<u>Ορισμοί</u> (i) Δύο σ-πεδία $\mathcal{G} \subset \mathcal{F}$ κ' $\mathcal{H} \subset \mathcal{F}$ είναι ανεξ εαν $\forall A \in \mathcal{G}$ κ' $\forall B \in \mathcal{H}$ τα ενδεχόμενα A κ' B είναι ανεξ

(ii) Δύο τμ X κ' Y είναι ανεξ εαν κ' μόνον εαν $\sigma(X) \subset \mathcal{F}$ κ' $\sigma(Y) \subset \mathcal{F}$ ανεξάρτητα.

(iii) Η τμ X κ' το σ-πεδίο $\mathcal{G} \subset \mathcal{F}$ είναι ανεξάρτητα εαν $\sigma(X) \subset \mathcal{F}$ κ' $\mathcal{G} \subset \mathcal{F}$ ανεξάρτητα.

CONDITIONING ON AN EVENT

<u>Παρ 2.1</u> 3 νομίσματα των 10, 20, 50 λεπτών ρίπτονται. Κερδίζει κάποιος την συνολική τιμή ξ των νομισμάτων που έρχονται Heads. Ποιο το αναμενόμενο ποσό ξ δοθέντος οτι 2 νομίσματα έδωσαν κεφαλές?

$$\Omega = \{ W_1 W_2 W_3 : w_i \in \{H, T\},\ i=1,2,3 \} \ni \omega, \quad P(\{\omega\}) = 1/8, \ \forall \omega \in \Omega$$

$$B = \{HHT, HTH, THH\}, \quad P(B) = 3/8$$

$$\boxed{\mathbb{E}(\xi | B) = \frac{1}{P(B)} \int_B \xi(\omega) P(d\omega)} = \frac{1}{3/8} \Big(\underbrace{\xi(HHT)}_{30} P(HHT) + \underbrace{\xi(HTH)}_{60} P(HTH) + \underbrace{\xi(THH)}_{70} P(THH) \Big)$$

<u>Παρ</u> (i) $\mathbb{E}(\xi|\Omega) = \dfrac{1}{P(\Omega)} \displaystyle\int_\Omega \xi(\omega) P(d\omega) = \int_\Omega \xi(\omega) P(d\omega) = \mathbb{E}(\xi)$

(ii) $\mathbb{E}(\mathbb{1}_A | B) = \dfrac{1}{P(B)} \displaystyle\int_B \mathbb{1}_A(\omega) P(d\omega) = \dfrac{1}{P(B)} \int_{AB} P(d\omega) = \dfrac{1}{P(B)} P(AB) = P(A|B)$

CONDITIONING ON A DISCRETE RV

<u>ορσ</u> $\xi \in L^1(\Omega, \mathcal{F}, P)$, $\eta(\Omega) = \{y_1, y_2, \dots\}$ τότε η υπό συνθήκη μέση τιμή της ξ δοθείσας της η είναι η τμ $\mathbb{E}(\xi|\eta) : \Omega \to \mathbb{R}$ τώ

$$\mathbb{E}(\xi|\eta)(\omega) = \mathbb{E}(\xi | \{\eta = y_i\}), \quad \eta(\omega) = y_i$$

$$\Rightarrow \mathbb{E}(\xi|\eta)(\omega) = \sum_{i=1}^{\infty} \mathbb{E}(\xi|\{\eta=y_i\}) \mathbb{1}_{\{\eta=y_i\}}(\omega)$$

<u>Παρ 2.2</u> Έστω ξ η τμ του παρ 2.1 σελ 9 κ' η το συνολικό κέρδος μόνο από το νόμισμα των 10 κ' 20 λεπτών. Να ορισθεί η τμ $\mathbb{E}(\xi|\eta)$.

$$\begin{aligned} &HHH \xrightarrow{\eta} 30 \\ &HHT \to 30 \\ &HTH \xrightarrow{\eta} 10 \\ &HTT \to 10 \\ &THH \xrightarrow{\eta} 20 \\ &THT \to 20 \\ &TTH \xrightarrow{\eta} 0 \\ &TTT \to 0 \end{aligned}$$

$$\{\eta=30\} = \{HHH, HHT\}$$
$$\{\eta=10\} = \{HTH, HTT\}$$
$$\Rightarrow \{\eta=20\} = \{THH, THT\}$$
$$\{\eta=0\} = \{TTH, TTT\}$$

$$\mathbb{E}(\xi|\{\eta=0\}) = \frac{1}{P\{\eta=0\}} \int_{\{\eta=0\}} \xi(\omega) P(d\omega) =$$

$$= \frac{1}{2/8}\Big[\underbrace{\xi(TTH)}_{50}\underbrace{P(TTH)}_{1/8} + \underbrace{\xi(TTT)}_{0}\underbrace{P(TTT)}_{1/8}\Big] = 25$$

όμοια υπολογίζουμε

$$\mathbb{E}(\xi|\eta)(\omega) = \begin{cases} 25, & \eta(\omega)=0 \\ 35, & \eta(\omega)=10 \\ 45, & \eta(\omega)=20 \\ 55, & \eta(\omega)=30 \end{cases}$$

$$\begin{aligned} &\{\mathbb{E}(\xi|\eta)=25\} = \{\eta=0\} \\ &\{\mathbb{E}(\xi|\eta)=35\} = \{\eta=10\} \\ &\{\mathbb{E}(\xi|\eta)=45\} = \{\eta=20\} \\ &\{\mathbb{E}(\xi|\eta)=55\} = \{\eta=30\} \end{aligned} \Rightarrow \sigma(\mathbb{E}(\xi|\eta)) = \sigma(\eta)$$

$$\boxed{\text{Προφανώς γενικότερα ισχύει } \sigma(\mathbb{E}(\xi|\eta)) \subseteq \sigma(\eta)}$$

<u>Ασκ</u> Βρείτε την τμ $\mathbb{E}(\xi|\eta)$ εάν

(i) $\eta = c \,, \forall \omega \in \Omega$

(ii) $\eta = \mathbb{1}_B$

(iii) $\eta = \mathbb{1}_B \,;\ \xi = \mathbb{1}_A$, (iv) Δο' $\mathbb{E}[\mathbb{E}(\xi|\eta)] = \mathbb{E}(\xi)$

(i) $\mathbb{E}(\xi|\eta)(\omega) = \mathbb{E}(\xi|\overbrace{\{\eta = c\}}^{\Omega}) = \mathbb{E}(\xi) = \text{const.}$

(ii) $$\mathbb{E}(\xi|\mathbb{1}_B)(\omega) = \begin{cases} \mathbb{E}(\xi|\overbrace{\{\mathbb{1}_B = 1\}}^{B}), & \omega \in B \\ \mathbb{E}(\xi|\underbrace{\{\mathbb{1}_B = 0\}}_{B'}), & \omega \in B' \end{cases} = \begin{cases} \frac{1}{P(B)} \int_B \xi \, dP, & \omega \in B \\ \frac{1}{P(B')} \int_{B'} \xi \, dP, & \omega \in B' \end{cases}$$

(iii) $$\mathbb{E}(\mathbb{1}_A|\mathbb{1}_B)(\omega) = \begin{cases} \frac{1}{P(B)} \int_B \mathbb{1}_A(\omega) P(d\omega), & \omega \in B \\ \frac{1}{P(B')} \int_{B'} \mathbb{1}_A(\omega) P(d\omega), & \omega \in B' \end{cases} = \begin{cases} P(A|B), & \omega \in B \\ P(A|B'), & \omega \in B' \end{cases}$$

$$= P(A|B)\mathbb{1}_B(\omega) + P(A|B')\mathbb{1}_{B'}(\omega) =$$

$$= \{P(A|B) - P(A|B')\}\mathbb{1}_B(\omega) + P(A|B')$$

(iv) (α) $$\mathbb{E}[\mathbb{E}(\xi|\mathbb{1}_B)] = \mathbb{E}\left[\frac{1}{P(B)} \int_B \xi \, dP \, \mathbb{1}_B + \frac{1}{P(B')} \int_{B'} \xi \, dP \, \mathbb{1}_{B'}\right] =$$

$$= \int_B \xi \, dP + \int_{B'} \xi \, dP = \int_\Omega \xi \, dP = \mathbb{E}[\xi]$$

(b) $$\mathbb{E}[\mathbb{E}[\mathbb{1}_A|\mathbb{1}_B]] = P(A|B)P(B) + P(A|B')P(B') =$$

$$= P(AB) + P(AB') = P(A) = \mathbb{E}[\mathbb{1}_A]$$

<u>Ασκ 2.5</u> Εαν η = διακριτή τ/μ δ.ό. $\mathbb{E}[\mathbb{E}(\xi|\eta)] = \mathbb{E}(\xi)$

$$\eta(\Omega) = \{y_1, \dots, y_n\},\ n \le \infty \Rightarrow \Omega = \bigcup_{i=1}^{n} \{\eta = y_i\} : \text{partition}$$

$$\mathbb{E}[\mathbb{E}(\xi|\eta)] = \int_\Omega \mathbb{E}(\xi|\eta)(\omega) P(d\omega) = \sum_{i=1}^{n} \int_{\{\eta = y_i\}} \mathbb{E}(\xi|\eta)(\omega) P(d\omega) =$$

$$= \sum_{i=1}^{n} \mathbb{E}(\xi | \{\eta = y_i\}) \int_{\{\eta = y_i\}} P(d\omega) = \sum_{i=1}^{n} \frac{1}{\cancel{P\{\eta = y_i\}}} \int_{\{\eta = y_i\}} \xi(\omega) P(d\omega) \times \cancel{P\{\eta = y_i\}}$$

$$= \int_\Omega \xi(\omega) P(d\omega) = \mathbb{E}(\xi)$$

<u>Παρατήρηση</u>: Για την συνεχή περίπτωση, πρακτικά έχουμε

$$\mathbb{E}(\mathbb{E}(X|Y)) = \int_Y \mathbb{E}(X|y)\pi(y)dy = \int_Y \left(\int_X x\pi(x|y)dx \right) \pi(y)dy \overset{\text{Fubini}}{=}$$

$$= \int_X x \left(\int_Y \pi(x|y)\pi(y)dy \right) dx = \int_X x\pi(x)dx = \mathbb{E}(X)$$

<u>Prop 2.1</u> $\xi \in L^1(\Omega, \mathcal{F}, P)$, η = discrete rv then:

(i) $\mathbb{E}(\xi|\eta)$ είναι $\sigma(\eta)$ μετρήσιμη

(ii) $\forall A \in \sigma(\eta)$ $\int_A \mathbb{E}(\xi|\eta)dP = \int_A \xi dP$

(i) $\{\mathbb{E}(\xi|\eta) = \mathbb{E}(\xi|\{\eta = y_i\})\} \xrightarrow{\varphi} \{\eta = y_i\}$

κ' το φ είναι many to one $\Rightarrow \sigma(\mathbb{E}(\xi|\eta)) \subseteq \sigma(\eta)$

(ii) $A \in \sigma(\eta)$ επειδή $\sigma(\eta) = \sigma\{\{\eta = y_i\} : i = 1, \dots, n\} \Rightarrow$

$\exists I : A = \bigcup_{i \in I} \{\eta = y_i\} \Rightarrow \int_A \mathbb{E}(\xi|\eta)dP = \sum_{i \in I} \mathbb{E}(\xi|\{\eta = y_i\}) \int_{\{\eta = y_i\}} dP =$

$$= \sum_{i\in I} \frac{1}{\cancel{P\{\eta=y_i\}}} \int_{\{\eta=y_i\}} \xi \, dP \times \cancel{P\{\eta=y_i\}} = \int_A \xi \, dP$$

<u>Παρ 2.3</u> Δίνεται $\Omega=[0,1]$, $\mathcal{F}=\mathcal{B}([0,1])$, $P=\lambda=$ the Lebesgue measure

$$\xi(\omega)=2\omega^2, \quad \eta(\omega)=\begin{cases} 1, & \omega\in[0,1/3] \\ 2, & \omega\in(1/2,2/3] \\ 0, & \omega\in(2/3,1] \end{cases}$$

$$\Omega=\{\eta=1\}\cup\{\eta=2\}\cup\{\eta=0\}$$

$$\omega\in[0,1/3] \Rightarrow \mathbb{E}(\xi|\eta)(\omega)=\mathbb{E}(\xi|\{\eta=1\})=\frac{1}{P\{\eta=1\}}\int_{\{\eta=1\}} \xi(\omega)\,\lambda(d\omega)$$

$$=\frac{1}{1/3}\int_0^{1/3} 2\omega^2 d\omega = 2/27$$

$$\omega\in(1/2,2/3] \Rightarrow \mathbb{E}(\xi|\eta)(\omega)=\mathbb{E}(\xi|\{\eta=2\})=\frac{1}{P\{\eta=2\}}\int_{\{\eta=1\}} \xi(\omega)\,\lambda(d\omega)$$

$$=\frac{1}{1/3}\int_{1/3}^{2/3} 2\omega^2 d\omega = 14/27$$

ομοίως για $\omega\in(2/3,1] \Rightarrow \mathbb{E}(\xi|\eta)(\omega)=38/27$

<u>έτσι</u> $$\mathbb{E}(\xi|\eta)(\omega)=\begin{cases} 2/27, & \omega\in[0,1/3] \\ 14/27, & \omega\in(1/2,2/3] \\ 38/27, & \omega\in(2/3,1] \end{cases}$$

<u>Ορ6</u> Εάν $\xi\in L^1(\Omega,\mathcal{F},P)$, $\eta:\Omega\to\mathbb{R}$ τμ, τότε η μέση τιμή της ξ δοθείσας της η ορίζεται σαν η τμ $\mathbb{E}(\xi|\eta)$ τω:

(i) $\mathbb{E}(\xi|\eta)$ = $\sigma(\eta)$-μετρήσιμη $\Leftrightarrow \sigma(\mathbb{E}(\xi|\eta))\subset\sigma(\eta)$

(ii) $\forall A\in\sigma(\eta)$, $\int_A \mathbb{E}(\xi|\eta)\,dP=\int_A \xi\,dP$ $\left(\overset{P(A)>0}{\Leftrightarrow} \mathbb{E}\{\mathbb{E}(\xi|\eta)|A\}=\mathbb{E}\{\xi|A\}\right)$

<u>Ορσ:</u> Η δεσμευμένη πιθ ενός ενδεχομένου $A \in \mathcal{F}$ δοθείσης της τμ η ορίζεται

$$P(A|\eta) = \mathbb{E}[\mathbb{1}_A|\eta] : \Omega \to \mathbb{R}$$

(discrete):

$$P(A|\eta)(\omega) = \mathbb{E}[\mathbb{1}_A|\eta](\omega) \underset{\eta(\omega)=y_i}{=} \mathbb{E}[\mathbb{1}_A|\{\eta=y_i\}] =$$

$$= \frac{1}{P\{\eta=y_i\}}\int_{\{\eta=y_i\}} \mathbb{1}_A(\omega)P(d\omega) = \frac{P(A\cap\{\eta=y_i\})}{P\{\eta=y_i\}} = P(A|\{\eta=y_i\})$$

<u>Ασκ:</u> Εάν ξ, η συνεχείς τμ να βρεθεί η $\mathbb{E}(\xi|\eta=y)$

επειδή $\mathbb{E}(\xi|\eta)$ είναι $\sigma(\eta)$ μετρήσιμη $\Rightarrow \mathbb{E}(\xi|\eta=y)=F(y)$

Έστω $A\in\sigma(\eta) \Leftrightarrow A=\{\eta\in B\}$ κ'

$$\int_{\{\eta\in B\}} \mathbb{E}(\xi|\eta)\,dP = \int_{\{\eta\in B\}} \xi\,dP \Leftrightarrow$$

$$\Leftrightarrow \int_{\{\eta\in B,\ \xi\in\xi(\Omega)\}} \mathbb{E}(\xi|\eta)\,dP = \int_{\{\eta\in B,\ \xi\in\xi(\Omega)\}} \xi\,dP \Leftrightarrow$$

$$\Leftrightarrow \int_{B\ni y}\left(\int_{\xi(\Omega)\ni x} F(y) f_{\xi\eta}(x,y)\,dx\right)dy = \int_{B\ni y}\left(\int_{\xi(\Omega)\ni x} x\, f_{\xi\eta}(x,y)\,dx\right)dy$$

$\forall B\in\mathcal{B}(\mathbb{R})$

$$\Rightarrow F(y) = \frac{\int_{\xi(\Omega)\ni x} x\, f_{\xi\eta}(x,y)\,dx}{\int_{\xi(\Omega)\ni x} f_{\xi\eta}(x,y)\,dx} \quad (14.1)$$

Παρατήρηση $(14.1) \Rightarrow F(y) = \int_{\xi(\Omega)\ni x} x \, \frac{f_{\xi\eta}(x,y)}{f_\eta(y)} \, dx = \underbrace{\int_{\xi(\Omega)\ni x} x \, f_{\xi|\eta}(x|y) \, dx}_{E(\xi|\eta=y)}$

Λεμμα 2.1 Έστω $(\Omega, \mathcal{F}, P)$ χώρος π.π. κ' $\mathcal{G}$ σ-field με $\mathcal{G} \subset \mathcal{F}$. Εαν ξ τ.μ. που είναι $\mathcal{G}$-μετρήσιμη κ' ισχύει ότι $\forall B \in \mathcal{G}$, $\int_B \xi \, dP = 0$ τότε $\xi = 0$ P-a.s.

Πρώτα δείχνουμε ότι $|\xi| < \epsilon$, $\forall \epsilon > 0$ P-a.s.

$$\left.\begin{array}{l} 0 \le \epsilon \cdot P\{\xi \ge \epsilon\} = \int_{\{\xi \ge \epsilon\}} \epsilon \cdot P(d\omega) \le \int_{\{\xi \ge \epsilon\}} \xi(\omega) P(d\omega) \underset{\{\xi \ge \epsilon\} \in \mathcal{G}}{=} 0 \\ 0 \ge -\epsilon P\{\xi \le -\epsilon\} = \int_{\{\xi \le -\epsilon\}} (-\epsilon) \cdot P(d\omega) \ge \int_{\{\xi \le -\epsilon\}} \xi(\omega) P(d\omega) \underset{\{\xi \le -\epsilon\} \in \mathcal{G}}{=} 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow P\{|\xi| \ge \epsilon\} = 0 \Leftrightarrow P\{|\xi| < \epsilon\} = 1 \quad (15.1)$$

$$A_n = \left\{-\tfrac{1}{n} < \xi < \tfrac{1}{n}\right\} \searrow \Rightarrow \bigcap_{n=1}^{\infty} \left\{-\tfrac{1}{n} < \xi < \tfrac{1}{n}\right\} = \{\xi = 0\}$$

$$\left.\begin{array}{l} \lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} \left\{-\tfrac{1}{n} < \xi < \tfrac{1}{n}\right\}\right) = P\{\xi = 0\} \\ \epsilon = 1/n \overset{(15.1)}{\Rightarrow} P(A_n) = 1 \ \forall n \end{array}\right\} \Rightarrow \boxed{P\{\xi = 0\} = 1}$$

Θέτω $\zeta = \xi - \xi'$ τότε εαν $\zeta = 0$ P-a.s $\Rightarrow E(\zeta|\eta) = 0$ P-a.s

$$\forall B \in \sigma(\eta) = \mathcal{G} \subset \mathcal{F} \overset{def}{\Rightarrow} \int_B E(\zeta|\eta) \, dP = \int_B \zeta \, dP = 0 \quad \forall B \in \sigma(\eta) \overset{(\text{Lem}2.1)}{\Rightarrow}$$

$$\Rightarrow E(\zeta|\eta) = 0 \ P\text{-a.s} \Leftrightarrow E(\xi|\eta) = E(\xi'|\eta) \ P\text{-a.s}$$

Ασκ (i) Έστω $\xi, \eta \in L^1(\Omega, \mathcal{F}, P)$ με joint
$f_{\xi\eta}(x,y) = \frac{3}{2}(x^2+y^2)\mathbb{1}((x,y)\in[0,1]^2)$ να βρεθεί η $\mathbb{E}(\xi|\eta)$

(ii) Δίνεται $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$, $\mathcal{F} = \mathcal{B}(\Omega)$ κ/
$\forall A\in\mathcal{F}$ $P(A) = \pi^{-1}\iint_A dxdy$ (το κανονικοποιημένο
$\xi(\omega) = x$, $\eta(\omega) = y$, $\forall \omega=(x,y)\in\Omega$ Lebesgue measure)
Βρείτε $\mathbb{E}(\xi^2|\eta)$

$\Updownarrow$

$(\xi,\eta)\sim\mathcal{U}(\cdot|S^1)$

(i) $\sigma(\mathbb{E}(\xi|\eta))\subseteq\sigma(\eta) \Rightarrow \mathbb{E}(\xi|\eta) = F(\eta)$

$$\forall A = \{\eta\in B\}\in\sigma(\eta),\ \int_{\{\eta\in B\}}\mathbb{E}(\xi|\eta)\,dP = \int_{\{\eta\in B\}}\xi\,dP \Rightarrow$$

$$\Rightarrow \int_{\{\eta\in B,\ \xi\in\xi(\Omega)\}}\mathbb{E}(\xi|\eta)\,dP = \int_{\{\eta\in B,\ \xi\in\xi(\Omega)\}}\xi\,dP \Leftrightarrow \int_{B\ni y}\int_0^1 F(y)\Big(\underbrace{\tfrac{3}{2}(x^2+y^2)\,dx\,dy}_{P_{\xi\eta}(dx,dy)}\Big) =$$

$$= \int_{B\ni y}\int_0^1 x\left(\tfrac{3}{2}(x^2+y^2)\,dx\,dy\right),\ \forall B\in\mathcal{B}(\mathbb{R}) \Rightarrow$$

$$\Rightarrow F(y) = \frac{\int_0^1 \frac{3}{2}(x^3+xy^2)\,dx}{\int_0^1\frac{3}{2}(x^2+y^2)\,dx} = \frac{3+6y^2}{4+12y^2}$$

(ii) $\sigma(\mathbb{E}(\xi^2|\eta))\subseteq\sigma(\eta) \Rightarrow \mathbb{E}(\xi^2|\eta) = F(\eta)$

$$\forall A = \{\eta\in B\}\in\sigma(\eta),\ \int_{\{\eta\in B\}}\mathbb{E}(\xi^2|\eta)\,dP = \int_{\{\eta\in B\}}\xi^2\,dP \Leftrightarrow$$

$$\Leftrightarrow \int_{B\ni y}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} F(y)\left(\tfrac{1}{\pi}\,dx\,dy\right) = \int_{B\ni y}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2\left(\tfrac{1}{\pi}\,dx\,dy\right) \Rightarrow$$

$$\Rightarrow F(y) = \frac{\frac{1}{3\pi}\left[x^3\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}}{\frac{1}{\pi}\left[x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}} = \frac{\frac{2}{3\pi}(1-y^2)^{3/2}}{\frac{2}{\pi}(1-y^2)^{1/2}} = \frac{1}{3}(1-y^2)$$