Asymptotic analysis and estimates of blow-up time for the radial symmetric semilinear heat equation in the open-spectrum case

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Abstract

We estimate the blow-up time for the reaction diffusion equation $u_t = \Delta u + \lambda f(u)$, for the radial symmetric case, where f is a positive, increasing and convex function growing fast enough at infinity. Here $\lambda > \lambda^*$, where λ^* is the "extremal" (critical) value for λ, such that there exists an "extremal" weak but not a classical steady-state solution at $\lambda = \lambda^*$ with $||w(\cdot,\lambda)||_\infty \to \infty$ as $0 < \lambda \to \lambda^*-$. Estimates of the blow-up time are obtained by using comparison methods. Also an asymptotic analysis is applied when $f(s) = e^s$, for $\lambda - \lambda^* \ll 1$, regarding the form of the solution during blow-up and an asymptotic estimate of blow-up time is obtained. Finally some numerical results are also presented.

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1 Introduction

It is known, [15], that the solution to the initial-boundary value problem

$$u_t(x,t) = \Delta u(x,t) + \lambda f(u(x,t)), \quad x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \tag{1.1a}$$

$$u(x,t) = 0\,,\ x \in \partial\Omega, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1.1c}$$

where $u_0 \in L^\infty(\Omega)$, Ω is a bounded domain with smooth boundary, and f satisfies

$$f(s) > 0, \quad f'(s) > 0, \quad f''(s) > 0, \quad \text{for} \quad s \geq 0, \tag{1.2a}$$

$$\int_a^\infty \mathrm{d}s/f(s) < \infty, \tag{1.2b}$$

for some a in $\mathbb{R}$, blows up in finite time, say t^*, when $\lambda > \lambda^* > 0$. Typical examples of functions satisfying these properties and arising in applications, see [8, 11] and the references there in, are: $f(s) = e^s$, $f(s) = (1+s)^p$ for $p > 1$. In this introduction we present a number of known results, some of which are needed for our analysis in the following sections.

The "extremal" critical value of the parameter λ, say λ^*, is such that the associated steady-state problem

$$\Delta w(x) + \lambda f(w(x)) = 0, \ \ x \in \Omega \subset \mathbb{R}^N, \quad w(x) = 0, \ \ x \in \partial\Omega, \tag{1.3}$$

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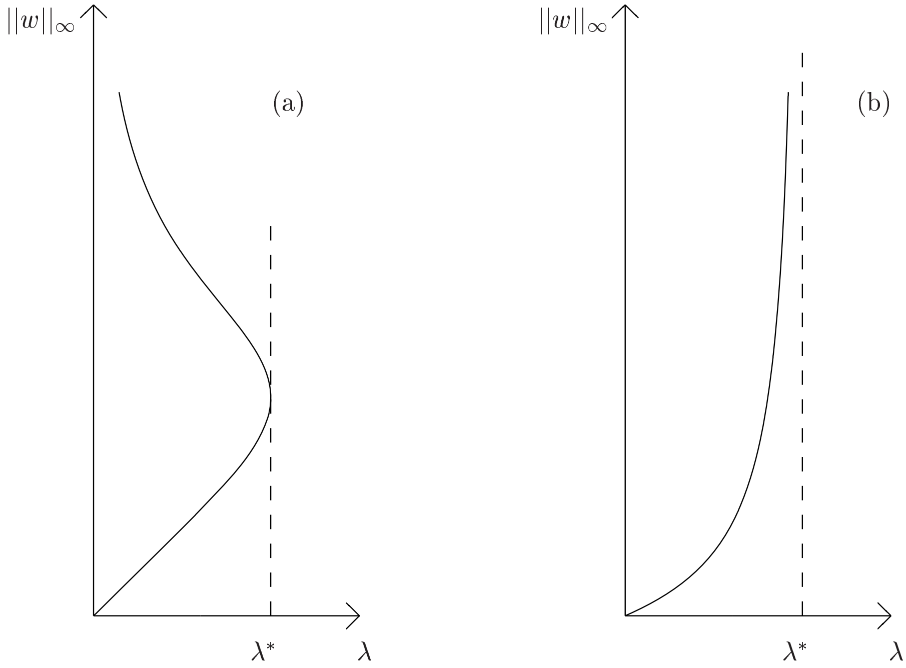

Figure 1: *The response diagram for problem* (1.3)*:*
(a) the case of a closed spectrum, (b) the case of an open spectrum.

for some f satisfying (1.2a), has a unique (classical) solution for $0 < \lambda < \lambda^*$ and no solution of any kind for $\lambda > \lambda^*$, [4, 13].

The behaviour of the extremal solution $w^*(x) = w(x; \lambda^*)$ depends on the dimension N. In particular, in case where $\Omega = B(0,1)$, $N \leq 9$ and f satisfies (1.2a)-(1.2b) then $w^* \in L^\infty(\Omega)$, i.e. $w^*(x)$ is a classical solution of (1.3), [6]. Moreover, for any smooth bounded domain Ω and $f(s) = \mathrm{e}^s$ the extremal solution w^* is bounded if $N < 10$, see [8, 18]. This is the so called closed-spectrum case, see Fig. 1(a). In this work we consider the case that there does not exist a classical steady-state solution at $\lambda = \lambda^*$ and $||w(\cdot, \lambda)||_\infty \to \infty$ as $\lambda \to \lambda^*-$, Fig. 1(b), (see [2, 5, 8, 11, 13]). Actually this is the case for the radially symmetric problem and for $f(s) = e^s$ when $N = \dim\{\Omega\} \geq 10$ or for $f(s) = (1+s)^p$, when $N > 10$ and $p \geq \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$, [5, 11]. Recently it was proved, see [9], that the same result holds for $N \geq 11$ even in the case of a smooth bounded domain diffeomorphic to unit ball $B(0,1)$.

In this case, the so called open-spectrum case, see Fig. 1(b), it has been proved that problem (1.3) has a weak solution ("limiting" singular solution) of the form,

$$w^*(x) = \lambda^* \int_\Omega G(x,y) f(w^*(y))\,\mathrm{d}y, \tag{1.4}$$

where G is the Green's function for the Laplacian (actually of $-\Delta$) with Dirichlet boundary conditions, [16].

A weaker formulation, introduced in [4], for the problem satisfied by w^*, taking it to be an $L^1(\Omega)$-function such that $f(w^*)\delta \in L^1(\Omega)$ for $\delta = \delta(x) = \mathrm{dist}(x, \partial\Omega)$, is

$$-\int_\Omega w^* \Delta\zeta \,\mathrm{d}x = \lambda^* \int_\Omega f(w^*)\,\zeta\,\mathrm{d}x \tag{1.5}$$

for every $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$.

The linearized problem corresponding to (1.3) for the eigenfunction $\phi = \phi(x; \lambda)$, $\lambda < \lambda^*$, has the form

$$\Delta\phi + \lambda f'(w)\phi + \mu\phi = 0 \text{ for } x \in \Omega\,, \qquad \phi = 0 \text{ for } x \in \partial\Omega\,. \tag{1.6}$$

Following (1.4), a weak formulation of this problem is

$$\phi(x) = \int_\Omega G(x,y)[\lambda f'(w(y)) + \mu]\,\phi(y)\,\mathrm{d}y, \quad \lambda \le \lambda^*, \tag{1.7}$$

while, according to (1.5), a weaker formulation than (1.7) is

$$-\int_\Omega \phi\Delta\zeta\,\mathrm{d}x = \int_\Omega [\lambda\, f'(w) + \mu]\,\phi\,\zeta\,\mathrm{d}x, \quad \lambda \le \lambda^*, \tag{1.8}$$

for every $\zeta \in C^2(\overline{\Omega})$, with $\zeta = 0$ on $\partial\Omega$ such that $(\lambda f'(w) + \mu)\,\phi\,\delta \in L^1(\Omega)$. For each $\lambda \in (0, \lambda^*)$ the corresponding principal eigenvalue $\mu = \mu(\lambda)$ of (1.6) is positive, [8, 13], and $\mu(\lambda) \to \mu(\lambda^*) = \mu^*+ \ge 0$ as $\lambda \to \lambda^*-$, actually $\mu'(\lambda) < 0$, see [8, 13]. In the open-spectrum case, when f satisfies (1.2a), it has been proved that the spectrum of the linearized problem contains a continuum part of the form $[0, \mu^*]$, where $\mu^* > 0$ is the principal eigenvalue, which corresponds to eigenfunction $\phi^* > 0$, and a part consisting of discrete eigenvalues $\mu^* = \mu_1^* < \mu_2^* < \mu_3^*, \cdots$, see [5, 7]. Moreover there exists a positive weak eigenfunction, in the sense of (1.8), say $\widetilde{\phi}(x)$, corresponding to the zero weak eigenvalue. Regarding the regularity of $\widetilde{\phi}(x)$ it is known that $\widetilde{\phi} \in L^q(\Omega)$ for $1 \le q < \frac{N}{N-2}$ with $N \ge 3$, [7].

In the radially symmetric case and for $\lambda > 0$, problem (1.1) takes the form

$$\frac{\partial u}{\partial t} = \Delta_r u + \lambda f(u), \quad 0 < r < 1, \quad t > 0, \quad u = u(r,t), \tag{1.9a}$$

$$u_r(0,t) = u(1,t) = 0, \quad t > 0, \tag{1.9b}$$

$$u(r,0) = u_0(r), \quad 0 < r < 1, \tag{1.9c}$$

with $u_0(r) \ge 0$ and (for simplicity) $u_0'(r) < 0$. Now problem (1.3), for $0 < \lambda < \lambda^*$, is written

$$\Delta_r w(r) + \lambda f(w(r)) = 0,\ \ 0 < r < 1,\ \ w'(0) = w(1) = 0, \tag{1.10}$$

where $\Delta_r(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{N-1}{r}\frac{\partial(\cdot)}{\partial r}$.

The related linearized eigenvalue problem to (1.10) is

$$\Delta_r\phi(r) + \lambda f'(w(r))\phi(r) + \mu\phi(r) = 0,\ \ 0 < r < 1,\ \ \phi'(0) = \phi(1) = 0. \tag{1.11}$$

The problem (1.11) plays an important role (at least in our methodology in which spectral properties are used) in the investigation of upper estimates of the blow-up time, (see [3] for another method of finding upper bounds of t^*). By using monotonicity and positivity, see [16], we obtain the following limits: as $\lambda \to \lambda^*-$ then $u(r,t;\lambda) \to u(r,t;\lambda^*) = u^*(r,t)-$ (this solution is regular for $t < \infty$, while $||u^*(r,t)||_\infty = u^*(0,t) \to \infty$ as $t \to \infty$) and $w(r;\lambda) \to w(r;\lambda^*) = w^*(r)-$ for $r \in (0,1]$, where $w^*(r)$ is the singular steady-state solution of (1.10) ($||w(\cdot;\lambda)||_\infty = w(0;\lambda) \to \infty$ as $\lambda \to \lambda^*-$); the same is true for the solution of problem (1.11). In addition, we have to notice that $|w'(0;\lambda)|,\ |\phi'(0;\lambda)| \to \infty$ as $\lambda \to \lambda^*-$ while $w'(0;\lambda) = 0$ and $\phi'(0;\lambda) = 0$ for $\lambda < \lambda^*$, with the latter relations indicating that solutions are bounded and hence regular. Taking $\lambda \to \lambda^*-$, we obtain the "limiting" problems denoted by $(\ )^*$: $(1.9)^*, (1.10)^*$ and $(1.11)^*$, (of course without the boundary condition at $r = 0$ for problems $(1.10)^*$, $(1.11)^*$). Furthermore, in the radially symmetric case and for functions f such that $f'(w^*) = c/r^2$, for some constant c, the principal eigenvalue μ^* is provided as the square of the first zero of the Bessel function of the first kind, i.e. $J_\nu(\sqrt{\mu^*}) = 0$ for $\nu = \sqrt{\frac{(N-2)^2}{4} - \lambda^*}$, see [7]. More precisely, in the case that $f(s) = \mathrm{e}^s$ and $N \ge 10$, we have $w^*(r) = -2\ln r$ and $f'(w^*) = c/r^2 = 1/r^2$, for $\lambda^* = 2(N-2)$. While for $f(s) = (1+s)^p$, $N > 10$, $p \ge \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$, we have $w^*(r) = -1 + r^{-2/(p-1)}$ and $f'(w^*) = c/r^2 = p/r^2$, for $\lambda^* = \frac{2}{p-1}(N - \frac{2p}{p-1})$, $p > 1$. Moreover, in the case where $f'(w^*) = c/r^2$, the asymptotic form near zero of the eigenfunction ϕ^* for the following problem is known, see [7]:

$$\frac{d^2\phi^*}{dr^2}(r) + \frac{N-1}{r}\frac{d\phi^*}{dr}(r) + \left(\frac{\lambda^* c}{r^2} + \mu^*\right)\phi^*(r) = 0\,, \quad 0 < r < 1, \quad \phi^*(1) = 0\,.$$

In particular, for $\lambda^* c < (N-2)^2/4$, (i.e. $N > 10$ for $f(s) = \mathrm{e}^s$) the asymptotic behaviour of $\phi^*(r)$ is given by

$$\phi^*(r) \sim r^{1-\frac{N}{2}+\nu}, \quad 0 < r \ll 1, \tag{1.12}$$

while for $\lambda^* c = (N-2)^2/4$, (i.e. $N = 10$ for $f(s) = e^s$), $\phi^*(r)$ is bounded above by const.$\times r^{1-N/2}|\ln r|$ for $0 < r \ll 1$.

Although cases of dimension $N \geq 10$ are artificial regarding, for example, combustion theory, higher-dimensional models can arise in other areas, such as mathematical finance. It is now well known that systems of parabolic semilinear equations are closely related to backward stochastic differential equations, see [17], which have a wide field of applications, *e.g.* in stochastic optimal control, stochastic games, and the theory of hedging and non-linear pricing theory of imperfect markets. Moreover, the methods developed in this work can be adapted to other open-spectrum models involving second-order nonlinear equations as well as to higher-order semilinear equations, see [14]. The particular case of $N = 10$ for $f(s) = e^s$ turns out to have a new type of asymptotic behaviour which is expected to occur in other nonlinear problems with more obvious physical applications. Furthermore, the asymptotic matching, in this case, is more complicated than others recently studied, see [14], so it is worth studying as a separate case.

In the following, we seek estimates of the blow-up time in the open-spectrum case. Our purpose is to find analytical and asymptotic estimates of the blow-up time in terms of the difference $\lambda - \lambda^* > 0$. The tools we mainly use are comparison techniques and asymptotic methods (e.g. boundary-layer theory). More precisely, by constructing proper upper and lower solutions for some auxiliary problems we obtain rigorous upper and lower bounds for t^* in the form

$$C_1 \ln\left(\ln\left[(\lambda - \lambda^*)^{-1}\right]\right) < t^* < C_2(\lambda - \lambda^*)^{-1/2},$$

while using some formal arguments we derive the asymptotic estimates for t^* of the form: $t^* \sim K_1 \ln\left[(\lambda - \lambda^*)^{-1}\right]$ $+K_2$ for $N > 10$ and $t^* \sim K_1 \ln(1/(\lambda - \lambda^*)) - K_2 \ln|\ln(\lambda - \lambda^*)| + M$ for $N = 10$, as $\lambda \to \lambda^*+$; where C_i, K_i, $i = 1, 2$, M are some positive constants. Our estimates are very different from the estimates were obtained in [15] for the closed-spectrum case and derived by a substantial modification of the methods existing in [15]. Indeed, [15] provides analytic estimates of the form

$$C_3\,(\lambda - \lambda^*)^{-1/2} < t^* < C_4\,(\lambda - \lambda^*)^{-1/2}, \tag{1.13}$$

and the following asymptotic estimate

$$t^* \sim K(\lambda - \lambda^*)^{-1/2} \quad \text{as} \quad \lambda \to \lambda^*+$$

for some positive constants C_3, C_4, K.

The paper is arranged as follows. In Section 2, we prove the existence of an upper bound of the form (1.13) for a general f satisfying (1.2) and of a lower bound of the form (1.13) when $f(s) = \mathrm{e}^s$, for the blow-up time t^*, by using comparison techniques as in [15]; our estimates are very different from those derived in [15] though, see above. In Section 3 we apply formal asymptotic expansion and matching techniques when $f(s) = \mathrm{e}^s$ and $\lambda - \lambda^* \ll 1$ in order to obtain the asymptotic form of the solution during blow-up and hence an asymptotic estimate of the blow-up time. For $N = 10$, it seems so far that formal asymptotics provide the only way to get good estimates of the blow-up time, see also Section 5. A numerical estimate is also given in Section 4. Finally we conclude in Section 5, with a discussion regarding the results demonstrated in this work and remaining open problems.

2 Upper and lower bounds for the blow-up time, $\lambda > \lambda^*$

We attempt to obtain lower and upper bounds of the blow-up time t^* for the radially symmetric problem (1.9). In the following we assume that $u_0(r) = u_0^*(r) \leq w^*(r)$ for $0 < r < 1$ and $u_0(1) = 0$.

2.1 Upper bound for t^*, $\lambda > \lambda^*$

In order to obtain an upper bound for t^* we estimate the time that the solution of the problem for $v = u - w^*$ blows up. Note that, before the blow up time, as $r \to 0$, $v = u - w^* \to -\infty$ because u is still bounded near $r = 0$ and $w^*(r) \to \infty$ as $r \to 0$. Following the same ideas as in [15] we obtain

$$\begin{aligned} v_t = u_t &= \Delta_r u + \lambda f(u) - \Delta_r w^* - \lambda^* f(w^*) \\ &= \Delta_r v + \lambda^* f'(w^*)v + (\lambda - \lambda^*) f(u) + \lambda^* \left[f(w^* + v) - f(w^*) - f'(w^*)v \right] \\ &= \Delta_r v + \lambda^* f'(w^*)v + (\lambda - \lambda^*) f(u) + \lambda^* f''(z)\frac{v^2}{2}, \quad 0 < r < 1, \quad t > 0, \end{aligned} \tag{2.1}$$

for $u < z < w^*$. Set $A(t) = \int_{B(0,1)} \widetilde{\phi}(x) v(x,t)\,\mathrm{d}x = \int_0^1 \widetilde{\phi}(r) v(r,t) \omega_N r^{N-1}\,\mathrm{d}r$ where $\omega_N = \pi^{\frac{N}{2}} / \Gamma(\frac{N}{2})$ is the area of the unit sphere $B(0,1)$ in $\mathbb{R}^N$ and Γ is the Gamma function. Multiplying now (2.1) by $\widetilde{\phi} = \widetilde{\phi}(r)$, using Green's identity and the eigenpair $(0, \widetilde{\phi})$ of the linearized problem, we derive

$$A'(t) \;\; \geq \;\; (\lambda - \lambda^*) I + C_0\, \omega_N \lambda^* \int_0^1 r^{N-1}\, \widetilde{\phi}(r) v^2(r,t)\,\mathrm{d}r, \tag{2.2}$$

for

$$I = \omega_N\, f(0) \int_0^1 r^{N-1} \widetilde{\phi}(r)\,\mathrm{d}r \leq \int_0^1 \omega_N r^{N-1} f(u) \widetilde{\phi}(r)\,\mathrm{d}r \tag{2.3}$$

and $C_0 = \inf_{[0,\infty)} \{ f''(s)/2 \} > 0$. In this case, in contrast to what happens to the closed-spectrum case, [15], it is not obvious that $A(t)$ and I are well defined. Hence we have to impose some additional assumptions on f, see below, as well as needing some regularity results for $\widetilde{\phi}(x)$. In fact I is bounded since $\widetilde{\phi}(r) \in L^q(B(0,1))$ for $1 \leq q < N/(N-2)$. We require the not very restrictive assumption on f:

$$f'(s) \geq s^2 \;\; \text{for} \;\; s \gg 1. \tag{2.4}$$

From the weak formulation of the linearized problem (1.8), see also [18], we have the estimate:

$$\int_\Omega \widetilde{\phi}(x) f'(w^*(x))\,\mathrm{d}x = \int_0^1 \omega_N r^{N-1} \widetilde{\phi}(r) f'(w^*(r))\,\mathrm{d}r < \infty. \tag{2.5}$$

Since $\lim_{r\to 0} w^*(r) = \infty$, there exists $0 < r_0 \ll 1$ such that $w^*(r) \gg 1$ for $r \in [0, r_0]$. Combining assumption (2.4) and estimate (2.5) we infer that

$$\begin{aligned} \int_0^{r_0} r^{N-1} \widetilde{\phi}(r) w^*(r)\,\mathrm{d}r &< \int_0^{r_0} r^{N-1} \widetilde{\phi}(r) (w^*(r))^2\,\mathrm{d}r < \int_0^{r_0} r^{N-1} \widetilde{\phi}(r) f'(w^*(r))\,\mathrm{d}r \\ &< \int_0^1 r^{N-1}\, \widetilde{\phi}(r) f'(w^*(r))\,\mathrm{d}r < \infty \end{aligned} \tag{2.6}$$

and hence

$$\int_0^1 r^{N-1} \widetilde{\phi}(r) w^*(r)\,\mathrm{d}r, \;\; \int_0^1 r^{N-1} \widetilde{\phi}(r) (w^*(r))^2\,\mathrm{d}r < \infty. \tag{2.7}$$

Also, as long as $t < t^*$, u is bounded and hence $\int_0^1 r^{N-1} \widetilde{\phi}(r) u(r,t)\,\mathrm{d}r < \infty$. Using now Jensen's inequality and an appropriate scaling of $\widetilde{\phi}$ so that $\int_0^1 \omega_N r^{N-1} \widetilde{\phi}(r)\,\mathrm{d}r = 1$, we obtain

$$A'(t) \geq (\lambda - \lambda^*)\, I + \Lambda\, A^2(t), \quad t > 0 \quad \text{for} \quad \Lambda = C_0 \lambda^*, \tag{2.8}$$

and

$$A(t) \geq \left[(\lambda - \lambda^*)\, \frac{I}{\Lambda} \right]^{1/2} \tan \left\{ t \left[(\lambda - \lambda^*) I \Lambda \right]^{1/2} - \frac{\pi}{2} \right\}, \tag{2.9}$$

from which we deduce that $A(t)$ must blow up at some time less than $t_u (\lambda - \lambda^*)^{-1/2}$, where $t_u = \pi (I\,\Lambda)^{-1/2}$. Hence it follows that an upper bound for the blow-up time t^* is $t_u (\lambda - \lambda^*)^{-1/2}$. This upper bound is the same as in the case where λ^* belongs to the spectrum of problem (1.10) ("closed-spectrum" case), see [15].

2.2 Lower bound for t^*, $\lambda > \lambda^*$

For the lower bound of t^* we have to construct an upper bound for $u(r,t)$ which stays below the singular steady state $w^*(r)$ for some time. Now the construction of this lower solution differs substantially from those of [15] since in this case $w^*(r)$ is a singular (*i.e.* unbounded) steady state. In the following we focus on the exponential case, $f(u) = \mathrm{e}^u$, although the same arguments can be adapted to the power-law case as well. We write $u(r,t) = u^*(r,t) + u_1(r,t)$ and since $u^*(r,t)$ is bounded from above (actually $u^* < w^*$) for any $t > 0$, see [10, 14, 16], it suffices to obtain an upper bound for $u_1(r,t)$. Also we take $u^*(r,0) = u(r,0)$ to ensure that $u_1(r,0) = 0$. Indeed, $u_1(r,t)$ satisfies

$$u_{1t} - \Delta_r u_1 = (\lambda - \lambda^*)\,\mathrm{e}^{u^*} + \lambda(\mathrm{e}^u - \mathrm{e}^{u^*}) \leq (\lambda - \lambda^*)\mathrm{e}^{u_E} + \lambda \mathrm{e}^u\, u_1,$$

since

$$u^*(r,t) \leq u_E(r,t) = \min\{\beta(t), w^*(r)\},$$

for $\beta(t)$ being $\max_{0\leq r\leq 1} u^*(r,t) = u^*(0,t)$.

Since $u_E(r,t) \leq \beta(t)$ we derive that $u_1(r,t)$ satisfies

$$u_{1t} - \Delta_r u_1 \leq (\lambda - \lambda^*)\mathrm{e}^{\beta(t)} + \lambda \mathrm{e}^{u_E} u_1 \mathrm{e}^{u_1} \leq (\lambda - \lambda^*)\mathrm{e}^{\beta(t)} + \lambda \mathrm{e}^{\beta(t)} u_1 \mathrm{e}^{u_1}.$$

However, when $N > 10$, see [10],

$$\beta(t) \sim Kt + O(1) \quad \text{as} \quad t \to \infty\,, \tag{2.10}$$

for $K = \mu^*/|\gamma_1| > 0$, where $\mu^* > 0$ is the principal eigenvalue of the linearized problem around the singular steady state w^*, while $\gamma_1 = \frac{1}{2}[-N + 2 + \sqrt{(N-2)(N-10)}]$ is a negative constant (see also Section 3 or [10]). In the critical dimension $N = 10$, see [14], we have

$$\beta(t) \sim K_0 t + O(\ln t), \quad \text{as} \quad t \to \infty, \tag{2.11}$$

with $K_0 = \rho_1^2/2$ where ρ_1 is the first zero of the zeroth-order Bessel function, i.e. $J_0(\rho_1) = 0$, since in this case $\mu^* = \rho_1^2$, see [14].

Thus we derive that

$$u_{1t} - \Delta_r u_1 \leq A_1(\lambda - \lambda^*)\mathrm{e}^{Kt} + A_2 \lambda \mathrm{e}^{Kt} u_1 \mathrm{e}^{u_1}, \quad \text{for} \quad 0 < t < \infty$$

for appropriate positive constants A_1, A_2. For the $N = 10$ case, because, of the presence of the $O(\ln)$ term in asymptotic estimate (2.11), a value $K > K_0$ is chosen, ensuring that $e^{Kt} \geq e^{K_0 t + O(\ln t)}$. Now we seek an upper solution of u_1, i.e. a function $z(r,t)$ satisfying

$$z_t - \Delta_r z \geq A_1(\lambda - \lambda^*)\mathrm{e}^{Kt} + A_2 \lambda \mathrm{e}^{Kt} z \mathrm{e}^z.$$

Motivated by the form of the solution of the corresponding spatially homogeneous problem, we try an upper solution of the form

$$z(r,t) = a\,(\lambda - \lambda^*) \exp\left(\frac{A_2 C \lambda}{K_1}\,\mathrm{e}^{K_1 t}\right) W(r),$$

where a and C are positive constants to be determined below, $K_1 > K$ and $W(r) = \ln(\frac{1+b}{r^2+b})$, for $b > 0$ sufficiently small. The spatial component $W(r)$ of $z(r,t)$ satisfies

$$\begin{aligned} \Delta_r W &= W_{rr} + \frac{N-1}{r} W_r = \frac{4r^2 - 2r^2 N - 2bN}{(r^2+b)^2} \leq \frac{4}{r^2+b} - \frac{2N}{r^2+b} \\ &\leq -\frac{2(N-2)}{1+b} \quad \text{for} \quad 0 < r < 1. \end{aligned} \tag{2.12}$$

Hence

$$
\begin{aligned}
& z_t - \Delta_r z - A_1(\lambda-\lambda^*)\mathrm{e}^{Kt} - A_2\lambda \mathrm{e}^{Kt} z \mathrm{e}^{z} \\
& \geq (\lambda-\lambda^*)\exp\left(\frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right)\left[\frac{a(2N-4)}{1+b} - A_1\exp\left(Kt - \frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right)\right] \\
& + A_2\lambda a(\lambda-\lambda^*)\exp\left(\frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right) W(r)\left[C\mathrm{e}^{K_1 t} - \mathrm{e}^{Kt}\exp\left(a(\lambda-\lambda^*)\exp\left(\frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right) W(r)\right)\right] \\
& \geq (\lambda-\lambda^*)\exp\left(\frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right)\left[\frac{a(2N-4)}{1+b} - A_1 C_1\right] \\
& + A_2\lambda a(\lambda-\lambda^*)\exp\left(\frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right) W(r)\left[C\mathrm{e}^{K_1 t} - C\mathrm{e}^{Kt}\right] \geq 0,
\end{aligned}
$$

provided that a is chosen so that $a > A_1 C_1(1+b)/(2N-4)$ where

$$C_1 = \sup_{t>0}\exp\left(Kt - \frac{A_2 C\lambda}{K_1}\,\mathrm{e}^{K_1 t}\right) > 0,$$

and

$$a(\lambda-\lambda^*) M \exp\left(\frac{A_2 C\lambda}{K_1}\mathrm{e}^{K_1 t}\right) \leq \widetilde{C} = \ln C,$$

for $M = \max_{[0,1]} W(r)$. The latter holds for

$$t \leq t_l = \frac{1}{K_1}\ln\left(C_2\ln\left(\frac{\widehat{C}}{(\lambda-\lambda^*)}\right)\right) \sim \frac{1}{K_1}\ln\left(\ln\left(\frac{1}{(\lambda-\lambda^*)}\right)\right),$$

for $(\lambda-\lambda^*) \ll 1$, where

$$C_2 = C_2(\lambda) = K_1/\lambda C A_2, \quad \widehat{C} = \widetilde{C}/aM \tag{2.13}$$

and $C > 1$. Clearly $z(r,t)$ satisfies the boundary conditions, i.e. $z_r(0,t) = z(1,t) = 0$. Moreover $z(r,0) = a(\lambda-\lambda^*)\mathrm{e}^{CA_2\lambda/K_1} W(r) \geq 0 = u_1(r,0) = u(r,0) - u^*(r,0)$. Therefore $z(r,t)$ is a bounded upper solution of u_1 for $t \leq t_l$ and hence t_l is a lower bound for the blow-up time t^*.

We have finally proved:

Theorem 2.1 *For $f(s)$ satisfying (2.4) the solution of problem (1.9) blows up at a time t^* with*

$$t^* \leq \pi(I\Lambda(\lambda-\lambda^*))^{-1/2},$$

where I and Λ are given by (2.3), (2.8) respectively.

For $f(s) = e^s$ the solution of (1.9) can not blow up before a time t_l :

$$t^* \geq t_l = \frac{1}{K_1}\ln\left(C_2\ln\left(\frac{\widehat{C}}{(\lambda-\lambda^*)}\right)\right),$$

where $C_2, \widehat{C}$ are given by (2.13) and $K_1 > K$.

3 Asymptotic estimate for t^*, $0 < \lambda - \lambda^* \ll 1$

We seek an asymptotic expansion of u satisfying problem (1.9) for λ close to λ^*, provided that $u_0(r) = u_0^*(r) \leq w^*(r)$ for $0 < r < 1$ and $u_0(1) = 0$, when $f(s) = \mathrm{e}^s$. This will allow us to obtain an asymptotic estimate for t^*. We know that in this case u performs a single-point blow-up at $r = 0$ and a boundary layer is formed near the origin $r = 0$.

3.1 Case $N > 10$

Motivated by [10, 15], for $N > 10$ we consider three important time-zones for the behaviour of the solution u as it blows up.

Zone I: In this zone the solution u starts at its initial condition u_0 and approaches u^* while time varies by $O(1)$. For $0 < \delta = \lambda - \lambda^* \ll 1$ we have that the problem takes the form

$$u_t = \Delta_r u + (\lambda^* + \delta)\mathrm{e}^u. \tag{3.1}$$

As in [10] we consider the problem near $r = 0$, seeking an inner approximation of u. Setting $u(r,t) \sim u_a(r,t) + \delta u_1(r,t) + \cdots$ in (3.1) we obtain

$$u_{a\,t} = \Delta_r u_a + \lambda^* \mathrm{e}^{u_a}. \tag{3.2}$$

We assume that $u_a(r,t) = a(t) + z(r\mathrm{e}^{a(t)/2}) = a(t) + z(\xi)$ for $\xi = r\mathrm{e}^{a(t)/2}$. In addition, $\frac{\partial u_a}{\partial r}(0,t) = 0$, $u_a(0,t) = a(t)$, where $a(t)$ is a function to be determined by the matching condition with the outer solution. Then as long as $da/dt \ll \mathrm{e}^a$, z, in terms of the variable ξ, should satisfy the equation

$$\frac{da(t)}{dt}\left(1 + \frac{r\mathrm{e}^{a(t)/2}}{2} z'(\xi)\right) = \mathrm{e}^{a(t)}\left(z''(\xi) + \frac{N-1}{\xi} z'(\xi) + \lambda^* \mathrm{e}^{z(\xi)}\right). \tag{3.3}$$

Thus, after dropping the $O(da/dt)$ term, we obtain the approximate problem

$$z''(\xi) + \frac{N-1}{\xi} z'(\xi) + \lambda^* e^{z(\xi)} = 0\,, \quad 0 < \xi < 1\,, \quad z'(0) = z(0) = 0, \tag{3.4}$$

with $z'(\xi) = \frac{dz(\xi)}{d\xi}$. The solution to (3.4), according to the analysis applied in [10], satisfies

$$z(\xi) \sim -2\ln\xi - b_0 \xi^{\gamma_1} \quad \text{as} \quad \xi \to \infty, \quad \text{or}$$

$$z(r,t) \sim -a(t) + \ln(1/r^2) - b_0 \mathrm{e}^{a(t)\gamma_1/2} r^{\gamma_1}, \quad \text{as} \quad \xi = r\mathrm{e}^{a(t)/2} \to \infty,$$

with b_0 a positive constant depending on the dimension N of the domain Ω. Here $\gamma_1 = \frac{1}{2}[(-N + 2 + \sqrt{(N-2)(N-10)}\,] < 0$, is the largest root of the equation

$$\gamma(\gamma - 1) + (N-1)\gamma + 2(N-2) = 0. \tag{3.5}$$

Then the inner solution of the problem in this zone gives

$$u \sim u_a \sim \ln\left(1/r^2\right) - b_0 \mathrm{e}^{a(t)\gamma_1/2} r^{\gamma_1} \quad \text{as} \ \ r\mathrm{e}^{a(t)/2} \to \infty. \tag{3.6}$$

Regarding the outer solution of equation (3.1), we take $u(r,t) \sim w^*(r) + v_1(r,t) + \cdots$, with $w^*(r) = \ln\left(\frac{1}{r^2}\right)$, while v_1 should satisfy

$$v_{1\,t} = \Delta_r v_1 + \lambda^* \mathrm{e}^{w^*} v_1. \tag{3.7}$$

Therefore, by using an eigenfunction approximation for v_1 ([10]), we deduce that

$$v_1 = v_1(r,t) = (-C_0 \mathrm{e}^{-\mu^* t} \phi_1^*(r))(1 + o(1)), \tag{3.8}$$

uniformly in $\{\epsilon \le r \le 1\}$, $\epsilon > 0$, for $\mu^* = \mu^*(N) > 0$ the principal eigenvalue of the problem

$$\Delta_r \phi^* + \lambda^* \mathrm{e}^{w^*} \phi^* + \mu\phi^* = 0, \ \ \phi^*(1) = 0\,,$$

and $\phi_1^* > 0$ the corresponding eigenfunction. Moreover, the constant $C_0 = C_0(u_0^*) > 0$ is such that $||u^*(\cdot,t) - w^*(\cdot)||_2 = \mathrm{e}^{-\mu^* t}(C_0 + o(1))$, [10]. Thus we derive the asymptotic form of u:

$$u(r,t) \sim \ln\left(1/r^2\right) - C_0 \mathrm{e}^{-\mu^* t} \phi_1^*(r) \ \ \text{as} \ \ r \to 0+, \tag{3.9}$$

where $\phi_1^*(r) \sim a_1 r^{\gamma_1}$ as $r \to 0+$, with $a_1 > 0$ a constant determined by an appropriate normalization.

In order to obtain a uniform expansion for $r \in [0,1]$ we match relations (3.6) and (3.9) and end up with

$$-b_0 e^{a(t)\gamma_1/2} r^{\gamma_1} \sim -C_0 a_1 e^{-\mu^* t} r^{\gamma_1} \quad \text{as} \quad r \to 0+ \quad (\text{and } \xi \to \infty). \tag{3.10}$$

Thus we must have, by equating the dominant terms, of $O(r^{\gamma_1})$,

$$a(t) = -\frac{2}{\gamma_1}\left(\mu^* t - \ln\left(\frac{C_0 a_1}{b_0}\right)\right), \tag{3.11}$$

This analysis holds as long as $e^{-\mu^* t} \gg \delta$ and the asymptotic expansion fails when $e^{-\mu^* t} \sim \delta$, i.e. for time $t \sim \frac{1}{\mu^*}\ln(\frac{1}{\delta})$. For such large times the form of the outer solution changes.

Zone II: In this zone we have that $e^{-\mu^* t} \sim O(\delta)$ and we use the new time variable $\tau = t - \frac{1}{\mu^*}\ln(\frac{1}{\delta})$.

For the inner solution we apply the same analysis as for zone I to obtain

$$u(r,\tau) \sim \ln(1/r^2) - b_0 e^{a(\tau)\gamma_1/2} r^{\gamma_1} \quad \text{as} \quad r e^{a(\tau)/2} \to \infty.$$

In the outer region we take $u(r,\tau) \sim w^*(r) + \delta v(r,\tau) + \cdots$. Then (3.1) becomes

$$\delta v_\tau \sim \Delta_r w^* + \delta \Delta_r v + \lambda^* e^{w^*} + \delta \lambda^* e^{w^*} v + \delta e^{w^*},$$

and from $O(\delta)$ terms we get

$$v_\tau = \Delta_r v + \lambda^* e^{w^*} v + e^{w^*}.$$

Setting $v = \theta + V$, with $\theta = \theta(r)$ satisfying the equation

$$\Delta_r \theta + \lambda^* e^{w^*}\theta + e^{w^*} = 0,$$

we deduce that $\theta = \frac{1}{\lambda^*}(r^{\gamma_1} - 1)$, and V should satisfy

$$V_\tau = \Delta_r V + \lambda^* e^{w^*} V, \tag{3.12}$$

with initial condition, for $\tau \to -\infty$, given by matching with (3.8):

$$V = V(r,\tau) \sim -C_0 \phi_1^*(r) e^{-\mu^* \tau} \qquad \text{as } \tau \to -\infty. \tag{3.13}$$

In order to match v with the inner solution, we must have as $r \to 0+$

$$-b_0 e^{a(\tau)\gamma_1/2} r^{\gamma_1} \sim \delta\left(\frac{1}{\lambda^*} r^{\gamma_1} + V(r,\tau)\right),$$

in other words,

$$V(r,\tau) \sim -\left(\frac{1}{\lambda^*} + \frac{b_0}{\delta} e^{a(\tau)\gamma_1/2}\right) r^{\gamma_1} \qquad \text{as } r \to 0+. \tag{3.14}$$

The solution to equation (3.12) with initial condition (3.13) satisfying the boundary conditions $V = 0$ on $r = 1$ and $V = O(r^{\gamma_1})$ as $r \to 0$ is simply

$$V(r,\tau) = -C_0 \phi_1^*(r) e^{-\mu^* \tau}. \tag{3.15}$$

For consistency between (3.14) and (3.15),

$$-C_0 \phi_1^*(r) e^{-\mu^* \tau} \sim -\left(\frac{1}{\lambda^*} + \frac{b_0}{\delta} e^{a(\tau)\gamma_1/2}\right) r^{\gamma_1}$$

so, to leading order,

$$\frac{b_0}{\delta} e^{a(\tau)\gamma_1/2} = a_1 C_0 e^{-\mu^* \tau} - \frac{1}{\lambda^*}.$$

It follows that

$$a(\tau) \sim -\frac{2}{\gamma_1}\ln\left(\frac{b_0/\delta}{C_0 a_1 \mathrm{e}^{-\mu^*\tau} - 1/\lambda^*}\right). \tag{3.16}$$

This clearly matches with the solution in Zone I, (3.11), since (3.16) gives $a \sim -\frac{2}{\gamma_1}(\ln\frac{b_0}{\delta} - \ln(C_0 a_1) + \mu^*\tau)$ as $\tau \to -\infty$. We also see that as $\tau \to \tau^* = \frac{1}{\mu^*}\ln(C_0 a_1 \lambda^*)$, so that $C_0 a_1 \mathrm{e}^{-\mu^*\tau} \sim (1 + \mu^*(\tau^* - \tau) + \ldots)/\lambda^*$, a blows up logarithmically:

$$a(\tau) \sim \frac{2}{\gamma_1}\ln\left(\frac{\mu^*\delta(\tau^* - \tau)}{b_0\lambda^*}\right) \to \infty \quad \text{as} \quad \tau \to \tau^* - .$$

We now see that in terms of the original time variable, t, the blow-up time is

$$t^* \sim \frac{1}{\mu^*}\ln\left(\frac{1}{\delta}\right) + \tau^* = \frac{1}{\mu^*}\ln\left(\frac{\lambda^* C_0 a_1}{\delta}\right). \tag{3.17}$$

The contribution to the estimate of t^* from the final time regime, zone III, where the relation $\frac{da}{dt} \ll \mathrm{e}^a$ does not hold any more, is less important as we can see from the following analysis.

Zone III: The expression for the inner solution in the analysis presented for zone I and II is not valid any more since in this zone the relation $\frac{\mathrm{d}a}{\mathrm{d}t} \sim \mathrm{e}^a$ (giving the blow-up) holds and hence

$$-\frac{2}{\gamma_1(\tau^* - \tau)} \sim \mathrm{e}^{c_1}(\tau^* - \tau)^{2/\gamma_1},$$

for $c_1 = \ln(\frac{\mu^*\delta}{b_0\lambda^*})^{\frac{2}{\gamma_1}}$. Then the form of the solution in the inner region of zone II becomes invalid for $(\tau^* - \tau) = O\big((\delta^{-2/\gamma_1}(\tau^* - \tau)^{-2/\gamma_1})\big)$ or for $(\tau^* - \tau) = O(\delta^{-\frac{2}{(2+\gamma_1)}})$. In zone III we have to rescale time by taking $\tau = \tau^* + \delta^{-\frac{2}{(2+\gamma_1)}}s$, with $s < 0$ the new time variable. (Note that $\gamma_1 < -2$ so that $\delta^{-\frac{2}{(2+\gamma_1)}} \ll 1$ as $\delta \ll 1$.) The corresponding scaling for r will be $r = \mathrm{e}^{-a/2}R = \delta^{-\frac{1}{(2+\gamma_1)}}R$.

We take, for the inner region,

$$u(r,\tau) \sim a(\tau) + W(r,\tau) \sim \frac{2}{2+\gamma_1}\ln\delta + W(r,\tau). \tag{3.18}$$

Substituting in the equation, we derive

$$\frac{\partial u}{\partial \tau} = \delta^{\frac{2}{2+\gamma_1}}\frac{\partial W}{\partial s} = \Delta_r u + \lambda^*\mathrm{e}^u = \delta^{\frac{2}{2+\gamma_1}}\Delta_R W + \lambda^*\delta^{\frac{2}{2+\gamma_1}}\mathrm{e}^W.$$

Thus W satisfies the equation

$$W_s = \Delta_R W + \lambda^*\mathrm{e}^W. \tag{3.19}$$

In the outer region we consider

$$\begin{aligned} u(r,\tau) &\sim w^*(r) + u_1(r,\tau) + \cdots \sim \ln\left(1/r^2\right) + \delta v(r,\tau) + \cdots \\ &\sim \ln\left(1/r^2\right) + \delta\left(-C_0\phi_1^*\mathrm{e}^{-\mu^*\tau} + (r^{\gamma_1} - 1)/\lambda^*\right) + \cdots \end{aligned}$$

and finally we obtain

$$u(R,s) \sim \frac{2}{2+\gamma_1}\ln\delta - 2\ln R + \frac{\mu^*\,s}{\lambda^*}R^{\gamma_1}. \tag{3.20}$$

Therefore from (3.18) and (3.20), as $R \to \infty$ and $r \to 0$, we obtain the matching condition for the inner and the outer solutions which is

$$W(R,s) \sim -2\ln R + \frac{\mu^* s}{\lambda^*}R^{\gamma_1} \quad \text{for } R \to \infty. \tag{3.21}$$

The initial condition for W is taken from zone II as $s \to -\infty$. Finally, from the equation (3.19) subject to the condition at infinity, (3.21), we deduce that $W(R,s)$ blows up at some finite time, s^*. (We might think of problem (3.19), (3.21) as giving the final blow-up behaviour.) This gives a contribution to the overall blow-up time, t^*, of $s^*\delta^{-\frac{2}{2+\gamma_1}} \ll 1$.

3.2 Case $N = 10$

As we have already mentioned in the Introduction, this is a delicate case and therefore it is treated separately. Since the asymptotics of the approach of $u^*(r,t)$ to the singular steady state $w^*(r)$ can not be fully justified via the perturbation theory of linear operators, see [14], a formal-asymptotics approach is the only way of getting a further understanding of the asymptotic behaviour of $u(r,t;\lambda)$ as $\lambda \to \lambda^*+$.

In the critical dimension $N = 10$, regarding equation (3.1), we again consider three time-zones.

Zone I: In this zone, where u approaches u^* from below and time is varying by $O(1)$, we consider, as in $N > 10$ case, an inner approximation of the form

$$u(r,t) \sim u_a(r,t) + \delta u_1(r,t) + \cdots$$

where $u_a(r,t) = a(t) + z(\xi)$ for $\xi = r\mathrm{e}^{a(t)/2}$. Again $a(t)$ will be determined by matching with the outer solution. Also, we easily get that $z(\xi)$ should satisfy the problem

$$z''(\xi) + \frac{9}{\xi} z'(\xi) + \lambda^* e^{z(\xi)} = 0\,, \quad 0 < \xi < 1\,, \quad z'(0) = z(0) = 0,$$

hence

$$z(\xi) \sim 2\ln\left(\frac{1}{\xi}\right) - \xi^{-4}(b_0 \ln \xi + b_1) \quad \text{as} \quad \xi \to \infty,$$

for some positive constants b_0, b_1, provided that $\frac{da}{dt} \ll \mathrm{e}^a$ holds. The occurrence of the logarithmic term is connected with the fact that $\gamma = 4$ is, in this case, a repeated root of the equation (3.5), see also [14]. Therefore, we end up with an inner approximation of the form

$$u(r,t) \sim \ln\left(\frac{1}{r^2}\right) - \left[\frac{b_0}{2} a(t)\mathrm{e}^{-2a(t)} r^{-4} + (b_0 r^{-4} \ln r + b_1 r^{-4})\mathrm{e}^{-2a(t)} + \cdots\right], \quad r\mathrm{e}^{a(t)/2} \to \infty. \tag{3.22}$$

As an outer approximation we regard

$$u(r,t) \sim \ln\left(\frac{1}{r^2}\right) - v_1(r,t) + \cdots, \tag{3.23}$$

with v_1 satisfying the problem

$$v_{1t} = \Delta_r v_1 + \lambda^* \mathrm{e}^{w^*} v_1. \tag{3.24}$$

In the following we proceed as in [14]. Applying a moderated (because of the appearance of the logarithmic correction term in (3.22)) version of the method of separation of variables we write

$$v_1(r,t) \sim \mathrm{e}^{-\rho^2 t}[\beta(t)\phi_0^*(r) + \dot{\beta}(t)\phi_1^*(r) + \cdots], \tag{3.25}$$

where the constant ρ and the "slowly varying" function $\beta(t)$ will be determined later. Substituting the above expression to (3.24) we obtain that $\phi_0^*(r)$, $\phi_1^*(r)$ satisfy

$$r^2(\phi_0^*)''(r) + 9\,r(\phi_0^*)'(r) + (16 + \rho^2 r^2)\phi_0^*(r) = 0$$

and

$$r^2(\phi_1^*)''(r) + 9\,r(\phi_1^*)'(r) + (16 + \rho^2 r^2)\phi_1^*(r) = r^2\phi_0^*(r).$$

Writing $\phi_0^*(r) = r^{-4}\,\sigma_0(r)$ and $\phi_1^*(r) = r^{-4}\,\sigma_1(r)$, we end up with the equations

$$r^2\sigma_0''(r) + r\sigma_0'(r) + \rho^2 r^2 \sigma_0(r) = 0, \tag{3.26}$$

$$r^2\sigma_1''(r) + r\sigma_1'(r) + \rho^2 r^2 \sigma_1(r) = r^2\sigma_0(r). \tag{3.27}$$

Equation (3.26) is Bessel's equation of order zero, so $\sigma_0(r) = J_0(\rho_1\, r)$ for ρ_1 being the first zero of the associated Bessel function, $J_0(\rho_1) = 0$. Matching relations (3.22) and (3.23) we derive

$$v_1(r,t) \sim \frac{b_0}{2} a(t)\mathrm{e}^{-2a(t)} r^{-4} + (b_0 r^{-4} \ln r + b_1 r^{-4})\mathrm{e}^{-2a(t)} \quad \text{as} \quad r \to 0+, \tag{3.28}$$

while leading-order matching of (3.25) and (3.28), taking also into account that $J_0(r) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2}(\frac{r}{2})^{2j}$, requires

$$\beta(t)\mathrm{e}^{-\rho_1^2 t} \sim \frac{b_0}{2} a(t)\mathrm{e}^{-2a(t)}. \tag{3.29}$$

It then follows from (3.26), (3.27) and some well known integral identities for Bessel functions, see for example [1] page 484, that

$$r(\sigma_0\,\sigma_1' - \sigma_1\,\sigma_0') = \int_r^1 s\,\sigma_0^2(s)\,ds = -\frac{1}{2}J_1^2(\rho_1) + \frac{1}{2}r^2\left(J_0^2(\rho_1 r) + J_1^2(\rho_1 r)\right),$$

hence

$$\sigma_1(r) = \frac{1}{2}\,J_1^2(\rho_1)\,\ln r + O(1) \quad \text{as} \quad r \to 0+$$

and matching with the logarithmic term in (3.28) yields

$$-\frac{1}{2}\mathrm{e}^{-\rho_1^2 t}\dot{\beta}(t)J_1^2(\rho_1) \sim b_0\mathrm{e}^{-2a(t)}. \tag{3.30}$$

Writing

$$a(t) = \frac{\rho_1^2}{2}t + a_1(t), \tag{3.31}$$

the dominant balances are provided by

$$\beta(t) \sim \frac{b_0\rho_1^2}{4}\,t\mathrm{e}^{-2\,a_1(t)}, \qquad \dot{\beta}(t) \sim \frac{-2b_0}{J_1^2(\rho_1)}\mathrm{e}^{-2\,a_1(t)}$$

and hence

$$\beta(t) \sim \beta_\infty\,t^{-\frac{8}{\rho_1^2\,J_1^2(\rho_1)}}, \tag{3.32}$$

for some constant β_∞ depending on the initial data. The latter implies that

$$a_1(t) \sim \frac{1}{2}\left(1 + \frac{8}{\rho_1^2\,J_1^2(\rho_1)}\right)\ln t + \frac{1}{2}\ln\left(\frac{b_0\rho_1^2}{\beta_\infty}\right). \tag{3.33}$$

Thus, from (3.31) and (3.33), we infer that $u(0,t)$ grows linearly as $t \to \infty$ but with a logarithmic correction term in contrast to what happens for $N > 10$.

The asymptotic behaviour of $\dot{\beta}(t)$ is given by

$$\dot{\beta}(t) \sim -\frac{\beta_\infty}{2\rho_1^2 J_1^2(\rho_1)}t^{-(1+\frac{8}{\rho_1^2\,J_1^2(\rho_1)})}\mathrm{e}^{-\rho_1^2 t}$$

and hence

$$v_1(r,t) \sim \beta_\infty\,t^{-\frac{8}{\rho_1^2\,J_1^2(\rho_1)}}\mathrm{e}^{-\rho_1^2 t}r^{-4} - \frac{\beta_\infty}{2\rho_1^2 J_1^2(\rho_1)}t^{-(1+\frac{8}{\rho_1^2\,J_1^2(\rho_1)})}\mathrm{e}^{-\rho_1^2 t}r^{-4}\ln r \quad \text{as} \quad r \to 0+\,. \tag{3.34}$$

In the above relation, note that the coefficient of r^{-4} decays slower than the one of $r^{-4}\ln r$. Thus as t increases the dominant term in the outer solution is $\beta_\infty\,t^{-\frac{8}{\rho_1^2\,J_1^2(\rho_1)}}\mathrm{e}^{-\rho_1^2 t}\phi_0^*(r)$. This analysis holds as long as $t^{-8/\rho_1^2\,J_1^2(\rho_1)}\mathrm{e}^{-\rho_1^2 t} \gg \delta$ and the outer correction (3.23) becomes invalid when $t^{-8/\rho_1^2\,J_1^2(\rho_1)}\mathrm{e}^{-\rho_1^2 t} \sim \delta$.

Zone II: In this zone we have $t^{-\frac{8}{\rho_1^2\,J_1^2(\rho_1)}}\mathrm{e}^{-\rho_1^2 t} \sim O(\delta)$, or $t \sim \frac{1}{\rho_1^2}\left(\ln(\frac{1}{\delta}) - \frac{8}{\rho_1^2 J_1^2(\rho_1)}\ln(|\ln\delta|)\right) + O(1)$, and we consider the new time variable $\tau = t - \sigma(\delta)$, where $\sigma(\delta) = \frac{1}{\mu^*}\left(\ln(\frac{1}{\delta}) - \frac{8}{\rho_1^2 J_1^2(\rho_1)}\ln(|\ln\delta|) + \frac{16\ln\rho_1}{\rho_1^2 J_1^2(\rho_1)}\right)$ (recall that $\mu^* = \rho_1^2$).

For the inner solution applying the same analysis as for zone I we consider

$$u(r,\tau) \sim \ln\left(\frac{1}{r^2}\right) - \frac{b_0}{2}a(\tau)\mathrm{e}^{-2a(\tau)}r^{-4} + \cdots \;\text{ as }\; r\mathrm{e}^{a(\tau)/2} \to \infty,$$

while in the outer region we assume $u(r,\tau) \sim w^*(r) + \delta v(r,\tau) + \cdots$ with

$$v_\tau = \Delta_r v + \lambda^* \mathrm{e}^{w^*} v + \mathrm{e}^{w^*}.$$

Then setting $v = \theta + V$, with θ satisfying the equation

$$\Delta_r \theta + \lambda^* \mathrm{e}^{w^*} \theta + \mathrm{e}^{w^*} = 0,$$

we obtain $\theta = (r^{-4} - 1)/\lambda^*$ and

$$V_\tau = \Delta_r V + \lambda^* \mathrm{e}^{w^*} V \tag{3.35}$$

with initial condition, for $\tau \to -\infty$, determined by matching to leading order with (3.25), with β given by (3.32):

$$\begin{aligned} V(r,\tau) &\sim -(\beta_\infty/\delta)\, t^{-8/\rho_1^2 J_1^2(\rho_1)}\, \mathrm{e}^{-\rho_1^2 t}\, \phi_0^*(r) \\ &\sim -\beta_\infty\, \mathrm{e}^{-\rho_1^2 \tau}\, \phi_0^*(r) \quad \text{as } \tau \to -\infty\,. \end{aligned} \tag{3.36}$$

Again it is easily checked that the solution to equation (3.35) with initial condition (3.36) satisfying the boundary conditions $V = 0$ on $r = 1$ and $V = O(r^{-4})$ as $r \to 0$ is precisely $V(r,\tau) = \beta_\infty\, \mathrm{e}^{-\rho_1^2 \tau}\, \phi_0^*(r)$.

Also matching with the inner solution requires

$$V(r,\tau) \sim -\left(\frac{1}{\lambda^*} + \frac{b_0}{2\delta}\, a(\tau)\, \mathrm{e}^{-2a(\tau)}\right) r^{-4} \qquad \text{as } r \to 0+\,.$$

(Note also that the behaviour of $a(\tau)$, given by the above relation, clearly matches (taking also into account (3.36)) with the behaviour of $a(t)$ in Zone I provided by $\frac{b_0}{2}\, a(t) e^{-2a(t)} \sim \beta_\infty\, t^{8/\rho_1^2 J_1^2(\rho_1)} e^{-\rho_1^2 t}$).

Following the $N > 10$ case, we have

$$V(r,\tau) \sim -\beta_\infty\, \mathrm{e}^{-\rho_1^2 \tau}\, \phi_0^*(r) \sim -\beta_\infty\, \mathrm{e}^{-\rho_1^2 \tau}\, r^{-4} \sim -\left(\frac{1}{\lambda^*} + \frac{b_0}{2\delta}\, a(\tau)\, \mathrm{e}^{-2a(\tau)}\right) r^{-4} \;\text{ as } r \to 0+\,.$$

This gives blow-up, with $a(\tau) \to \infty$ as $\tau \to \tau^*-$ or equivalently as $t \to t^*-$, with τ^* given by

$$\frac{1}{\lambda^*}\, \mathrm{e}^{\rho_1^2 \tau^*} = \beta_\infty\,.$$

This yields the asymptotic estimate

$$t^* \sim \frac{1}{\rho_1^2} \ln \frac{1}{\delta} - \frac{8}{\rho_1^4 J_1^2(\rho_1)} \ln|\ln \delta| + \frac{1}{\rho_1^2} \ln(\lambda^* \beta_\infty) + \frac{16 \ln \rho_1}{\rho_1^4 J_1^2(\rho_1)} + o(1) \quad \text{as} \quad \delta \to 0+\,.$$

Zone III: In this zone the relation $\frac{da}{dt} \sim \mathrm{e}^a$ (giving the blow-up) holds. Using the same arguments as in the case where $N > 10$ we derive that the contribution from this zone to the asymptotic estimate of the blow-up time t^* is small.

4 Numerical solutions

In order to get an overview of the blow-up process, we can obtain a numerical solution of the problem (1.9) by using for instance an explicit finite difference scheme.

Taking a uniform grid in $[0,1]$ of M points with step δr, such that $r_i = i\delta r$, $M\delta r = 1$, and a time step δt, the approximate solution of the problem $u_j^i \sim u(r_j, t_i)$ will satisfy

$$u_j^{i+1} = u_j^i + \frac{\delta t}{\delta r^2}(u_{j-1}^i - 2u_j^i + u_{j+1}^i) + \frac{N-1}{r}\frac{\delta t}{\delta r}(u_{j+1}^i - u_{j-1}^i) + \delta t \lambda \mathrm{e}^{u_j^i}, \tag{4.1}$$

for $j = 2, ..., M-1$, $u_M^i = 0$ and $u_1^{i+1} = u_1^i + \frac{\delta t}{\delta r^2}(N+1)(u_2^i - u_1^i) + \delta t \lambda \mathrm{e}^{u_1^i}$.

In Fig. 2 the solution of problem (1.9) is plotted against space and time for $\lambda > \lambda^*$. In this simulation the solution increases with time near the point $r = 0$ where blow-up occurs.

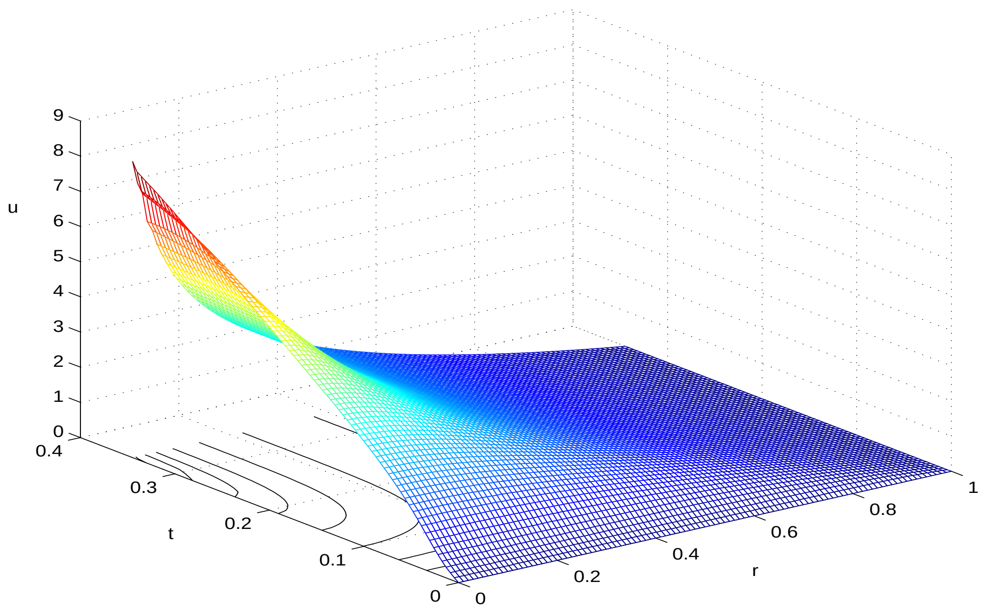

Figure 2: *The numerical solution of problem (1.9) is plotted against space and time for* $\lambda > \lambda^*$*. The values of the parameters used for the simulation are* $N = 12$, $\lambda = \lambda^* + 0.0025$, $\delta t = 4 * 10^{-6}$, $\delta r = 0.01$.

In Fig. 3 the solution is plotted against space at different values of time for $\lambda > \lambda^*$. The numerical scheme fails as $t \to t^*$ near $r = 0$ as the solution becomes unbounded and the scheme becomes unstable. Refinement of the partition (smaller δt and δx) will allow the numerical solution to be continued for slightly larger time.

In Fig. 4 the maximum of the solution, $u(0, t)$ is plotted against time for $\lambda > \lambda^*$ and the numerical estimate of the blow-up time is compared with the asymptotic estimate. Notice that the asymptotic estimate t_a is lower than the numerical one t_N. The numerical error and the restriction of the numerical scheme, i.e. its failure when the solution at $r = 0$ becomes very large, indicates that the asymptotic estimate given by analytical techniques is more accurate than the numerical one.

5 Discussion

In the present work we have estimated the blow-up time t^* of the solution of problem (1.9). It is important from the point of view of applications to estimate when the quantity u (representing, for example, temperature) becomes infinite. Similar estimates for blow-up time are also known for the semilinear heat equation for the closed-spectrum case [15] and also for non-local problems [12, 19].

Here a similar approach has been applied for $\lambda > \lambda^*$ in the open-spectrum case when there exists a steady-state solution $w^* = w(r; \lambda^*)$ in a weak form and we can assume radial symmetry. In order to get upper and lower estimates of the blow-up time we have used comparison methods, while formal asymptotics were used to obtain an asymptotic estimate of the blow-up time when $0 < \lambda - \lambda^* \ll 1$. Some numerical results have also been presented.

Our main estimates for given $\lambda > \lambda^*$ are of the form $C_1 \ln\left(\ln(\lambda - \lambda^*)^{-1}\right)$, for the lower bound, and $C_2(\lambda - \lambda^*)^{-1/2}$, for the upper bound. The asymptotic estimates, as $\lambda \to \lambda^*+$, are found to be of the form

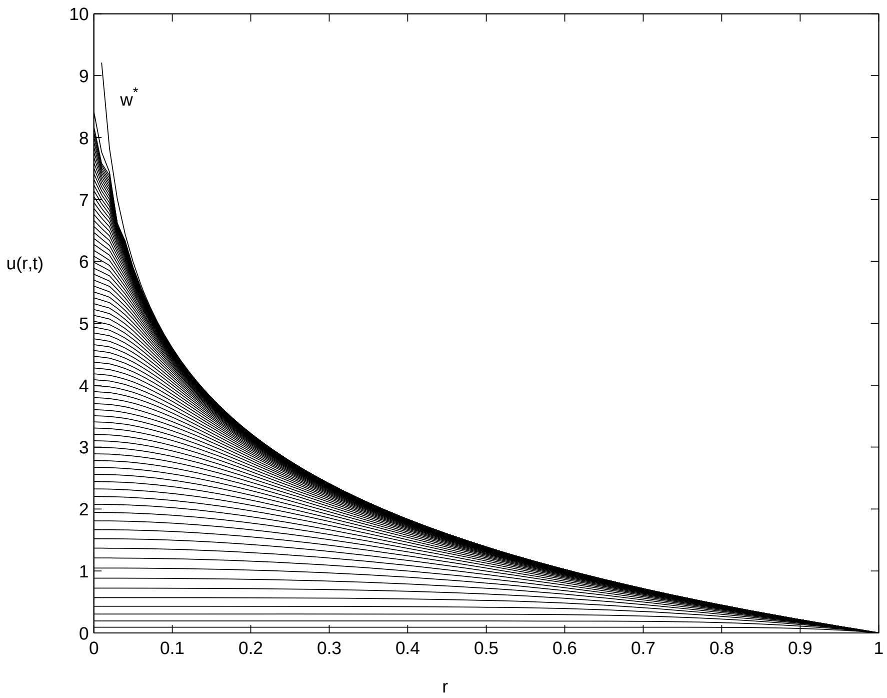

Figure 3: *The numerical solution of problem (1.9) is plotted against space for different values of time for* $\lambda > \lambda^*$. *The values of the parameters used for the simulation are* $N = 12$, $\lambda = \lambda^* + 0.0025$, $\delta t = 4 * 10^{-6}$, $\delta r = 0.01$.

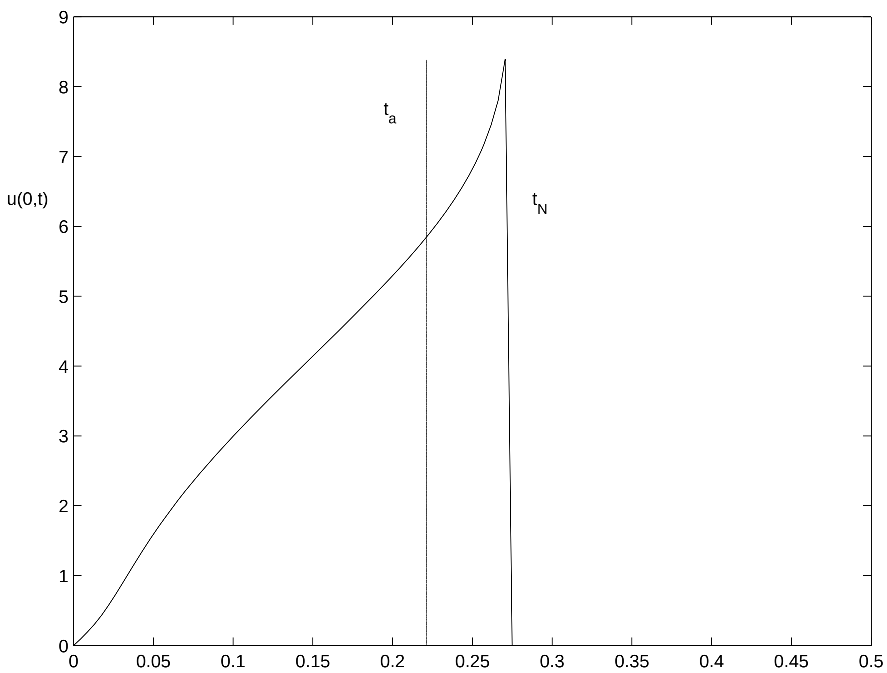

Figure 4: *The maximum of the solution,* $u(0,t)$*, of problem (1.9) is plotted against time for* $\lambda > \lambda^*$. *Also the asymptotic estimate, given by equation (3.17), for the blow-up time of the solution is plotted. The values of the parameters used for the simulation are* $N = 12$, $\lambda = \lambda^* + 0.005$, $\delta t = 4 * 10^{-6}$, $\delta r = 0.01$

$K_1 \ln(1/(\lambda - \lambda^*)) + K_2$, for $N > 10$, and $K_1 \ln(1/(\lambda - \lambda^*)) - K_2 \ln|\ln(\lambda - \lambda^*)| + M$, for $N = 10$. Here C_i, K_i, $i = 1, 2$ and M are some positive constants. These formal asymptotic estimates indicate that the rigorous bounds are far from best possible and it would be desirable to improve them.

We expect that for the case of $N > 10$ it should be possible to obtain rigorous proofs of the estimates in Section 3. However, for the case $N = 10$ this is likely to be harder to do since the rigorous asymptotic behaviour of $u^*(r,t)$ as $t \to \infty$ has not yet been fully justified by the perturbation theory of linear self-adjoint operators and nonlinear matching, see [14]. It also remains an open problem to generalize the analysis of the present paper to cases without radial symmetry.

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