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A model for optimal stopping in advertisement

C.V. Nikolopoulos [a,*], A.N. Yannacopoulos [b]

[a] *Department of Mathematics, University of the Aegean, Karlovasi, 83200 Samos, Greece*

[b] *Department of Statistics, Athens University of Economics and Business, Patission 76, 10434 Athens, Greece*

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ABSTRACT

In this work we propose a model for optimal advertisement in new product diffusion based on the Bass model and assuming that the effect of the environmental pressure in the diffusion of the product is subject to a stochastic dependence. The optimal stopping problem is reduced to a free boundary problem which is analyzed and solved numerically, in order to determine an optimal stopping rule for the advertisement campaign. The numerical solution is obtained through a policy iteration like contraction scheme, the convergence properties of which are studied in detail. Furthermore, the expected time until the optimal stopping of the campaign is estimated. Finally, a combined optimal stopping and control problem for the optimization of the advertisement effectiveness is also proposed and solved numerically. Our results are expected to provide useful guidelines for campaign managers, for the choice of effectiveness and duration of an advertisement campaign.

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1. The model

A very popular and successful model for new product diffusion in a market is the Bass model. According to the standard Bass model, the adoption of a new product is associated with two different procedures [1]. The first is the adoption of the product because of the effect of advertisement. This is assumed to be a process which may be modelled by a linear term, i.e. if $p(t)$ is the number of customers that have adopted the product up to time t, and N is the total market potential, then the rate of change of $p(t)$ with respect to time is likely to be proportional to the number of potential customers that have not yet adopted the product, $N-p(t)$. The proportionality factor A_0 is a measure of the effectiveness of the advertisement campaign for the product. The second process that plays an important effect is the so called *word of mouth* process which assumes that the various potential adopters are affected, among other factors, by the peer pressure of those that have already adopted the product. This is a second order effect and the rate of change of adopters due to this mechanism is likely to be proportional to the number of adopters $p(t)$ multiplied by the number of potential adopters $N-p(t)$. Thus the rate of change of adopters due to this mechanism will be of the form $B_0p(t)(N-p(t))$. The proportionality factor B_0 is a measure of the effectiveness of the word of mouth mechanism.

According to this model, the evolution of the number of customers, $p(t)$ that have adopted the product will follow the differential equation

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = A_0\,(N-p(t)) + B_0\,p(t)(N-p(t)).$$

Working with the scaled variable $x(t) = p(t)/N$ the equation becomes

* Corresponding author. Tel.: +30 22730 82156; fax: +30 22730 82007.
E-mail addresses: cnikolo@aegean.gr (C.V. Nikolopoulos), ayannaco@aueb.gr (A.N. Yannacopoulos).

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$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\,(1 - x(t)) + b\,x(t)(1 - x(t)),$$

where $a = A_0 N$ and $b = B_0 N^2$.

The Bass model has been used widely in the modelling of the diffusion of a wide range of products, with very good fit to real data, and has been proved very successful as a model. However, it is a fairly simple model which has to be further enhanced with realistic effects so as to provide a better understanding and approximation of the real world. One important step towards this direction is to introduce some uncertainty in the various parameters of the model. This uncertainty for the purpose of the present paper will be modelled with the use of white noise. The noise will model uncertainty concerning the various model parameters, such as the effectiveness of the advertisement campaign or the effectiveness of the word of mouth mechanism. The effectiveness of an advertisement campaign, which is related to the probability of a client adopting the product because of the effect of advertising, depends on a number of factors such as the intensity of the campaign, the quality and strategy adopted by the advertiser, the composition of the target group, the socio-economic circumstances etc. All these factors are subject to uncertainty. We will then assume that $a(t) = a_d(t) + a_s(t)$ where $a_d(t)$ is the deterministic (average) part of the parameter $a(t)$ and $a_s(t)$ is the stochastic part which models the fluctuations around the deterministic part. In order to keep the model as simple as possible, we will only consider the effectiveness a of the campaign in this paper, and assume that in principle it depends in a monotonic fashion on the intensity of the campaign. A reasonable first model for the fluctuating term would be to model them as a Gaussian process and a suitable model for that would be to take $a_s(t)$ to be given by an Itô integral. Reasons for the adoption of such a model can be based on considerations related to the law of large numbers and central limit theorem type arguments. Generalization to other types of noise, for example noise terms including jumps etc. is possible, under appropriate modifications of the modelling considerations and technical assumptions.

Under the given modelling assumptions, and denoting $a_d(t) = a$, and $a_s(t) = \sigma_0 a \mathrm{d}W_t$, where W_t is a Wiener process and σ_0 is a constant, the Bass model can be formulated as a stochastic differential equation of Itô type, of the form

$$\mathrm{d}x(t) = \{a\,(1 - x(t)) + b\,x(t)(1 - x(t))\}\mathrm{d}t + \sigma_0 a(1 - x(t))\mathrm{d}W(t), \tag{1.1}$$

Remark 1. Of course one could use similar assumptions and introduce uncertainty in $b(t)$ as well. This would lead to an extra diffusion term of the form $\sigma_1 bx(1 - x)$ for some constant σ_1.

Remark 2. The model, in the absence of uncertainty, has monotone trajectories, i.e. $x(t)$ is an increasing function. For the simple type of uncertainty we have introduced, we may lose the monotonicity of $x(t)$ in t. However, for small σ_0 the model still presents reasonable results, which may be interpreted as some customers deciding to abandon the product. One may introduce noise in such a way that monotonicity is still guaranteed; however, this would complicate the model considerably. Such an extension is outside the scope of the present paper.

Remark 3. The model remains well defined in the case where uncertainty is absent ($\sigma_0 = 0$). Then the dynamic programming approach is still valid, and the variational inequality holds, where of course the operator $\mathcal{A}$ is now the first order operator which is obtained by setting $\sigma_0 = 0$. The concept of viscosity solutions is still applicable, and similar results for the value function $\Phi(s, x)$ hold with necessary modifications (see e.g. [2] or [3]). Furthermore, results concerning the time until the optimal stopping of the campaign may be formulated for the deterministic problem (see e.g. [3,12]).

We now assume that the effectiveness of advertising a as well as its total duration, can be controlled through an advertising campaign for the product. Of course this is done at some cost. Let $c(a, t)$ be the cost of advertising the product for time t at intensity that guarantees effectiveness a. We shall consider the problem of determination of an optimal advertising campaign for a product, such that the firm receives the maximum expected profit from the sales of the product but at the minimum possible cost. Therefore, this problem can be redressed as an optimal stopping and control problem, the solution of which will provide us with the optimal advertising policy.

Assume that the product starts at time s with sales $x(s) = x$. The cost functional of the problem, will be of the form

$$\begin{aligned}\Theta(s, x; t) &= J(a)\\ &= E\left[\mathrm{e}^{-r(t-s)} f_0(x(t)) + \int_s^t \mathrm{e}^{-r(t-s')} f_1(x(s'))\mathrm{d}s' - \mathrm{e}^{-r(t-s)} c(a, t - s) \mid \mathcal{F}_s\right],\end{aligned}$$

where by $\mathcal{F}_s = \sigma(W(u), u \le s)$ we denote the σ-algebra which is generated by the Wiener process. We may consider this σ-algebra as containing the information about the history of the product up to time s. The term $\mathrm{e}^{-r(t-s)}$ is a discount factor that allows us to compare values obtained at different times. The function f_0 is a profit function that gives us the profit obtained by sales $x(t)$ at time t. The second term, which is expressed as an integral gives us the intertemporal profit from all the sales of the product within the interval $[s, t]$. For the sake of simplicity in this paper we consider that $f_1(x) = 0$, i.e. the firm receives no intertemporal profit but is only interested in the final condition of the sales at time t. However, our results are easily generalized to the case where $f_1(x) \neq 0$. The functions $f_0(x)$ and $f_1(x)$ must be increasing functions of their argument.

We will consider separately, the optimal stopping problem and the optimal stopping and control problem, in Sections 2–4 and Section 5 respectively.

2. The optimal stopping problem and an associated variational inequality

In this section, we assume that the value of a is given, and constant during the period of the campaign and the campaign manager may only choose the length of the campaign $\tau - s$. From the form of the previous cost functional we see that as time proceeds and the advertising campaign is still active, the firm derives increasing profit from the increasing sales of the product. However, at all times she must take into account the increasing cost of the advertisement campaign. At some point, the cost of the campaign becomes so large that it is no longer worth pursuing it any more. This is the optimal stopping time for the campaign. The purpose of this paper is to derive the optimal time when the firm should stop advertising the product and leave the market.

We now formulate the optimal stopping problem that was described above:

Find a stopping time τ^* such that

$$\Phi(s, x) = \Theta(s, x; \tau^*) = \sup_{\tau} \Theta(s, x; \tau)$$

where the supremum is taken over all stopping times τ (recall that a stopping time is a random time τ such that $\{\omega, \tau \leq t\} \in \mathcal{F}_t$).

The stopping time τ^* will be the optimal stopping time for the advertisement campaign.

This problem will be solved through the formulation of a free boundary value problem.

In the following we assume that the diffusion process $Y = (t + s, x(t))$ takes values in the domain $V \subset \mathbb{R}^2$. Furthermore, we assume that the optimal value of the cost functional $\Phi(s, x)$ is of the form $\Phi(s, x) = e^{-rs}\phi(s, x)$. Consider also the generator of the diffusion process $Y = (s + t, x(t))$ describing the sales of the product:

$$\mathcal{A} = \frac{\partial}{\partial s} + \{a(1 - x) + bx(1 - x)\}\frac{\partial}{\partial x} + \sigma_0^2 a^2 (1 - x)^2 \frac{\partial^2}{\partial x^2}$$

and the reduced operator

$$A = \{a(1 - x) + bx(1 - x)\}\frac{\partial}{\partial x} + \sigma_0^2 a^2 (1 - x)^2 \frac{\partial^2}{\partial x^2}.$$

The following result is standard in the theory of optimal stopping problems [4].

Proposition 1. *Under sufficient regularity conditions, the optimal value of the cost functional $\phi(s, x)$ satisfies the differential inequality*

$$\begin{aligned} &\mathcal{A}\phi - r\phi = 0 \quad \text{in } D = \{(s, x) \in V, : \phi > g\} \\ &\mathcal{A}\phi - r\phi \leq 0 \quad \text{with } \phi = g \text{ in } \partial D. \end{aligned} \tag{2.1}$$

where $g(s, x) = f_0(x) - c(a, s)$.
The continuation region D is determined by the solution of the free boundary value problem and the optimal stopping time is the first time when the sales process $x(t)$ hits the boundary D.

This problem will have to be completed with a final (initial) condition and a boundary condition. The final condition will be of the form $\phi(T, x) = f_0(x)$ (we omit the advertisement cost term, since the advertising campaign has had no time to run so it does not cost anything).

We have some freedom concerning the choice of boundary conditions. One possibility is to assume that $V = [0, 1] \times [0, T]$ and that the stopping times over which we allow the selection of the optimal stopping time to be made are $\tau \leq T_V$ where T_V is the first exit time of the process $Y(t) = (s + t, x(t))$ from the region V. That means that we stop the campaign as soon as the volume of the product which is sold hits either the boundary $x = 0$ or $x = 1$. This assumption is a reasonable modelling assumption, in the sense that when $x = 1$ the product sales have reached the full market potential and no further advertisement of the product is needed. On the other hand when $x = 0$, the product sales have reached 0 and therefore the company decides to stop all advertising activity concerning the product. In this case, the right boundary conditions would be non-homogeneous Dirichlet type boundary conditions of the form $\phi(s, 0) = f_0(0)$ and $\phi(s, 1) = f_0(1)$.

Another reasonable choice for the boundary condition may be Neumann type boundary conditions at $x = 0$ and $x = 1$. Such types of conditions model the fact that for these limiting situations we will not have a significant change in the profit by changing our initial position in the market x by a small amount.

Corollary 1. *The free boundary value problem of Proposition 1 can be written in the equivalent form*

$$H(s, x, \phi, D_s\phi, D_x\phi, D_x^2\phi) = \min\left[r\phi - \frac{\partial \phi}{\partial s} - A\phi,\ \phi(s, x) - g(s, x)\right] = 0, \tag{2.2}$$

$$\phi(x, T) = f_0(x), \quad x \in [0, 1],$$

for $g(s, x) = f_0(x) - c(s)$, with either non-homogeneous Dirichlet boundary conditions or homogeneous Neumann boundary conditions.

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Note also that a reasonable choice for the cost function $c(s) = c(s; a)$ would be to assume that it is proportional to the time s that the campaign has run and to the advertisement effectiveness a, i.e. $c \propto a\, s$.

Remark 4. The above problem is a final value problem, which may be turned into an initial value problem by the use of the time transformation $z = T - s$, which will change the sign of the time derivative and of course the function g to $g(z, x) = f_0(x) - c(T - z)$.

Note that the boundary D is unknown and it is completely determined only when the solution ϕ of the problem is determined. This is why problems of the above type are called free boundary value problems. Problems of this type are in general very difficult and do not admit solutions using analytic techniques. It is therefore necessary to address the problem using numerical methods. This is the aim of the following sections.

Before proceeding to the numerical solution we have to address an alternative formulation of the problem, and introduce a notion of generalized solution for the above free boundary value problem, the **viscosity solutions**.

Often, the regularity needed for $\phi(s, x)$ for Proposition 1 to hold is not obtained. In this case, we need to use weaker notions of the solution, which are called **viscosity solutions**.

We recall the definition of viscosity solution for a variational inequality.

Definition 1. Consider the variational inequality

$$\max(L\phi(y) + f(y), g(y) - \phi(y)) = 0, \quad y \in D \tag{2.3}$$
$$\phi(y) = g(y), \quad y \in \partial D$$

where L is a second order parabolic operator, and let $\phi \in C(\bar{D})$. Then, we have the following definitions:

(i) ϕ is a **viscosity subsolution** of (2.3) if $\phi(y) = g(y)$ for $y \in \partial D$ holds and for every $h \in C^2(\mathbb{R}^k)$ and every $y_0 \in D$ such that $h \geq \phi$ on D and $h(y_0) = \phi(y_0)$ we have

$$\max\,(Lh(y_0) + f(y_0), g(y_0) - \phi(y_0)) \geq 0.$$

(ii) ϕ is a **viscosity supersolution** of (2.3) if $\phi(y) = g(y)$ holds for $y \in \partial D$ and for every $h \in C^2(\mathbb{R}^k)$ and every $y_0 \in D$ such that $h \leq \phi$ on D and $h(y_0) = \phi(y_0)$ we have

$$\max\,(Lh(y_0) + f(y_0), g(y_0) - \phi(y_0)) \leq 0.$$

(iii) ϕ is a **viscosity solution** of (2.3) if ϕ is both a viscosity subsolution and a viscosity supersolution of (2.3).

The solution of the optimal stopping problem is related to the viscosity solution of the variational inequality, through the following proposition (see e.g. [5]):

Proposition 2. *Assume that the value function is a continuous function of s, t. Then $\phi(s, x)$ is a viscosity solution of the variational inequality.*

The continuity of the value function is guaranteed by the following lemma:

Lemma 1. *The value function $\Phi(s, x)$ is continuous.*

Proof. The continuity of the value function follows by adaptation of general arguments of Krylov [6]. These are sketched here for the sake of completeness of the paper.

We first need the following estimate: Consider two solutions of Eq. (1.1) one with "initial condition" $x(s) = x$ and one with initial condition $x(s) = y$. These will be denoted for ease of notation as $x(t)$ and $y(t)$ respectively. Also, on account of the possible degeneracy of the equation at $x = 0$ or $x = 1$, consider the equation with a perturbed drift and diffusion coefficient, by a factor of order ϵ which will remove the singularity. The solution of this equation will be denoted by $x(t; \epsilon)$ or $y(t; \epsilon)$ where the choice of notation for the process is such as to remind us of the initial condition we started from. We denote $x(t; 0) = x(t)$.

We will show that these diffusion processes satisfy the estimate

$$E\left[\sup_{t \leq \tau_0} \mathrm{e}^{-Mt}\, |x(t) - y(t; \epsilon)|^p\right] \leq C(|x - y|^p + \epsilon^p) \tag{2.4}$$

for every $p \geq 0$, where τ_0 is any stopping time which is smaller that the first exit time of either process from the interval $[0, 1]$, and $M = M(p) > 0$ is a positive constant that depends on the data of the problem.

To show estimate (2.4), following [6], apply Itô's rule on the process $\mathrm{e}^{-Mt}(|x(t) - y(t, \epsilon)|^p + \epsilon^p)$. Taking into account that $x(t)$ and $y(t; \epsilon)$ satisfy

$$x(t) = x + \int_s^t b(x(r))\mathrm{d}r + \int_s^t \sigma(x(r))\mathrm{d}W(r),$$
$$y(t; \epsilon) = y + \int_s^t b(y(r; \epsilon); \epsilon)\mathrm{d}r + \int_s^t \sigma(y(r; \epsilon); \epsilon)\mathrm{d}W(r),$$

respectively we see that

$$e^{-Mt}(|x(t)-y(t;\epsilon)|^p+\epsilon^p) = |x-y|^p+\epsilon^p+m(t)-M\int_s^t e^{-Mr}(|x(r)-y(r;\epsilon)|^p+\epsilon^p)\mathrm{d}r$$
$$+p\int_s^t e^{-Mr}|x(r)-y(r;\epsilon)|^{p-1}(b(x(r))-b(y(r;\epsilon);\epsilon))\mathrm{d}r$$
$$+\frac{p\,(p-1)}{2}\int_s^t e^{-Mr}|x(r)-y(r;\epsilon)|^{p-2}|b(x(r))-b(y(r;\epsilon);\epsilon)|^2\,\mathrm{d}r, \tag{2.5}$$

where $m(t)$ is a local martingale. The drift and diffusion coefficient in the interval [0, 1] satisfy the Lipschitz continuity conditions

$$|b(x)-b(y;\epsilon)|\le K(|x-y|+\epsilon),$$
$$|\sigma(x)-\sigma(y;\epsilon)|^2\le K(|x-y|+\epsilon).$$

Substituting these estimates in (2.5) we obtain that

$$e^{-Mt}U(t)\le |x-y|^p+\epsilon^p+m(t)+(-M+C)\int_s^t e^{-Mr'}U(r')\,\mathrm{d}r',$$

where $U(t)=|x(t)-y(t;\epsilon)|^p$ and C is constant depending on the Lipschitz constants and p. By the Gronwall inequality, we see that choosing M large the following holds

$$e^{-Mt}(|x(t)-y(t;\epsilon)|^p+\epsilon^p)\le |x-y|^p+\epsilon^p+m(t).$$

Using an inequality of Krylov (see Lemma 7.4 of [7]) we obtain the stated result, i.e estimate (2.4).

We are now in a position to obtain the continuity of the value function. By the definition of $\Phi(s,x)$ we see that

$$|\Phi(s,x)-\Phi(s,y)|\le\sup_\tau E\left[\int_s^t e^{-r(r'-s)}|f_1(x(r'))-f_1(y(r'))|\mathrm{d}r'+e^{-r(\tau-s)}|f(x(\tau))-f(y(\tau))|\right]$$

where in f we have included the cost term. We observe that

$$E\left[\int_s^\tau e^{-r(r'-s)}|f_1(x(r'))-f_1(y(r'))|\mathrm{d}r'\right]\le e^{-(r-M)(\tau-s)}\int_s^\tau E[e^{-M(r'-s)}|f_1(x(r'))-f_1(y(r'))|\mathrm{d}r']$$

where M is the same as in the estimate above. Now, we use the Lipschitz continuity conditions on f and f_1 and the estimate (2.4) to obtain the stated result. The final time term is treated similarly.

The continuity with respect to time proceeds accordingly. □

3. The expected time until the optimal stopping of the campaign

The previous section provides an interesting stopping rule in terms of the stopping region D, whose boundary can be parametrized as $x^*(t,x)$, i.e. given that we start at time t with a given volume of sales x, it is optimal to stop when the sales reach the level x^*. However, an interesting piece of information for the campaign manager would be an estimate of how long it would take on average for this to happen, that is, an estimate of the expected duration of the campaign. This is of course dependent on the parameters of the problem.

The theory of diffusion processes may be used to obtain an answer to this question.

Proposition 3. *The expected exit time for the process, from point x^*, given that it has started at point x at time t, is $\mathbb{E}[T]=t+T(x)$ where $T(x)$ is the solution to the boundary value problem*

$$\frac{1}{2}\sigma_0^2a^2(1-x)^2\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+\{a\,(1-x)+b\,x\,(1-x)\}\frac{\mathrm{d}T(x)}{\mathrm{d}x}=-1, \tag{3.1}$$
$$T(0)=T(x^*)=0.$$

Proof. This can be proved by a direct application of the Feynman–Kac representation (see e.g. [4]). The boundary condition at 0 reflects that we automatically stop the campaign if the customers of the product have reached the level 0. □

This ODE may be solved analytically by quadrature, as long as x^* is known to provide an estimate for the expected life of the campaign. To obtain this solution we define the new function $R(x):=\frac{\mathrm{d}T(x)}{\mathrm{d}x}$. In terms of this new function, the equation is a first order equation, for which an integrating factor may be found. More specifically, it is of the form

$$\frac{\mathrm{d}R}{\mathrm{d}x}+\left\{\lambda_1\frac{1}{1-x}+\lambda_2\frac{x}{1-x}\right\}R=\lambda_3\frac{1}{(1-x)^2}$$

where

$$\lambda_1 = \frac{2}{\sigma_0^2 a}, \qquad \lambda_2 = \frac{2b}{\sigma_0^2 a^2}, \qquad \lambda_3 = -\frac{2}{\sigma_0^2 a^2}.$$

Thus we have

$$R(x) = C_1(x-1)^{\lambda_1+\lambda_2} \mathrm{e}^{\lambda_2 x} - \lambda_3 (x-1)^{\lambda_1+\lambda_2} \mathrm{e}^{\lambda_2 x} \int (x-1)^{-2-\lambda_1-\lambda_2} \mathrm{e}^{-\lambda_2 x} \mathrm{d}x,$$

and by direct integration we obtain

$$T(x) = -C_1(x-1)^{1+\lambda_1+\lambda_2} \int_1^\infty \frac{\mathrm{e}^{-\lambda_2(1-x)t}}{t^{1+\lambda_1+\lambda_2}} \mathrm{d}t + C_2 + G(x), \tag{3.2}$$

where, $G(x)$ is, in terms of the Gamma function,

$$G(x) = \lambda_2^{1+\lambda_1+\lambda_2} \int \mathrm{e}^{\lambda_2 x}(x-1)\Gamma(-1-\lambda_1-\lambda_2, -\lambda_2(1-x))\mathrm{d}x.$$

Then using the boundary conditions $T(0) = T(x^*) = 0$ we determine the constants C_1, C_2 to be

$$C_1(\lambda_1, \lambda_2, x^*) = \left(G(0) - G(x^*)\right) \frac{(-1)^{1+\lambda_1+\lambda_2} \lambda_2^{1+\lambda_1+\lambda_2}}{\mathrm{e}^{\lambda_2}\left[\Gamma(1+\lambda_1+\lambda_2, \lambda_2(1-x^*)) - 1\right]},$$

$$C_2(\lambda_1, \lambda_2, x^*) = \frac{(G(0) - G(x^*))\,\Gamma(1+\lambda_1+\lambda_2, \lambda_2(1-x^*))}{\Gamma(1+\lambda_1+\lambda_2, \lambda_2) - \Gamma(1+\lambda_1+\lambda_2, \lambda_2(1-x^*))} - G(x^*).$$

Therefore for x^* determined by the solution of problem (2.2) we can have an estimate of the average time until the optimal stopping of the campaign.

4. Formulation of the numerical method

The free boundary value problem that will provide us with the optimal stopping rule cannot admit an analytic solution, therefore numerical treatment is necessary. In order to treat this problem numerically we will use a policy iteration-like method motivated by [8].

Our analysis is valid for both Dirichlet and Neumann boundary conditions. We only present the slightly more complex case of Neumann boundary conditions; the case of Dirichlet boundary conditions proceeds in a similar fashion.

4.1. A policy iteration scheme

The equations to be solved, in their complete form, can be written as follows

$$H(s, x, \phi, D_s\phi, D_x\phi, D_x^2\phi) = \min\left[r\phi - \frac{\partial \phi}{\partial s} - \alpha(x)\frac{\partial \phi}{\partial x} - \frac{\sigma_0^2}{2}\beta^2(x)\frac{\partial^2 \phi}{\partial x^2},\ \phi(s,x) - g(s,x)\right] = 0, \tag{4.1}$$

$$\frac{\partial \phi}{\partial x} = 0, \quad \text{at } x = 0 \quad \text{and} \quad x = 1, \quad s \in [0, T], \tag{4.2}$$

$$\phi(T, x) = f_0(x), \quad x \in [0, 1], \tag{4.3}$$

where $\alpha(x) = a(1-x) + bx(1-x)$, $\beta(x) = a(1-x)$. In addition we have that $g(s, x; a) = (f_0(x) - c(a, s))$.

A similar algorithm has been proposed and studied by Pem and Zhang [8] in the context of a model for optimal stock liquidation. The problem studied by these authors involved a linear diffusion process, with the inclusion of regime switching. Our analysis is inspired by their work, however, the problem studied here considers the optimal control of a nonlinear diffusion, which degenerates on the boundary. This leads to a variable coefficient, degenerate variational inequality, a fact which introduces several complications in the analysis.

Let $\mathcal{B}$ denote the space of continuous bounded functions $\phi(s, x)$ on $[0, T] \times [0, 1]$, endowed with the usual supremum norm (i.e. $\|\phi\|_\infty =: \|\phi\|$) and the ball $\mathcal{B}_M = \{\phi \in \mathcal{B} \mid \|\phi\| \le M\}$ for some $M > 0$, and let $h > 0$ denote the spatial step and $k > 0$ the time step. We consider the following discrete approximations for the derivatives:

$$\frac{\partial \phi}{\partial s} \simeq \delta_s \phi(s,x) := \frac{\phi(s+k, x) - \phi(s,x)}{k},$$

$$\frac{\partial \phi}{\partial x} \simeq \delta_x \phi(s,x) := \frac{\phi(s, x+h) - \phi(s,x)}{h},$$

$$\frac{\partial^2 \phi}{\partial x^2} \simeq \delta_x^2 \phi(s,x) := \frac{\phi(s, x+h) - 2\phi(s,x) + \phi(s, x-h)}{h^2}.$$

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Therefore the corresponding discrete version of Eq. (4.1), is

$$\min\left[r\phi(s,x)-\delta_s\phi(s,x)-\alpha(x)\delta_x\phi(s,x)-\frac{\sigma_0^2}{2}\beta^2(x)\delta_x^2\phi(s,x),\ \phi(s,x)-g(s,x)\right]=0,$$

for $x\in[h,1-h]$, or alternatively, including also the equations for the boundary points $x=0,1$,

$$\min\left[\phi(s,x)\left(r+\frac{1}{k}+\frac{\sigma_0^2\beta^2(x)}{h^2}+\frac{\alpha(x)}{h}\right)-\phi(s,x+h)\left(\frac{\sigma_0^2\beta^2(x)}{2h^2}+\frac{\alpha(x)}{h}\right)\right.$$
$$\left.-\frac{1}{k}\phi(s+k,x)-\phi(s,x-h)\frac{\sigma_0^2\beta^2(x)}{2h^2},\ \phi(s,x)-g(s,x)\right]=0, \tag{4.4}$$

$$\min\left[\phi(s,0)\left(r+\frac{1}{k}+\frac{\sigma_0^2\beta^2(0)}{h^2}+\frac{\alpha(0)}{h}\right)-\phi(s,h)\left(\frac{\sigma_0^2\beta^2(0)}{h^2}+\frac{\alpha(0)}{h}\right)\right.$$
$$\left.-\frac{1}{k}\phi(s+k,0),\ \phi(s,0)-g(s,0)\right]=0, \tag{4.5}$$

for $x=0$ and

$$\min\left[\phi(s,1)\left(r+\frac{1}{k}+\frac{\sigma_0^2\beta^2(1)}{h^2}+\frac{\alpha(1)}{h}\right)-\phi(s,1-h)\left(\frac{\sigma_0^2\beta^2(1)}{h^2}+\frac{\alpha(1)}{h}\right)\right.$$
$$\left.-\frac{1}{k}\phi(s+k,1),\ \phi(s,1)-g(s,1)\right]=0, \tag{4.6}$$

for $x=1$.

Then we can define the mapping $S:\mathbb{R}^+\times\mathbb{R}^+\times[0,1]\times\mathbb{R}\times\mathcal{B}\to\mathbb{R}$

$$S(k,h,x,y,\phi)=\min\left[yh\left(r+\frac{1}{k}+\frac{\sigma_0^2\beta^2(x)}{h^2}+\frac{\alpha(x)}{h}\right)-\frac{h}{k}\phi(s+k,x)\right.$$
$$\left.-\ \phi(s,x+h)\left(\frac{\sigma_0^2\beta^2(x)}{2h}+\alpha(x)\right)-\phi(s,x-h)\frac{\sigma_0^2\beta^2(x)}{2h},\ \phi(s,x)-g(s,x)\right]=0, \tag{4.7}$$

for $x\in[h,1-h]$, while for $x=0,1$, the mapping S has the form of the left hand side of Eqs. (4.5) and (4.6) respectively.

Therefore Eq. (4.4) is equivalent to $S(k,h,x,y,\phi)=0$, and Eqs. (4.5) and (4.6) are equivalent to $S(k,h,0,y,\phi)=0$ and $S(k,h,1,y,\phi)=0$ respectively.

It is also important to note that the scheme S is monotone. Indeed the coefficients of ϕ in Eq. (4.7) are all negative and hence S is monotone, i.e. for all $\phi,\psi\in\mathcal{B}$, $k,h\in\mathbb{R}^*$, $x\in[0,1]$, $y\in\mathbb{R}$ we have that the relation $\phi\geq\psi$ implies that

$$S(k,h,x,y,\phi)\leq S(k,h,x,y,\psi).$$

The scheme S is said to be **consistent** if for every $x\in[0,1]$, $s\in[0,T]$ and for every $w(\cdot,\cdot)\in C^{1,2}([0,T]\times[0,1])$ we have

$$\lim_{z\to x,\,k\to0,\,h\to0,\,\epsilon\to0}\frac{S(k,h,z,w(s,z)+\epsilon,w+\epsilon)}{h}=H(t,x,w,D_sw,D_xw,D_x^2w).$$

Lemma 2. *The scheme S is consistent for $w(\cdot,\cdot)\in C^{1,2}([0,T]\times[0,1])$.*

Proof. Indeed we have for $z\in[h,1-h]$

$$\frac{S(k,h,z,w(s,z),w)}{h}=\min\left[rw(s,z)-\frac{w(s+k,z)-w(s,z)}{k}-\frac{\sigma_0^2\beta^2(z)}{2}\frac{w(s,z+h)-2w(s,z)+w(s,z-h)}{h^2}\right.$$
$$\left.-\ \alpha(z)\frac{w(s,z+h)-w(s,z)}{k},\ w(s,z)-g(s,z)\right]=0.$$

Also at the point (s,z) we have

$$\lim_{s\to0}\delta_sw(s,z)=\lim_{s\to0}\frac{w(s+k,z)-w(s,z)}{k}=w_s(s,z),$$
$$\lim_{z\to0}\delta_zw(s,z)=\lim_{z\to0}\frac{w(s,z+h)-w(s,z)}{h}=w_z(s,z),$$
$$\lim_{z\to0}\delta_z^2w(s,z)=\lim_{z\to0}\frac{w(s,z+h)-2w(s,z)+w(s,z-h)}{h^2}=w_{zz}(s,z),$$

for $w(\cdot,\cdot)\in C^{1,2}([0,T]\times[0,1])$.

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Therefore for $z \to x,\ k \to 0,\ h \to 0,\ \epsilon \to 0$ we have

$$\frac{S(k,h,z,w(s,z)+\epsilon,w+\epsilon)}{h} \to H(t,x,w,D_s w,D_x w,D_x^2 w),$$

and it can be easily checked that the same holds for $z = 0, 1$. □

Note that the scheme $S(k,h,z,w(s,z),w) = 0$ can be rewritten in the form

$$w(s,z) = \max\left[\frac{1}{C(z)}\left(\frac{hw(s+k,z)}{k} + w(s,z+h)\left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right) + w(s,z-h)\frac{\sigma_0^2\beta^2(z)}{2h}\right),\ g(s,z)\right], \tag{4.8}$$

while

$$w(s,0) = \max\left[\frac{1}{C(0)}\left(\frac{hw(s+k,0)}{k} + w(s,h)\left(\frac{\sigma_0^2\beta^2(0)}{h} + \alpha(0)\right)\right),\ g(s,z)\right], \tag{4.9}$$

$$w(s,1) = \max\left[\frac{1}{C(1)}\left(\frac{hw(s+k,1)}{k} + w(s,1-h)\left(\frac{\sigma_0^2\beta^2(1)}{h} + \alpha(1)\right)\right),\ g(s,z)\right], \tag{4.10}$$

where

$$C(z) = \frac{1}{h}\left(rh + \frac{h^2}{k} + \sigma_0^2\beta^2(z) + h\alpha(z)\right).$$

We define the operator $T_{h,k} : \mathcal{B} \to \mathcal{B}$

$$T_{h,k}w(s,z) := \max\left[\frac{1}{C(z)}\left(\frac{hw(s+k,z)}{k} + w(s,z+h)\left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right) + w(s,z-h)\frac{\sigma_0^2\beta^2(z)}{2h}\right),\ g(s,z)\right],$$

where for $z = 0, 1$, $T_{h,k}$ has the form of the right hand side of Eqs. (4.9) and (4.10) respectively.

Lemma 3. *The operator $T_{h,k}$ is a contraction for every $k,\ h > 0$.*

Proof. Indeed, we have that

$$\begin{aligned}T_{h,k}w(s,z) - T_{h,k}u(s,z) = &\max\left[\frac{1}{C(z)}\left(\frac{hw(s+k,z)}{k} + w(s,z+h)\left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right) + w(s,z-h)\frac{\sigma_0^2\beta^2(z)}{2h}\right),\ g(s,z)\right] \\ &- \max\left[\frac{1}{C(z)}\left(\frac{hu(s+k,z)}{k} + u(s,z+h)\left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right) + u(s,z-h)\frac{\sigma_0^2\beta^2(z)}{2h}\right),\ g(s,z)\right].\end{aligned}$$

Thus

$$\begin{aligned}|T_{h,k}w(s,z) - T_{h,k}u(s,z)| \le &\frac{1}{C(z)}\left[\frac{h}{k}|w(s+k,z) - u(s+k,z)| + \left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right)|w(s,z+h) - u(s+h,z)| + \frac{\sigma_0^2\beta^2(z)}{2h}|w(s,z-h) - u(s-h,z)|\right] \\ \le &\frac{1}{C(z)}\left(\frac{h}{k} + \frac{\sigma_0^2\beta^2(z)}{h} + \alpha(z)\right)\|w-u\|,\end{aligned}$$

while the same relation also holds for the boundary points $z = 0, 1$.

It is easy to check that for every k, h and $z \in [0, 1]$

$$\frac{1}{C(z)}\left(\frac{h}{k} + \frac{\sigma_0^2\beta^2(z)}{h} + \alpha(z)\right) \le c < 1,$$

for some c, and therefore the mapping $T_{h,k}$ is a contraction. □

We also need to show that the scheme is **stable** i.e. that there exist a bounded solution of the scheme $\phi_h \in \mathcal{B}_M$ to the equation $S(k, h, x, \phi(s, x), \phi) = 0$.

Lemma 4. *The scheme S is stable.*

Proof. If $w \in \mathcal{B}_M$ then $\|T_{h,k}w\| \le M$ for some $M > 0$. Indeed, we have

$$T_{h,k}w(s, z) = \max\left[\frac{h}{k}w(s + k, z) + w(s, z + h)\left(\frac{\sigma_0^2\beta^2(z)}{2h} + \alpha(z)\right) + w(s, z - h)\frac{\sigma_0^2\beta^2(z)}{2h},\ g(s, z)\right]$$

and

$$T_{h,k}w(s, z) \le \frac{1}{C(z)}\left(\frac{h}{k} + \frac{\sigma_0^2\beta^2(z)}{h} + \alpha(z)\right) \times \max(\|w\|, g(s, z)).$$

Thus

$$\|T_{h,k}w\| \le \max(\|w\|, \|g\|) \le M.$$

In addition the solution of the equation $S(k, h, z, w(t, z), w) = 0$ is the unique fixed point of the map $T_{h,k}$ due to the contraction mapping theorem. Thus there is a unique solution ϕ_h of the scheme in $\mathcal{B}_M$ with $\|\phi_h\| \le M$. □

In the following we show that the scheme converges to the viscosity solution by applying the same steps as in [8].

Proposition 4. *The numerical solution ϕ_h converges to the unique viscosity solution of Eqs. (4.1)–(4.3) as $k \to 0$ and $h \to 0$.*

Proof. We define the function

$$\psi_h(s, x) := \begin{cases} \phi_h(s, x), & s \in [0, T), \\ g(s, x) & s = T. \end{cases}$$

and in addition

$$\bar{\psi}(s, x) := \limsup_{y\to x,\, k,h\to 0} \psi_h(s, y), \qquad \underline{\psi}(s, x) := \liminf_{y\to x,\, k,h\to 0} \psi_h(s, y).$$

We want to prove that $\bar{\psi}$ is a subsolution, and $\underline{\psi}$ is a supersolution of problem (4.1)–(4.3). For the function $\bar{\psi}$ to be subsolution of the problem it is enough to prove, according to the definition of a subsolution for the problem, that if for every $\Psi \in C^2(\mathbb{R}^2)$ and every $(s_0, x_0) \in D$ such that $\Psi \ge \bar{\psi}$ on D and $\Psi(s_0, x_0) = \bar{\psi}(s_0, x_0)$ we have

$$\max\left[\mathcal{A}\Psi(s_0, x_0) - r\bar{\psi}(s_0, x_0), g(s_0, x_0) - \bar{\psi}(s_0, x_0)\right] \ge 0.$$

This relation is equivalent to

$$H(s_0, x_0, \bar{\psi}, D_s\Psi, D_x\Psi, D_x^2\Psi) = \min\left[r\bar{\psi} - \mathcal{A}\Psi,\ \bar{\psi}(s, x) - g(s, x)\right] \le 0$$

for every test function $\Psi \in C^{1,2}([0, T] \times [0, 1])$ such that (s_0, x_0) is a strict local maximum of $\bar{\psi}(s, x) - \Psi(s, x)$.

We assume that we have a function, $\Psi \in C^2(\mathbb{R}^2)$ such that $\bar{\psi}(s_0, x_0) = \Psi(s_0, x_0)$ and that $\bar{\psi}(s, x) - \Psi(s, x) \le 0$ in the ball $B((s_0, x_0), R_0)$ for some $R_0 > 0$.

Due to the fact that the scheme S is stable, i.e. that the solution of the scheme $\phi_h \in \mathcal{B}_M$ we may take Ψ so that $\Psi \ge 2\sup_{k,h}\|\phi_h\|$ outside the ball $B((s_0, x_0), R_0)$.

Therefore we can find a sequence $k_n > 0$, $h_n > 0$, $(s_n, y_n) \in [0, T] \times [0, 1]$ such that for $n \to \infty$, $k_n \to 0$, $h_n \to 0$, $(s_n, y_n) \to (s_0, x_0)$, and $\psi_h(s_n, y_n) \to \bar{\psi}(s_0, y_0)$ by the definition of $\bar{\psi}$.

In addition, the point (s_n, y_n) is a global maximum point of $\psi_h - \Psi$, by the definition of $\bar{\psi}$ and the relations $\bar{\psi}(s, x) - \Psi(s, x) \le 0$ in $B((s_0, x_0), R_0)$ and $\Psi \ge 2\sup_{k,h}\|\phi_h\|$ outside $B((s_0, x_0), R_0)$.

Thus for $\delta_n = \psi_h(s_n, y_n) - \Psi(s_n, y_n)$ we have $\delta_n \to 0$ and $\psi_h(s, y) = \Psi(s, y) + \delta_n$ for every $(s, x) \in [0, T] \times [0, 1]$.

Then by the monotonicity of S and the fact that $\Psi(s_n, y_n) + \delta_n \ge \psi_h(s, y)$ we have

$$S(k_n, h_n, y_n, \Psi(s_n, y_n) + \delta_n, \Psi + \delta_n) \le S(k_n, h_n, y_n, \psi_h(s, y), \psi_h) = 0,$$

and that

$$\lim_{n\to\infty} \frac{S\left(k_n, h_n, y_n, \Psi(s_n, y_n) + \delta_n, \Psi + \delta_n\right)}{h_n} \le 0.$$

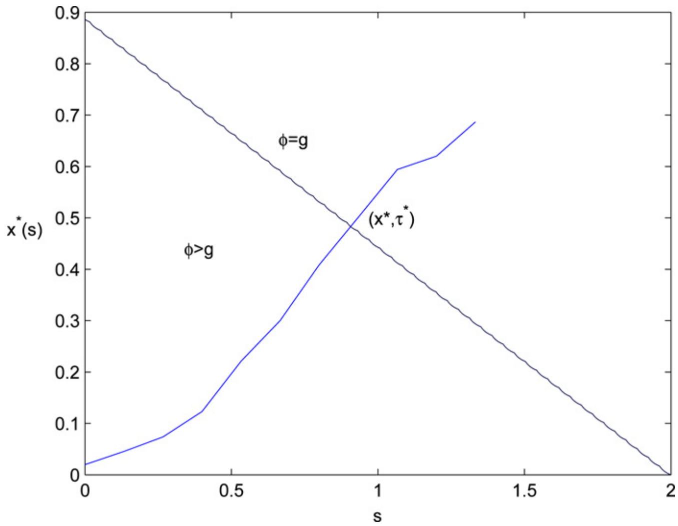

Fig. 1. The free boundary, $x^*(s)$, plotted against time, arising by the numerical solution of problem (4.1)–(4.3) for $M_h = 81$, $h = 0.0125$, $k = 7.8 \times 10^{-5}$, and accuracy $\epsilon = 10^{-6}$.

Thus finally we have

$$H(s_0, x_0, \bar{\psi}, D_s\Psi, D_x\Psi, D_x^2\Psi) = \lim_{y\to x_0, k_n, \delta, h\to 0} \frac{S(k, h, y, \Psi(s_n, y_n) + \delta, \Psi + \delta)}{h} \le 0,$$

and therefore that $\bar{\psi}$ is a viscosity subsolution.

Similarly we can deduce that $\underline{\psi}$ is a viscosity supersolution. By the uniqueness of the viscosity solution, we have that $\bar{\psi} = \psi = \underline{\psi}$ and $\lim_{h,k\to 0} \phi_h \to \phi$. □

4.2. Numerical simulations

We will demonstrate the numerical solution of the problem by choosing an arbitrary example. The numerical solution is obtained by applying the iteration scheme (4.7) for Dirichlet boundary conditions ($\phi(s, 0) = 0, \phi(s, 1) = f_0(1)$). For the simulations shown in this section we assume that $c(a, s) = c_0(T - s)$, where the dependence of c_0 on a, which is taken to be a constant, is assumed to be absorbed in the constant c_0 and that f_0 has a simple linear form $f_0 = \gamma x$, for γ constant. Note that different forms of the functions $c(a, s)$ and $f_0(x)$ can also be considered.

For the discretization we consider a partition of $[0, T] \times [0, 1]$ consisting of points $(s_n x_j,)$, $s_n = nk$, with $n = 1, \ldots, N_h$, $kN_h = T$ and $x_j = (j - 1)h, j = 1, \ldots, M_h$, $h(M_h - 1) = 1$ where h is the spatial step and k the time step. The iteration starts with initial value of $\phi_h^1 = 1$ for each point $(s_n x_j,)$ of the partition, and it is stopped when $\|\phi_h^{n-1} - \phi_h^n\| \le \epsilon$ for ϵ being the accuracy we require.

The values of the parameters used are $\sigma_0 = 0.5$, $r = 0.1$, $a = 1$, $b = 1$, $T = 2$, $\gamma = 0.9$, and $c_0 = 0.4$. For the accuracy we take $\epsilon = 10^{-6}$.

In Fig. 1 we plot the free boundary $x^*(s)$ by solving the problem using the iteration method of Eq. (4.7). The optimal stopping time for the campaign will be the first (random) time that a path of the process $x(t)$, starting at x at time s (jagged line in Fig. 1) hits the free boundary x^* (smooth line in Fig. 1). Note that due to the linear form of the functions f_0 and $c(s)$ we have also a linear form in the graph of the free boundary $x^*(s)$.

We can have also a different choice of the cost function, e.g. $c(s) = c_0(T - s)^m$, $0 < m < 1$ expressing the fact that adverting a product for a longer time one is able to achieve better prices and thus lower cost. Note also that an alternative choice would be $c(s) = c_0 \ln(T - s)$, etc. and that similar forms of dependence can be taken also for the function f_0. The nonlinearity in the cost function has an effect on the form of the free boundary. In Fig. 2 we plot the free boundary $x^*(s)$ by taking $c(s) = c_0(T - s)^m$, for $m = 0.7$.

Note that if instead of the nonhomogeneous Dirichlet Boundary conditions we pose Neumann boundary conditions at the points $x = 0$ and $x = 1$, numerical simulations indicate that the form of the free boundary $x^*(s)$ changes very little, if at all. This is an interesting observation, since the change of boundary conditions implies different modelling assumptions concerning the campaign policy and may be interpreted as some form of robustness of our results with respect to changes in the model.

In addition, the expected time until the optimal stopping of the campaign, as this is determined by Eq. (3.1) can by estimated by the use of the solution (3.2). This is done and the results are demonstrated for various times in Fig. 3. The values used for this simulation are $T = 2$, $m = 0.9$, $r = 0.1$, $c_0 = 0.4$, $a = b = 1$, $M_h = 81$.

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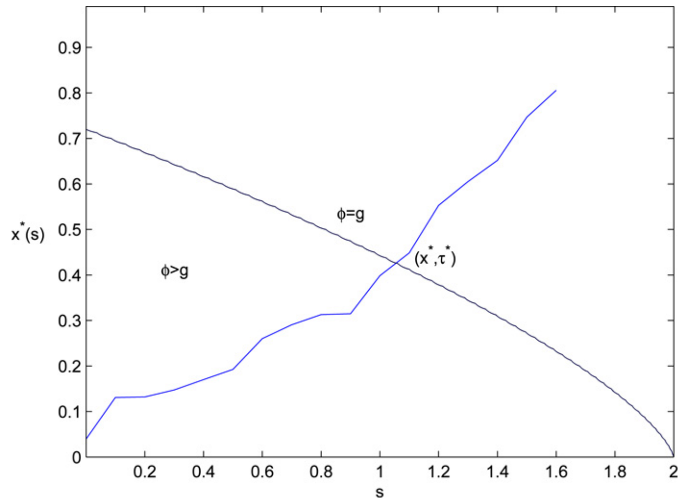

Fig. 2. The free boundary, $x^*(s)$, plotted against time, arising by the numerical solution of problem (4.1)–(4.3) for $c(s) = c_0(T - s)^m$, $m = 0.7$, $c_0 = 0.4$, $M_h = 81$, $h = 0.0125$, $k = 7.8125 \times 10^{-5}$, and accuracy $\epsilon = 10^{-6}$.

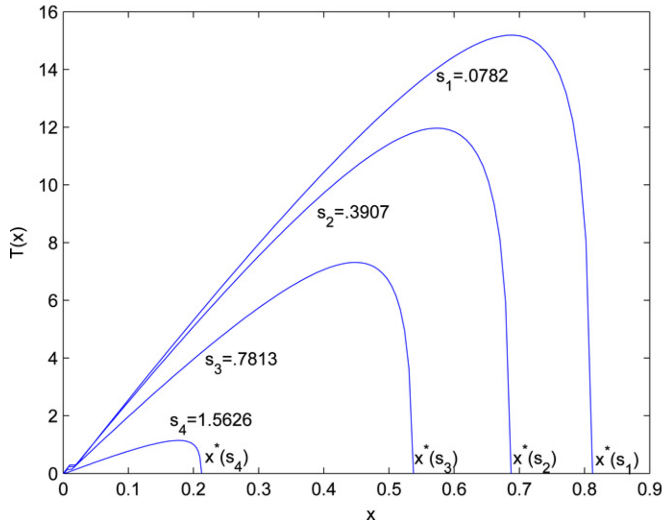

Fig. 3. Plot of the average time until stopping time, $T(x)$ for different values of $x^* = x^*(s)$, $s = 0.782,\ 0.3907,\ 0.7813,\ 1.5626$. Also $c(s) = c_0(T - s)^m$, $m = 0.9$, $c_0 = 0.4$, $M_h = 81$, $h = 0.0125$, $k = 7.8125 \cdot 10^{-5}$, and accuracy 10^{-6}.

5. A combined optimal stopping and control problem: Optimization of the advertisement effectiveness *a*

In the previous model, we have focused on choosing the optimal time for stopping the advertising campaign while keeping the advertising effectiveness parameter a at a steady state. This model gave us a feeling for how the campaign effectiveness parameter a affects the decision of when to stop the campaign.

We now move to the next level of complication, where we allow the campaign manager to change the advertising effectiveness parameter a, for example by increasing the intensity of advertising, i.e. we allow a to be a control variable. Therefore, now the campaign manager has the discretion to stop the campaign whenever she thinks it is optimal to do so, but while the campaign is running she is also free to change the effectiveness parameter in an optimal way. This turns the problem into a combined optimal stopping and control problem.

The cost functional now depends explicitly on the advertising policy a, i.e.

$$\Theta(s, x; t, a) = J(a) = E\left[\left(e^{-r(t-s)} f_0(x(t)) - \int_s^t e^{-r(t-s')} c(a(s'), s')\mathrm{d}s'\right) \mid \mathcal{F}_s\right],$$

Note, that in this problem we have introduced the cost of the advertisement campaign as an intertemporal term. This is a reasonable modelling assumption, since we now allow the effectiveness of the campaign to be a control variable, so that it is changing in time in an optimal way. Therefore, it does not make sense any longer to evaluate the cost of the campaign as a terminal term, and the correct cost evaluation must be made as an intertemporal term.

The problem now becomes:

Find a stopping time τ^* and an advertising policy $a^* \in U$ such that

$$\Phi(s,x) = \Theta(s,x;\tau^*,a^*) = sup_{\tau,a}\Theta(s,x;\tau,a \in U)$$

where the supremum is taken over all stopping times τ and all acceptable advertising policies $a \in U$. Of course the controlled process $x(t)$ is subject to the Bass model dynamics, i.e. it follows the stochastic differential equation (1.1).

The following proposition, relates the solution of this combined optimal stopping and control problem to the solution of a variational inequality [9].

Proposition 5. *The value of the optimal cost functional $\phi(s,x)$ is the solution of the variational inequality*

$$\max\left[\sup_{a\in U}\left(\mathcal{A}^a\Phi(s,x) - c(s,a)\right), g - \Phi(s,x)\right] = 0$$

where $\mathcal{A}^a$ is the generator of the diffusion process determined by (1.1) and $g(x) = f_0(x)$.

The operator $\mathcal{A}^a$ has the form

$$\mathcal{A}^a := \mathcal{A} = \frac{\partial}{\partial s} + \{a(1-x) + bx(1-x)\}\frac{\partial}{\partial x} + \frac{\sigma_0^2}{2}a^2(1-x)^2\frac{\partial^2}{\partial x^2}.$$

Thus for $\Phi(s,x) = \mathrm{e}^{-r(t-s)}\phi(s,x)$ the variational inequality becomes

$$\max\left[\sup_{a\in U}\left(\mathcal{A}^a\phi(s,x) - r\phi(s,x) - c(s,a)\right), g - \phi(s,x)\right] = 0. \tag{5.1}$$

In the following, as a preliminary approach, we can consider $U = [a_1, a_2]$ as a closed interval expressing the range of acceptable advertising policies.

5.1. Numerical formulation

As an initial approach to solve problem (5.1) we will apply a simple discretization of the problem resulting in a finite difference scheme similar to that for problem (4.1)–(4.3). Such a scheme can be considered as a generalization of standard explicit finite difference schemes applied to optimal stopping problems (see [10]). More specifically we have in $D = \{(s,x) \in V, : \phi > g\}$,

$$\max\left[\sup_{a\in U}\left(\mathcal{A}^a\phi(s,x) - r\phi(s,x) - c(a,s)\right), g - \phi(s,x)\right] = 0,$$
$$\text{with } \phi = g \quad \text{in } \partial D,$$

or, given that $U = [a_1, a_2]$, $a_1 > 0$ is the interval of acceptable advertising policies we have

$$\max\left[\max_{a\in[a_1,a_2]}\left(\phi_s(s,x) + \alpha(x;a)\phi_x(s,x) + \frac{\sigma_0^2}{2}\beta^2(x;a)\phi_{xx}(s,x) - r\phi(x,s) - c(a,s),\right) g(s,x) - \phi(s,x)\right] = 0,$$
$$\phi(s,0) = 0, \quad \text{at } x = 0, \qquad \phi(s,1) = f_0(1), \quad \text{at } x = 1,$$
$$\phi(T,x) = f_0(x), \quad x \in [0,1].$$

Therefore, applying a discretization in the interval $[a_1, a_2]$, by taking the values a^i, $a^i = (i-1)\delta a + a_1$, $i = 1, \ldots, I_h$, $a^{I_h} = a_2$, we derive an explicit finite difference scheme, for this problem

$$\phi_j^n = \max\left[\max_{a^i}\left(\phi_j^{n+1} + k\alpha(x_j;a^i)\frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2h} + \frac{\sigma_0^2}{2}\beta^2(x_j;a^i)\frac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{h^2}\right.\right.$$
$$\left.\left. -\; r\phi_j^{n+1} - c(s_{n+1}, a^i)\right), g(s_{n+1}, x_j)\right], \tag{5.2}$$

at the points (s_n, x_j), with $s_n = nk$, $n = 1, \ldots, N_h$, $kN_h = T$, $x_j = (j-1)h$, $j = 1, \ldots, M_h$ and $i = 1, \ldots, I_h$.

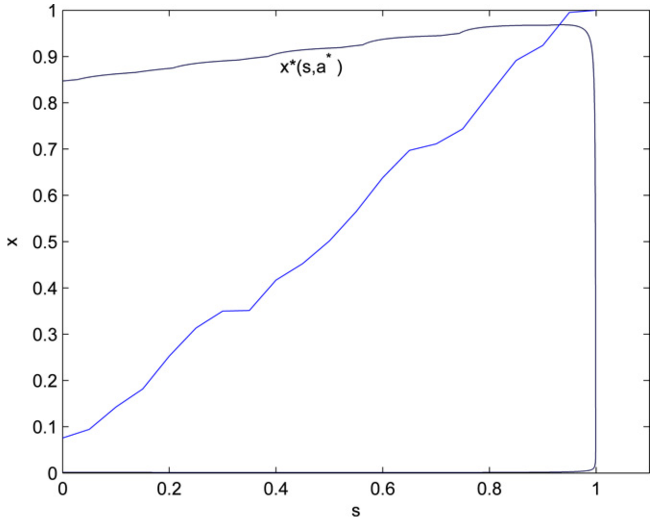

Fig. 4. The free boundary, $x^*(s)$, by the solution of the problem (5.1), plotted against time, for $\delta a = 0.005$, $M_h = 81$, $h = 0.0125$, $k = 7.8 \times 10^{-5}$.

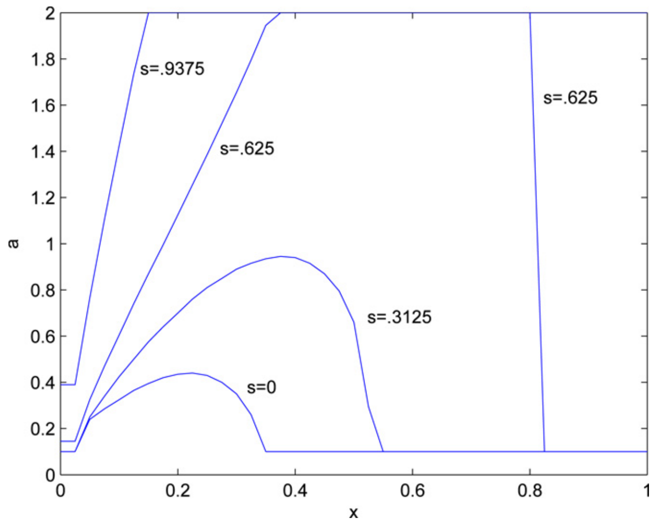

Fig. 5. Profile of the optimized value, $a^* = a^*(x, s)$, taken by the solution of the problem (5.1), against space, for various times. Here $a \in [a_1, a_2]$ with $a_1 = .1$, $a_2 = 2$ for $M_h = 81$, $h = .0125$, $k = 7.8 \times 10^{-5}$.

5.2. Numerical simulation

The numerical solution of the problem can be obtained by applying the finite difference scheme (5.2). We assume that the cost function depends linearly on the advertising policy a. More specifically we take $c(s; a) = c_0\, a\, (T - s)^m$, and that $f_0(x) = \gamma x$ for some constants m and γ.

In Fig. 4 we plot the free boundary $x^*(s)$ by solving numerically the problem (5.1). The values taken for this simulation are $T = 1$, $\sigma_0 = 0.5$, $r = 0.05$, $b = 0.5$, $c_0 = 0.1$, $m = 0.9$, $\gamma = 0.5$. Also the set U is taken to be $U = [a_1, a_2]$, for $a_1 = 0.1$ and $a_2 = 3.2$.

Note that the process of optimizing the value of a (the effectiveness of advertising) results in a shift of the free boundary to the right. Varying a in an optimal way means that the ability to lower a, and thus lowering the cost of the campaign, allows for the product to be advertised longer to obtain more profit.

The form of the optimized value of the parameter $a^* = a^*(x, s)$ can be seen in Fig. 5 where the form of a^* is plotted against x for various times. The parameter values here are the same as for Fig. 4, but with $U = [a_1, a_2]$, for $a_1 = 0.1$ and $a_2 = 2$.

We refrain from presenting a detailed analysis for the combined optimal stopping and optimal control problem, as it proceeds along similar lines as the optimal stopping problem.

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6. Conclusions

In this work we have derived a model for the determination of the optimal stopping time in the advertisement of a new product. A relevant free boundary problem, which attains a viscosity solution, is derived and an iteration algorithm is proposed for the solution of the free boundary problem. This numerical scheme is analyzed regarding its consistency, stability and convergence. In addition, the average time until the optimal stopping of the campaign is estimated. Finally the behavior of the model with respect to the various parameters is illustrated through the use of numerical experiments.

Furthermore, a combined optimal stopping and control problem for the optimization of the advertising effectiveness is also presented and solved numerically.

The parameters of the Bass model may be estimated using standard statistical tests from data readily available in the market. Such statistical estimates will provide the campaign manager with an idea of the effectiveness of the advertisement a and the peer pressure parameter b. The estimation of the cost of the campaign $c(a, t)$ is straightforward as well as the profit function related to the sales of the product. The noise parameter σ_0 can be related either to the errors involved in the statistical estimation of the parameter a, which is readily available from the design of the statistical test, or intrinsic uncertainty involved in the effectiveness of the campaign which can be modelled as noise and whose amplitude can be estimated by expert opinion. Its introduction, therefore, makes the model more robust and allows it to model reality more closely. When these parameters are estimated, the variational inequality and the approximation of the free boundary can offer the campaign manager an easy way to decide when the campaign should terminate. The stopping criterion is very simple: Given the exact shape of the free boundary all the manager has to do is to observe when the sales reach the boundary and then is the optimal time to terminate the campaign. The proposed numerical method can produce easy and fast results for the free boundary for different parameter ranges. Therefore, our approach offers the campaign manager an easy to use and effective decision tool, upon which different market scenarios can be generated and evaluated, and used to sharpen and enforce the practitioners' intuition and experience concerning the market and the effect that different mechanisms have on the evolution of the process. Of course, we should always bear in mind that the present model, as any model, is an idealized version of reality and at best can offer benchmarks concerning the situation.

It is also interesting, and a subject of future work, to examine the expansion of the model by including more general models for the random perturbations of the model parameters such as, for instance (see [11]), jump diffusion models.

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