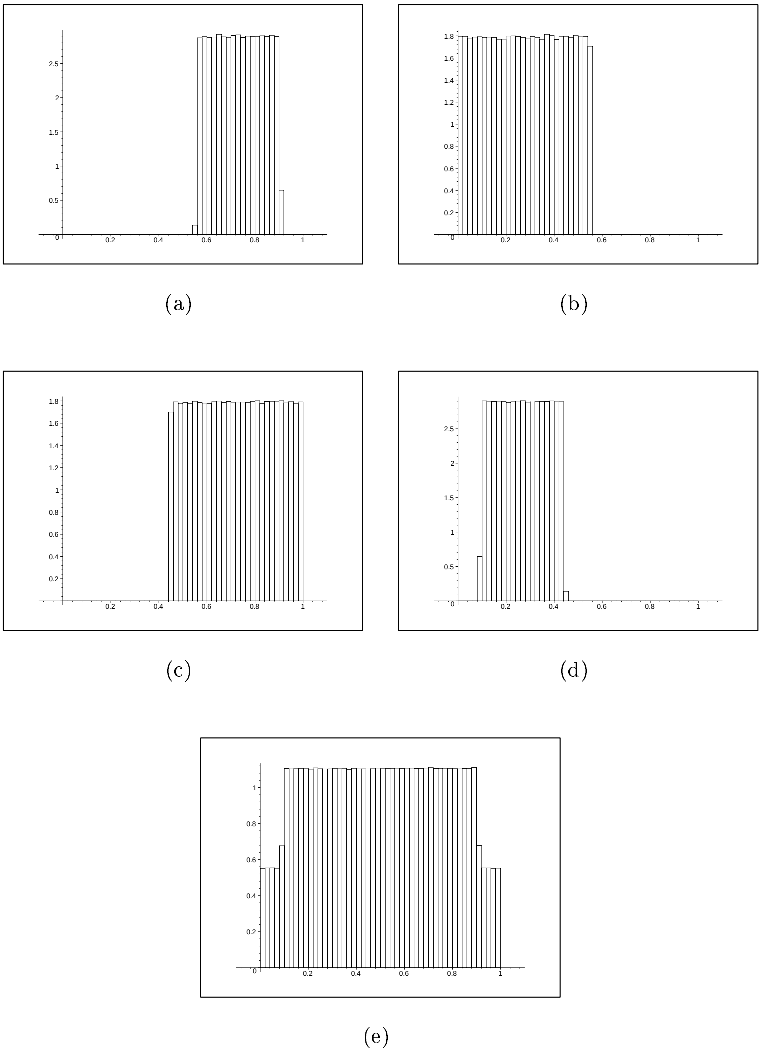
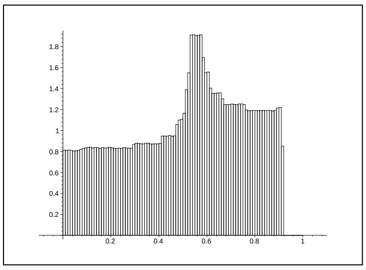
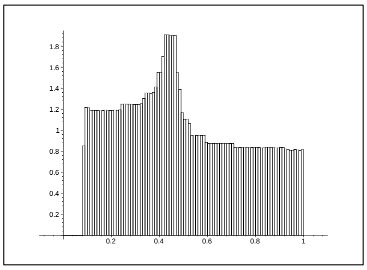
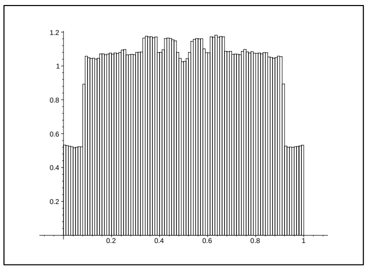


Figure 2: Approximations to the invariant measure of the piecewise linear deterministic maps Θ^d and Θ^u in figures (a)-(b), and, (c)-(d), respectively. The switcing conditions are not satisfied and there are two invariant sets for $\lambda = \lambda^*$. In figure (e) we approximate the invariant measure of the random map h in relation (27) for $\text{prob}\{\eta = 1\} = 0.5$ and $\lambda = \lambda^*$.



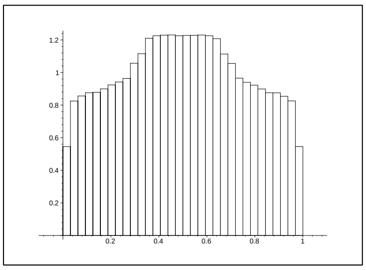


(a) (b)

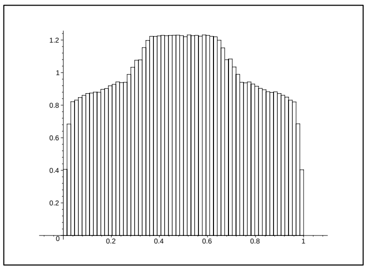


(c)

Figure 3: Approximations to the invariant measure of the piecewise linear deterministic maps Θ^d and Θ^u in figures (a)-(b), here $\lambda = \lambda^* + 0.1$. Here the switcing conditions are satisfied and there is only one invariant set. In figure (c) we approximate the invariant measure of the random map h in relation (27) for prob$\{\eta = 1\} = 0.5$ and $\lambda = \lambda^* + 0.1$.



(a)



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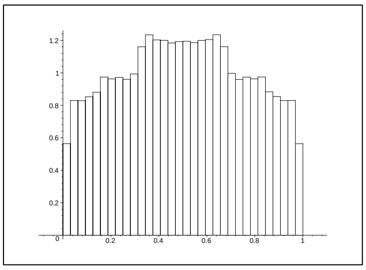
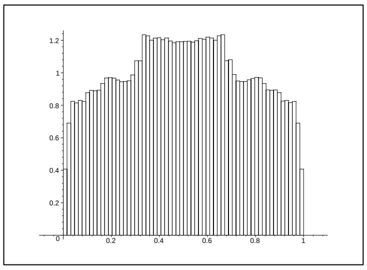
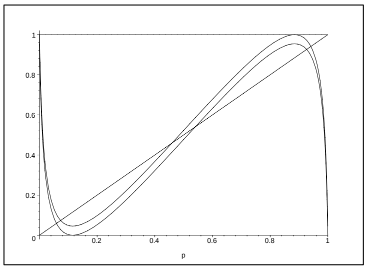


(c)

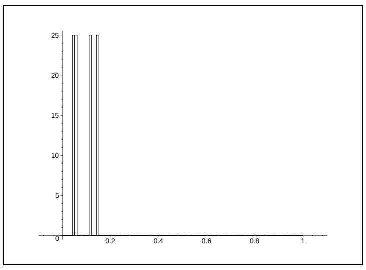


(d)

Figure 4: Approximations to the invariant measure of the piecewise linear random map h for $\lambda = \lambda^* + 0.6$ and $\sigma = 1$. In figures (a) and (b) we have employed a long test orbit, and, we have equipartitioned the inteval $[0, 1]$ into 32 and 64 subintervals respectively. In figures (c) and (d) we have calculated the approximation to the invariant measure as a solution to the eigenvalue equation in relation (56) for $n = 32$ and $n = 64$ respectively.



(a)



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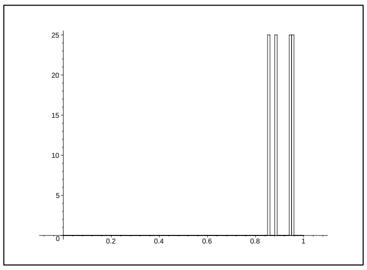
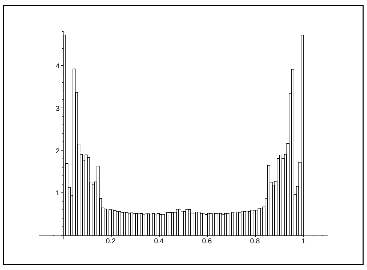


(c)



(d)

Figure 5: In figure (a) we plot the members of the piecewise monotone IFS h in relation (36) for $a = 0.20$, $b = 0.75$ and $\sigma = \sigma^* = 0.0192$. In figures (b) and (c) the invariant measure of the deterministic maps Θ^u and Θ^d is a sum of four δ-functions. In figure (d), for the same values of the parameters, we plot an approximation to the invariant measure of the random map h for $r = 0.5$.

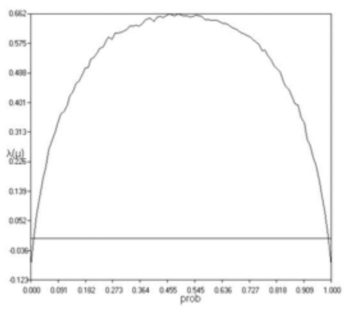
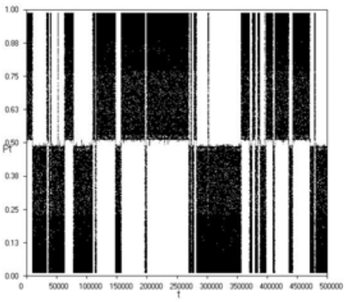
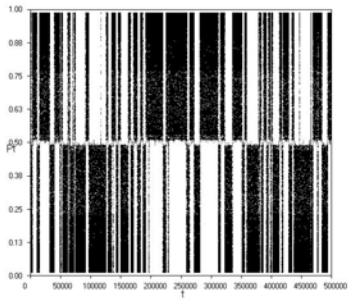


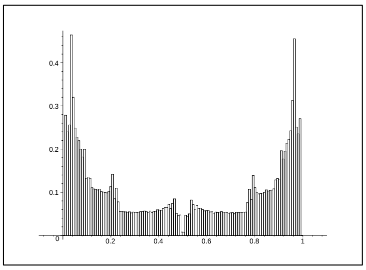
Figure 6: The liapunov exponent $\lambda(r)$ of the piecewise monotone random map h in relation (36), as a function of the probability r, for parameter values $a = 0.20$, $b = 0.75$, and, $\sigma = \sigma^* = 0.0192$.



(a)

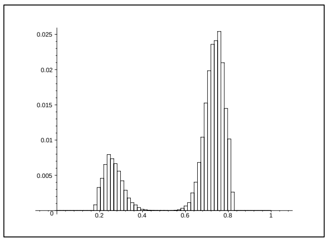


(b)

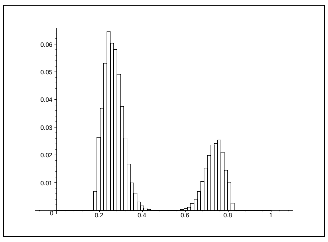


(c)

Figure 7: In figures (a) and (b) we plot two trajectory realizations of the piecewise monotone random map h in relation (36) for $a = 0.20$, $b = 0.75$, $\sigma = 0.00780$ and $\sigma = 0.00785$ respectively. For $\sigma = 0.00785$ in figures (c) we plot an approximation to the corresponding invariant measure.



(a)



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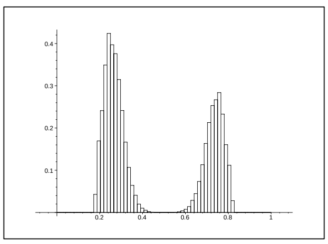
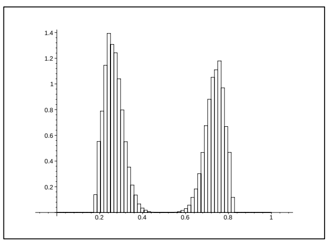
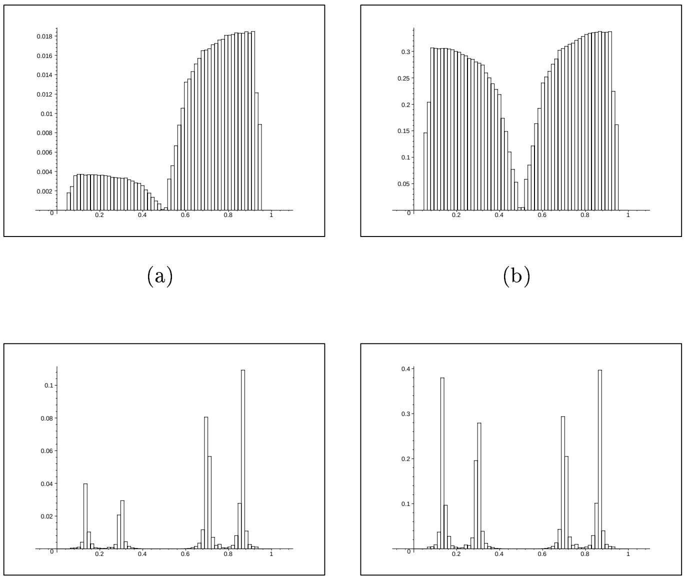


(c)



(d)

Figure 8: In figures (a) to (d) we plot a series of histograms generated by a long test orbit of the piecewise monotone random map h for $a = 0.20$, $b = 0.45$ and $\sigma = 0.0375$. Slow switching between the bear and bull zones of Θ^d and Θ^u, which contain stable 1-cycles, causes a slow convergence to the invariant measure.



(a) (b)

(c) (d)

Figure 9: In figures (a) and (b), we observe slow switching in the piecewise linear random map for $\lambda = \lambda^* - 0.2$ and $\sigma = 0.210$, case of a bimodal invariant measure. In figures (c) and (d), we observe slow switching in the piecewise linear random map for $\lambda = \lambda^* - 1.27$ and $\sigma = 0.527$, case of a multimodal invariant measure.

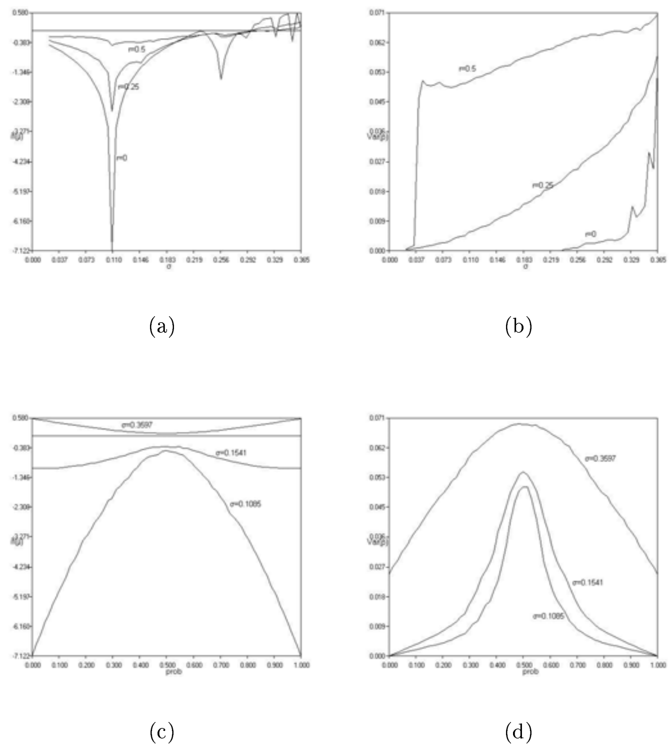


Figure 10: In figures (a) and (b) we plot σ cross sections of the λ_μ and $\mathrm{Var}(p)$ functions respectively. In figure (a) we observe that as r tends to 0.5, the finite variability of the Liapunov exponent function decreases. In figure (b) the variance of the price for $r = 0.5$ has the smallest variability (for values of $\sigma > 0.05$ when intermittent behaviour is much weaker). In figures (c) and (d) we plot r cross sections of the λ_μ and $\mathrm{Var}(p)$ functions respectively. In figure (c) we observe that when the underlying deterministic dynamics are unstable (curve for $s = 0.3597$) the increase of the noise makes the system more stable. Nevertheless when the underlying deterministic dynamics are stable (curves $\sigma = 0.1085$ and $\sigma = 0.1541$) the increase of the noise makes the system less stable.

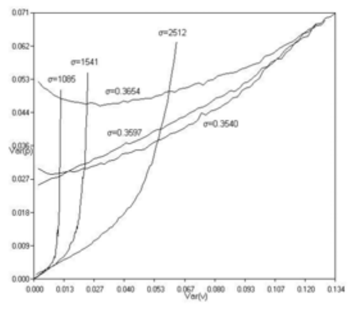
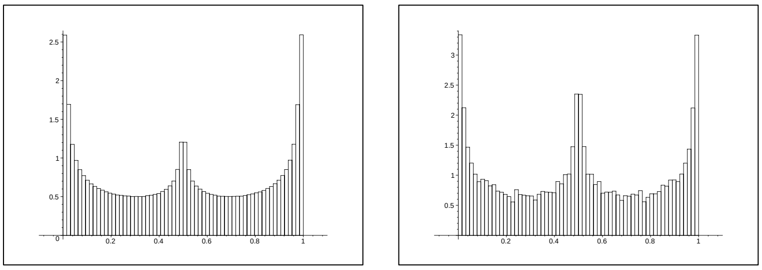


Figure 11: The $\mathrm{Var}_\mu(p)$ variance of the price parametrized with respect to the variance $\mathrm{Var}(v)$ of the fundamental for different values of the σ parameter.



(a) (b)

Figure 12: In figures (a) and (b) we exhibit finite dimensional approximations ($n = 64$) of the invariant measure of the piecewise monotone random map h for $a = 0.1295$, $b = 0.5$ and $\sigma = \sigma^*$, where the switching conditions are satisfied. In figure (a) we give a test orbit approximation, and, in figure (b) the associated eigenvalue approximation.