A HYPERBOLIC NON-LOCAL PROBLEM MODELLING MEMS TECHNOLOGY

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Abstract. In this work we study a non-local hyperbolic equation of the form

$$u_{tt} = u_{xx} + \lambda \frac{1}{(1-u)^2 \left(1 + \alpha \int_0^1 \frac{1}{1-u} dx\right)^2},$$

with homogeneous Dirichlet boundary conditions and appropriate initial conditions. The problem models an idealised electrostatically actuated MEMS (Micro-Electro-Mechanical System) device. Initially we present the derivation of the model. Then we prove local existence of solutions for $\lambda > 0$ and global existence for $0 < \lambda < \lambda_-^*$ for some positive λ_-^*, with zero initial conditions; similar results are obtained for other initial data. For larger values of the parameter λ, *i.e.* when $\lambda > \lambda_+^*$ for some constant $\lambda_+^* \geq \lambda_-^*$, and with zero initial conditions, it is proved that the solution of the problem quenches in finite time; again similar results are obtained for other initial data. Finally the problem is solved numerically with a finite difference scheme. Various simulations of the solution of the problem are presented, illustrating the relevant theoretical results.

1. Introduction

The aim is to study the problem

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2 \left(1 + \alpha \int_0^1 \frac{1}{1-u} dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{1.1a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 < x < 1\,, \tag{1.1c}$$

in the case where λ, α are positive parameters. When $\alpha = 0$ problem (1.1) reduces to the local hyperbolic problem

$$u_{tt} = u_{xx} + \lambda/(1-u)^2, \quad 0 < x < 1, \quad t > 0, \tag{1.2a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{1.2b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 < x < 1\,, \tag{1.2c}$$

which has been studied in [3, 4, 14, 22]. For $\alpha > 0$ we obtain a non-local hyperbolic problem which, to our knowledge, has not been previously studied.

The equation describes the operation of an idealised electrostatically actuated MEMS (Micro-Electro-Mechanical System) device, which consists of a membrane and a rigid plate placed in a parallel position, and connected in series with a capacitor. The membrane is taken to be rectangular with two fixed parallel sides, while the other sides are free. A potential difference applied between the membrane and the plate causes the membrane to be deformed.

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The deformation u satisfies equation (1.1a) assuming that dissipation, which might result from either an electric current flowing through a resistance or viscous effects on the moving membrane, can be neglected. The parameter α represents the ratio of a reference capacitance to the fixed capacitance, in particular, the reference will be the capacitance of the device when its displacement vanishes, while λ is a control parameter proportional to the square of the applied voltage. A deformation of $u = 1$ at some point (x, t), represents the membrane having been moved so that it touches the plate (the zero-potential equilibrium separation is 1, in dimensionless variables).

More specifically in the derivation of the model, [9, 24], it is possible to consider a dissipative, u_t, term which can possibly dominate the acceleration, u_{tt}, term. In this case the version of equation (1.1a) is parabolic:

$$u_t = u_{xx} + \frac{\lambda}{(1-u)^2\left(1+\alpha\int_0^1 \frac{1}{1-u}dx\right)^2}, \quad 0 < x < 1, \quad t > 0\,, \tag{1.3a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{1.3b}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1\,. \tag{1.3c}$$

This problem has been extensively studied ([15, 16, 20, 24, 25], *etc.*), although there are still a number of open questions.

Including both the dissipative and inertial terms we would get equation (1.1a) but with an additional term ϵu_t, for some constant ϵ, on the left-hand side. For ϵ small, *i.e.* negligible dissipation, equation (1.1a) is obtained as an approximation of the model.

Note that for the non-local problem (1.1) there are no results regarding the behaviour of the solution. However, there are some results for the local version of the problem in [3], for the one-dimensional case, and in [27] for the two- and three-dimensional cases.

The local parabolic problem

$$u_t = \Delta u + \lambda f(x)/(1-u)^2, \quad x \in \Omega, \quad t > 0, \tag{1.4a}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.4b}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4c}$$

where Ω is a smooth domain of $\mathbb{R}^N$, $N \geq 1$, also describes the operation of a MEMS device for a simpler regime. Here the positive parameter λ is again proportional to the square of a fixed applied voltage, while $f(x)$ represents spatial variation of the electrical permittivity of the elastic membrane; by the physics of the problem, f is a non-negative function. A dielectric profile which is often encountered in applications is $f(x) = |x|^p$, $p > 0$. For such a case, there exists a critical value of λ, λ^*, [12, 21], usually called pull-in voltage in MEMS literature, such that the steady-state problem corresponding to (1.4) has no solution for $\lambda > \lambda^*$. Moreover, for such values of λ, the solution $u(x,t)$ of (1.4) "quenches" in finite time, *i.e.* $\min_x\{1 - u(x,t)\} \to 0$ as $t \to T- < \infty$, see [13, 17, 21]. The quenching behaviour physically corresponds to the phenomenon of "touch-down", *i.e.* the elastic membrane touches the rigid plate. For a thorough study of the structure of the solutions of the steady-state problem corresponding to (1.4), see [7, 12, 21], while for a further study of (1.4) see [18, 19].

Regarding problem (1.3), it has also been proved that quenching occurs for large enough values of the parameter λ, [15, 20], as well as for big enough initial datum $u_0(x)$, see [16]. Furthermore, some estimates of the pull-in voltage λ^* as well as the study of the stability, through the Morse index, of the steady states of (1.3) for higher dimensions are presented in [16]. Note that finite time of existence has been more frequently studied for blow-up in equations such as $u_t = u_{xx} + \lambda f(u)/(\int f(u)\,dx)^p$, where now $||u||_\infty \to \infty$ as $t \to T < \infty$; such equations can model Ohmic heating or shear-band formation, see for example [1] and [2].

The present paper is organized as follows. In Section 2 we present the derivation of the model. The local existence of (1.1) is studied in Section 3. The corresponding steady-state problem is studied in Section 4, while global existence of (1.1) is established for small enough values of the parameter λ in Section 5. Section 6 is devoted to the study of the quenching of solutions of problem (1.1). More precisely, quenching of solution $u(x,t)$ is proved for a large enough value of the parameter λ first in the case of zero initial data, and second for non-zero data. Finally in Section 7 a numerical solution of the problem is given via a finite difference (two-step Crank-Nicolson) scheme and some simulations illustrating the theoretical results of this work are presented.

The study of the local behaviour near quenching will be investigated in a subsequent paper, using a self-similar approach as has been done for the blow-up behaviour of standard semilinear wave equations, see [5, 6, 10].

2. Derivation of the Model

We consider an idealised electrostatically actuated MEMS device. It consists of a membrane and a rigid plate which are placed parallel to each other. The membrane has two parallel sides fixed, while the other sides are free. For simplicity we only look at the homogeneous case, $f \equiv 1$.

A potential difference, not *a-priori* known, is applied between the top surface (the membrane) and the rigid plate. Both membrane and plate have width w and length L, and in the undeformed state (for the membrane) the distance between the membrane and the plate is l.

We assume here that the gap between the plate and membrane, typically of order l, is small, $l \ll L$ and $l \ll w$, and is occupied by some inviscid material with dielectric constant one, so permittivity is that of free space, ϵ_0.

Taking the potential difference across the device to be V, and assuming that the plate is earthed, the small aspect ratio of the gap gives potential, ϕ, to leading order,

$$\phi = V(l - z')/(l - u') \tag{2.1}$$

where u' is the displacement of the membrane towards the plate ($u' = l$ corresponds to touch-down) and z' is the distance measured from the undisturbed membrane position towards the plate. The electrostatic force per unit area on the membrane (in the z' direction) is then

$$\tfrac{1}{2} \times \text{surface charge density} \times \text{electic field} = \tfrac{1}{2}\epsilon_0\phi_{z'}^2 = \tfrac{1}{2}\epsilon_0 V^2/(l-u')^2\,.$$

Taking the sides of length w, say $x' = 0$ and $x' = L$, to be fixed, with those of length L, say $y' = 0$ and $y' = w$, to be free, there to be no variation in the y' direction, so $u' = u'(x', z', t')$ for time t', the surface density of the membrane to be ρ, and there to be a constant surface tension T_m in the membrane, its displacement satisfies the forced wave equation

$$\rho u'_{t't'} = T_m u'_{x'x'} + \frac{1}{2}\epsilon_0 V^2/(l-u')^2\,. \tag{2.2}$$

The boundary conditions

$$u'(0,t') = u'(L,t') = 0 \tag{2.3}$$

also hold.

Partially scaling variables, $u' = lu$, $x' = Lx$, $t' = L\sqrt{\rho/T_m}\,t$, these become

$$u_{tt} = u_{xx} + (\epsilon_0 V^2 L^2/T_m l^2)/(1-u)^2\,, \tag{2.4}$$

and

$$u(0,t) = u(1,t) = 0\,. \tag{2.5}$$

Now we suppose that the MEMS device, which will have a capacitance C depending on displacement, is connected in series with a capacitor of fixed capacitance C_f and a source of fixed voltage V_s. Then $V_s = \frac{Q}{C_c} = Q\left(\frac{1}{C} + \frac{1}{C_f}\right)$, where Q is the charge on the device and fixed capacitor, and C_c the series capacitance of the two. Thus the potential difference, V, across the MEMS device will be

$$V = \frac{V_s}{1 + C/C_f}\,. \tag{2.6}$$

In addition

$$Q = \epsilon_0 \int_0^w \int_0^L \phi_{z'}(x,y,0)\,dx'\,dy' = V\frac{wL\epsilon_0}{l}\int_0^1 \frac{1}{1-u}\,dx\,,$$

using (2.1),

$$\text{so} \quad C = C_0 \int_0^1 \frac{1}{1-u}\,dx\,,$$

where $C_0 = wL\epsilon_0/l$ is the capacitance of the undeflected device.

Substituting this into (2.6) and the result into (2.4) leads to

$$u_{tt} = u_{xx} + \lambda \frac{1}{(1-u)^2\left(1 + \alpha\int_0^1 \frac{1}{1-u}\,dx\right)^2}\,, \tag{2.7}$$

with $\lambda = \epsilon_0 V_s^2 L^2/T_m l^2$ and $\alpha = wL\epsilon_0/lC_f$. Conditions (2.5) continue to apply.

Cases of interest often have vanishing initial conditions,

$$u(x,0) = u_t(x,0) = 0\,,$$

but we shall also look at cases with more general conditions $u_0(x)$, $v_0(x)$.

3. Local Existence of Solutions

In this section we establish local existence of the solution of problem

$$u_{tt} = u_{xx} + \lambda \frac{1}{(1-u)^2\left(1 + \alpha\int_0^1 \frac{1}{1-u}\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0 \tag{3.1a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{3.1b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 < x < 1. \tag{3.1c}$$

where $u_0, v_0 \in C^1\big((0,1)\big)$ and $u_0(0) = u_0(1) = 0$, by modifying appropriately the proof given for the local problem in [3], see also [11].

Definition 3.1. *We say that u is a weak solution of (3.1) in $Q_T \equiv (0,1)\times(0,T)$ if:*

- *(i) u is continuous in $\bar{Q}_T$ and satisfies the initial and boundary conditions.*
- *(ii) There exists some $\delta > 0$ such that $|u| \leq 1 - \delta$ in $\bar{Q}_T$.*
- *(iii) u has weak derivatives u_x, u_t in $\bar{Q}_T$ and for all $t \in (0, T)$, $u_x, u_t \in L^2\big((0,1)\big)$.*
- *(iv) For any function $\zeta(x,t) \in C^2(\bar{Q}_T)$ satisfying the boundary conditions and for $0 \leq t \leq T$,*

$$\begin{aligned}\int_0^1 \zeta(x,t) u_t(x,t)\, dx &= \int_0^t \int_0^1 [\zeta_\tau(x,\tau) u_\tau(x,\tau) - \zeta_x(x,\tau) u_x(x,\tau)]\, dx\, d\tau \\ &+ \lambda \int_0^t \int_0^1 \frac{\zeta(x,\tau)\, dx\, d\tau}{(1-u(x,\tau))^2 \left(1 + \alpha \int_0^1 \frac{1}{1-u(y,\tau)}\, dy\right)^2}. \end{aligned} \tag{3.2}$$

- *(v) The total energy associated with (3.1) is preserved (see also Section 5), i.e.*

$$E_T(t) = \frac{1}{2} \int_0^1 (u_x^2 + u_t^2)\, dx + \frac{\lambda}{\alpha} \frac{1}{\left(1 + \alpha \int_0^1 \frac{1}{1-u}\, dx\right)} = E_T(0) := E_0. \tag{3.3}$$

We consider a fixed $\delta \in (0,1)$ and assume that the initial data satisfy the condition

$$||u_0||_\infty + T||v_0||_\infty < 1 - 2\delta \tag{3.4}$$

for a positive T. Define the odd periodic (with period two) extensions with respect to x on $\mathbb{R} \times [0,T]$ of u, u_0, v_0 which are denoted, without any confusion, again as u, u_0, v_0.

We also define the function $G_p : \mathbb{R} \times [0, +\infty) \times (-1,1) \to \mathbb{R}$, as

$$G_p(x,t,u) = \begin{cases} F(u), & x \in [2n, 2n+1), \\ F(-u), & x \in [2n-1, 2n), \end{cases} \tag{3.5}$$

for $n = 0, \pm 1, \pm 2, \ldots$, where

$$F(u) = \frac{1}{(1-u)^2 \left(1 + \alpha \int_0^1 \frac{1}{1-u} dx\right)^2}.$$

Then by standard arguments applied to wave equations, we have that u is a solution of (3.1) if and only if u solves in $\mathbb{R} \times [0,T]$ the integral equation

$$u(x,t) = u_1(x,t) + \frac{\lambda}{2} \int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y, \tau, u(y,\tau))\, dy\, d\tau, \tag{3.6}$$

where

$$u_1(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2} \int_{x-t}^{x+t} v_0(z)\, dz.$$

Note that (3.4) implies that

$$||u_1||_T = \sup_{t \in [0,T]} ||u_1(t)||_\infty < 1 - 2\delta. \tag{3.7}$$

Let B_T be the Banach space of continuous odd periodic (with period two) functions with respect to x, defined in $\mathbb{R} \times [0,T]$, vanishing on $x = n$, $n \in \mathbb{Z}$, with the norm $||(\cdot)||_T$. Also let $\bar{B}(u_1,\delta)$ be the closed ball of radius δ centred on u_1 in B_T. Consider the operator

$$\mathcal{T}[u(x,t)] = u_1(x,t) + \frac{\lambda}{2}\int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y,\tau,u(y,\tau))\,dy\,d\tau,$$

which, due to the definition of the function G_p, is well-defined.

In order to show that (3.1), or equivalently (3.6), has a local-in-time solution, it is enough to show that the operator $\mathcal{T}$ is a contraction from $\bar{B}(u_1,\delta)$ to $\bar{B}(u_1,\delta)$. (Note that $u_1 \in B_T$.) In particular, we have to show that

$$\big\|\mathcal{T}u - u_1\big\|_T < \delta\,, \qquad \big\|\mathcal{T}u - \mathcal{T}v\big\|_T < K\big\|u-v\big\|_T\,,$$

for $u, v \in \bar{B}(u_1,\delta)$ and $0<K<1$.

In the following, for convenience, we write $f(u) := \frac{1}{1-u}$ and $I(u) := \int_0^1 f(u)dx$, hence $F(u) := \frac{f^2(u)}{[1+\alpha I(u)]^2}$. Then we have

$$\big\|\mathcal{T}v(x,t) - \mathcal{T}u(x,t)\big\|_T = \frac{\lambda}{2}\Big\|\int_0^t \int_{x-t+\tau}^{x+t+\tau} \left[G_p(y,\tau,v(y,\tau)) - G_p(y,\tau,u(y,\tau))\right]\,dy\,d\tau\Big\|_T\,,$$

where for $x-t+\tau>0$

$$\begin{aligned}[G_p(y,\tau,v(y,\tau)) - G_p(y,\tau,u(y,\tau))] &= \frac{f^2(v)}{[1+\alpha I(v)]^2} - \frac{f^2(u)}{[1+\alpha I(u)]^2}\\ = f^2(v)\left(\frac{1}{[1+\alpha I(v)]^2} - \frac{1}{[1+\alpha I(u)]^2}\right) &+ \frac{1}{[1+\alpha I(u)]^2}\left(f^2(v)-f^2(u)\right).\end{aligned}$$

Note that $f^2(v)-f^2(u) = (f(v)+f(u))(f(u)-f(v))$ and $\|f(u)-f(v)\|_T \le |f'(1-\delta)|\,\|u-v\|_T$, by taking into account (3.7) and the fact that f is a convex function.

Moreover we have

$$\begin{aligned}\frac{1}{[1+\alpha I(v)]^2} - \frac{1}{[1+\alpha I(u)]^2} &= \frac{[1+\alpha I(u)]^2 - [1+\alpha I(v)]^2}{[1+\alpha I(v)]^2\,[1+\alpha I(v)]^2}\\ &< \alpha[2+\alpha(I(u)+I(v))][I(u)-I(v)]\,.\end{aligned}$$

since $I(u), I(v) > 0$. Using again (3.7) and the fact that f is an increasing and convex function we derive $\|I(u)-I(v)\|_T \le f'(1-\delta)\|u-v\|_T$ and $\|2+\alpha(I(u)+I(v))\|_T \le 2[1+\alpha f(1-\delta)]$. Thus finally

$$\begin{aligned}\big|G_p(y,\tau,v(y,\tau)) - G_p(y,\tau,u(y,\tau))\big| &\le 2\,\alpha f^2(1-\delta)f'(1-\delta)[1+\alpha f(1-\delta)]\big\|u-v\big\|_T\\ &\quad +2f(1-\delta)f'(1-\delta)\big\|u-v\big\|_T,\end{aligned}$$

$$\text{or} \qquad \big|G_p(y,\tau,v(y,\tau)) - G_p(y,\tau,u(y,\tau))\big| \le C(f,\delta)\|u-v\|_T\,,$$

for $C(f,\delta) = 2\alpha f^2(1-\delta)f'(1-\delta)[1+\alpha f(1-\delta)] + 2f(1-\delta)f'(1-\delta)$. The same final estimate also holds when $x-t+\tau<0$.

Therefore

$$\begin{aligned}\big\|\mathcal{T}v(x,t) - \mathcal{T}u(x,t)\big\|_T &= \frac{\lambda}{2}\Big\|\int_0^t \int_{x-t+\tau}^{x+t+\tau} \big[G_p(y,\tau,v(y,\tau)) - G_p(y,\tau,u(y,\tau))\big]\,dyd\tau\,\Big\|_T\\ &\le \frac{\lambda}{2}t^2 C(f,\delta)\|u-v\|_T \le \frac{\lambda}{2}\sigma^2 C(f,\delta)\big\|u-v\big\|_T\,, \qquad (3.8)\end{aligned}$$

for $0\le t\le\sigma$. Note that $K = \frac{\lambda}{2}\sigma^2 C(f,\delta) < 1$ if $\sigma < \sqrt{\frac{2}{\lambda}}\frac{1}{\sqrt{C(f,\delta)}}$.

It remains to show that $\|\mathcal{T}u - u_1\|_T < \delta$. We have again for $x-t+\tau > 0$,

$$\begin{aligned}
\|\mathcal{T}u(x,t) - u_1\|_T &= \frac{\lambda}{2}\Big\|\int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y,\tau,u(y,\tau))dy\,d\tau\Big\|_T \\
&\leq \Big\|\frac{\lambda}{2}\int_0^t \int_{x-t+\tau}^{x+t+\tau} \frac{f^2(u)}{[1+\alpha I(u)]^2}dy\,d\tau\Big\|_T \\
&\leq \frac{\lambda}{2}f^2(1-\delta)\Big\|\int_0^t \int_{x-t+\tau}^{x+t+\tau} dyd\tau\,\Big\|_T \\
&\leq \frac{\lambda}{2}t^2 f^2(1-\delta) \leq \frac{\lambda}{2}\sigma^2 f^2(1-\delta).
\end{aligned}$$

Since the same final estimate is obtained for $x-t+\tau < 0$ (when the first inequality in (3.8) becomes strict), we end up with $\|\mathcal{T}u(x,t) - u_1\|_T \leq \delta$ as long as $\frac{\lambda}{2}\sigma^2 f^2(1-\delta) \leq \delta$, *i.e.* if $\sigma \leq \sqrt{\frac{2}{\lambda}}\sqrt{\frac{\delta}{f^2(1-\delta)}}$.

Thus finally, if we choose σ such that

$$\sigma < \min\left\{T, \sqrt{\frac{2}{\lambda}}\left(\frac{\delta}{f^2(1-\delta)}\right)^{\frac{1}{2}}, \sqrt{\frac{2}{\lambda}}\left(\frac{1}{C(f,\delta)}\right)^{\frac{1}{2}}\right\},$$

we conclude that the operator $\mathcal{T} : B(u_1,\delta) \to B(u_1,\delta)$ is a contraction and hence the Banach fixed point theorem guarantees the existence of a unique fixed point for $\mathcal{T}$.

We have thus established local existence of solution of problem (1.1) in an interval $[0,\sigma]$. In particular:

Theorem 3.2. *If the initial data $u_0(x), v_0(x) \in C^1((0,1))$ satisfy condition (3.4), then, for any $\lambda > 0$, problem (3.1) has a unique weak C^1-solution on $Q_T = (0,1)\times[0,T]$ if T is sufficiently small.*

Remark 3.3. *By the proof of the above theorem we also obtain that the solution u to (3.1) is piecewise C^2 in Q_T. Furthermore, the solution of (3.1) could be extended to any interval of the form $[0, T+\tau]$ for τ sufficiently small and positive as long as $|u| < 1$ on $\bar{Q}_T$.*

Remark 3.4. *In the case of zero initial datum we could obtain, by differentiating relation (3.6), that $u(x,t)$ is a regular solution to (3.1) except on the point set*

$$\{(x,t) \in \mathbb{R}\times[0,T] | x, x-t, x+t \text{ are integers}\},$$

see also [3].

Definition 3.5. *The solution $u(x,t)$ of problem (1.1) quenches at point $x^* \in (0,1)$ in finite time $0 < T < \infty$ if there exist sequences $\{x_n\}_{n=1}^{\infty} \in (0,1)$ and $\{t_n\}_{n=1}^{\infty} \in (0,\infty)$ with $x_n \to x^*$, $t_n \to T-$ and $u(x_n,t_n) \to 1-$ as $n \to \infty$. In the case where $T = \infty$ we say that $u(x,t)$ quenches in infinite time at x^*.*

By Remark 3.3 we conclude that the solution $u(x,t)$ to problem (1.1) ceases to exist only by quenching. In MEMS terminology quenching is usually called touch-down since it describes the phenomenon when the elastic membrane touches the rigid plate on the bottom of the MEMS device. In the physical problem this is usually followed by destruction of the MEMS device. The study of the quenching phenomenon is also important from mathematical point of view, because when it occurs there is singular behaviour of $u(x,t)$.

4. The Steady State

The steady-state problem corresponding to (3.1) is

$$w'' + \frac{\lambda}{(1-w)^2\left(1+\alpha\int_0^1 \frac{1}{1-w}dx\right)^2} = 0\,, \quad 0<x<1\,; \qquad w(0)=0\,, \quad w(1)=0\,. \tag{4.1}$$

If we set $W = 1-w$ then (4.1) becomes

$$W'' = \mu/W^2\,, \quad 0<x<1\,; \qquad W(0)=1\,, \quad W(1)=1\,, \tag{4.2}$$

where

$$\mu = \lambda \bigg/ \left(1+\alpha\int_0^1 \frac{1}{W}dx\right)^2 . \tag{4.3}$$

Then multiplying both sides by W' and integrating from $m = \min\{W(x),\ x\in[0,1]\} = W(1/2)$ to W we derive

$$\int_0^{W'} W'dW' = \int_{\frac{1}{2}}^{x} W''W'dx = \mu\int_{\frac{1}{2}}^{x}\frac{W'}{W^2}\,dx = \mu\int_m^W \frac{dW}{W^2}\,,$$

hence

$$\frac{1}{2}(W')^2 = \mu\left(\frac{1}{m}-\frac{1}{W}\right).$$

This gives equivalently

$$\frac{dx}{dW} = \sqrt{\frac{m}{2\mu}}\sqrt{\frac{W}{W-m}}\,,$$

which implies

$$x-\tfrac{1}{2} = \sqrt{\frac{m}{2\mu}}\left[\sqrt{W(W-m)} - \frac{1}{2}m\ln(m) + m\ln\left(\sqrt{W}+\sqrt{W-m}\right)\right].$$

This yields, on setting $x=1$ so that $W=1$,

$$\mu = 2m\left[\sqrt{1-m} - \frac{1}{2}m\ln(m) + m\ln\left(1+\sqrt{1-m}\right)\right]^2.$$

Furthermore we have

$$\begin{aligned}
\int_0^1 \frac{1}{W}dx &= \int_0^1 \frac{dx}{dW}\frac{dW}{W} = \sqrt{\frac{2m}{\mu}}\int_m^1 \frac{1}{\sqrt{W(W-m)}}dW \\
&= \frac{1}{\sqrt{1-m}-\frac{1}{2}m\ln(m)+m\ln\left(1+\sqrt{1-m}\right)}\int_m^1 \frac{1}{\sqrt{W(W-m)}}dW \\
&= \frac{1}{\sqrt{1-m}-\frac{1}{2}m\ln(m)+m\ln\left(1+\sqrt{1-m}\right)}\ln\left(\frac{2-m+2\sqrt{1-m}}{m}\right).
\end{aligned}$$

On using (4.3), we finally establish the following relation between λ and m,

$$\lambda = 2m\left[\sqrt{1-m} + m\ln\left(\frac{\left(1+\sqrt{1-m}\right)}{\sqrt{m}}\right) + \alpha\ln\left(\frac{2-m+2\sqrt{1-m}}{m}\right)\right]^2. \tag{4.4}$$

From the above relation, we can obtain the response (bifurcation) diagram of problem (4.1), see Figure 1 for the case of $\alpha=1$. In particular, (4.4) implies that $\lambda \sim 2\alpha^2 m(\ln m)^2$ as $m\to 0$.

Note that the numerical results in Section 7 indicate that the long-time behaviour of the solution, when it exists for all times, is periodic and the relevant steady-state solution is, from stability point of view, a centre. This means that although we have existence of steady-state solutions for every $\lambda < \lambda^*$, it may happen that for λ quite close but less than λ^* the solution of the evolutionary problem could oscillate around the steady solution and approach 1, *i.e.* it retains significant deformation and speed, and therefore it is more likely to quench. In conclusion, the critical value for the parameter λ, say λ^*_-, for which the solution of the hyperbolic problem exists for all times (with vanishing initial data), is expected to be lower than λ^*.

For a thorough study of the structure of the set of the steady-state solutions to (1.1) in higher dimensions see [16].

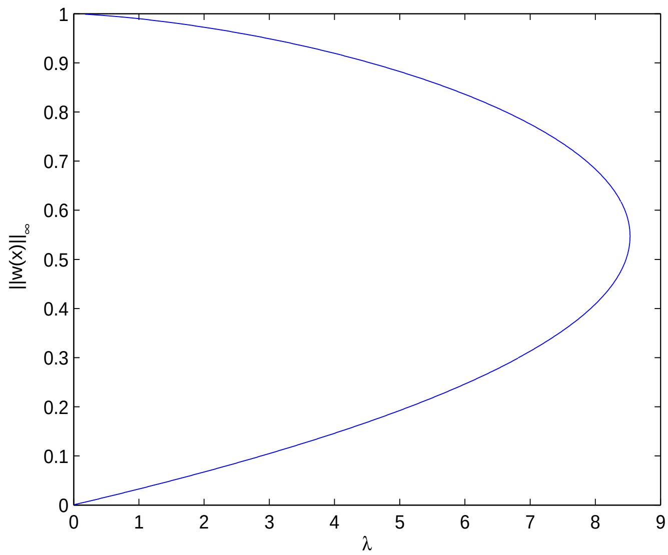

FIGURE 1. *Bifurcation diagram for problem (4.1) with* $\alpha = 1$. *Note that for this case,* $\lambda^* \approx 8.533$.

5. Global Existence

In order to prove global existence of problem (1.1), we need as a first step to determine the corresponding energy functional, see also Section 3. Unless otherwise stated, $||\cdot||$ now denotes $||\cdot||_2$.

To find the energy, we multiply equation (1.1a) by u_t and integrate over $[0,1]$. Thus we obtain

$$\int_0^1 (u_t u_{tt} - u_t u_{xx})\, dx = \lambda \int_0^1 \frac{u_t\, dx}{(1-u)^2} \frac{1}{(1+\alpha \int_0^1 \frac{1}{1-u} dx)^2},$$

or

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left(u_t^2+u_x^2\right)dx\right) &= \lambda\frac{d}{dt}\left(\int_0^1 \frac{1}{1-u}dx\right)\frac{1}{(1+\alpha\int_0^1 \frac{1}{1-u}dx)^2}\\ &= -\frac{\lambda}{\alpha}\frac{d}{dt}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u}dx}\right).\end{aligned}$$

Thus finally we get

$$\frac{1}{2}\int_0^1 \left(u_t^2+u_x^2\right)dx+\frac{\lambda}{\alpha}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u}dx}\right)=E_0\,, \tag{5.1}$$

where E_0 is a constant representing the initial energy of the system which is conserved and is given by

$$E_0=\frac{1}{2}\int_0^1 \left(v_0^2+{u_{0x}}^2\right)dx+\frac{\lambda}{\alpha}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u_0}dx}\right). \tag{5.2}$$

From equation (5.1) we deduce that

$$\|u_t\|^2+\|u_x\|^2+\frac{2\lambda}{\alpha}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u}dx}\right)=2E_0\,,$$

where $\|u_t\|=\|u_t(\cdot,t)\|$, $\|u_x\|=\|u_x(\cdot,t)\|$, and that

$$\|u_x\|^2+\frac{2\lambda}{\alpha}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u}dx}\right)\le 2E_0\,. \tag{5.3}$$

At this point we can use the one-dimensional Sobolev embedding and the Poincaré inequality to obtain

$$4u^2(x,t)\le\|u_x\|^2\,,\quad \text{for}\quad (x,t)\in[0,1]\times[0,T]. \tag{5.4}$$

We also define M, $M:=\max\{u(x,t),\,(x,t)\in[0,1]\times[0,T]\}$. For cases where $M<1$ irrespective of the value of T, the maximal time of existence of u is infinite.

Therefore we have that $u\le M$, $1-u\ge 1-M$, $1+\alpha\int_0^1\frac{1}{1-u}dx\le 1+\frac{\alpha}{1-M}$, and that

$$\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u}dx}\ge\frac{1}{1+\frac{\alpha}{1-M}}.$$

Thus we obtain by (5.3)

$$\|u_x\|^2+\frac{2\lambda}{\alpha}\left(\frac{1}{1+\alpha/(1-M)}\right)\le 2E_0$$

and due to (5.4) we conclude that

$$4u^2(x,t)+\frac{2\lambda}{\alpha}\left(\frac{1}{1+\alpha/(1-M)}\right)\le 2E_0\quad\text{for}\quad(x,t)\in[0,1]\times[0,T]. \tag{5.5}$$

Considering the simplest case where $u_0=0$, $v_0=0$ we have that $E_0=\lambda/\alpha(\alpha+1)$. For simplicity in the following we take $\alpha=1$ so $E_0=\lambda/2$.

Note that the solution u ceases to exist if $M=1$, while for $\alpha=1$ the inequality (5.5) yields

$$h(M)=h(M\,;\lambda):=2M^2+\lambda(1-M)/(2-M)\le E_0\,. \tag{5.6}$$

We have that $h(1) = 2$ and $h(0) = \frac{\lambda}{2}$, hence for $\lambda < 4$ we have that $h(1) > h(0)$. Also $h'(M) = 4M - \lambda/(2-M)^2$ with $h'(1) = 4 - \lambda > 0$ for $\lambda < 4$.

Hence we have the following global-in-time existence result.

Theorem 5.1. *Problem (1.1) with zero initial data has a global-in-time solution, i.e.* $0 < u(x,t) < 1$ *for all* $(x,t) \in [0,1] \times [0,\infty)$, *provided that* $\lambda < 4$.

With λ_-^* the largest possible for every $\lambda < \lambda_-^*$ to give a global solution of (1.1), in other words λ_-^* is the infimum of those λ which produce quenching, $\lambda_-^* \geq 4$.

Now consider the case of non-zero initial data, *i.e.* $u_0 \neq 0$ or $v_0 \neq 0$. In this case the initial energy is given by the expression

$$E_0 = E_0(\lambda, u_0, v_0) = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_{0x}\|^2 + \frac{\lambda}{\alpha}\left(\frac{1}{1+\alpha\int_0^1 \frac{1}{1-u_0}dx}\right).$$

Following the above, for u to exist it would then be enough to show that

$$4M^2 + \frac{2\lambda}{\alpha}\left(\frac{1-M}{1+\alpha-M}\right) \leq \|v_0\|^2 + \|u_{0x}\|^2 + \frac{2\lambda}{\alpha}F(u_0), \tag{5.7}$$

for $F(u_0) = 1/(1+\alpha\int_0^1 \frac{1}{1-u_0}dx)$, guarantees that $M < 1$. Clearly inequality (5.7) fails at $M = 1$ if

$$4 > \|v_0\|^2 + \|u_{0x}\|^2 + 2\lambda/\alpha \tag{5.8}$$

since $F(u_0) \leq 1$. Thus the solution exists for all time as long as $\lambda < \alpha\left(2 - \frac{1}{2}(\|v_0\|^2 + \|u_{0x}\|^2)\right)$. (*N.B.* With $u_0 = 0$, $F(u_0) = 1/(1+\alpha)$, and a stronger estimate, as of Thm. 5.1, is achieved.)

6. Quenching

6.1. **Quenching for zero initial conditions.** In the first part of this section we impose zero initial conditions, *i.e.* we consider the following problem

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2\left(1+\alpha\int_0^1 \frac{1}{1-u}dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{6.1a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{6.1b}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < 1. \tag{6.1c}$$

Our purpose is to show that there exists a critical value $\lambda_+^* \geq \lambda_-^*$ of λ such that the solution of problem (6.1) quenches in finite time for any $\lambda \geq \lambda_+^*$. Thus λ_+^* is the supremum of those λ for which (6.1) has a global solution.[1]

For convenience and due to the expected symmetry of the solution $u(x,t)$ at the point $x = \frac{1}{2}$ we will consider the equivalent problem

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2\left(1+2\alpha\int_0^{\frac{1}{2}} \frac{1}{1-u}dx\right)^2}, \quad 0 < x < \frac{1}{2}, \quad t > 0, \tag{6.2a}$$

$$u(0,t) = 0, \quad u_x\left(\frac{1}{2}, t\right) = 0, \quad t > 0, \qquad u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < \frac{1}{2}. \tag{6.2b}$$

We restrict our analysis for times $0 < t < 1/2$ so that the characteristic line $t = x$ remains in the strip $0 < x < 1/2$.

[1] Because (6.1) lacks clear monotonicity properties, it is not obvious that $\lambda_+^* = \lambda_-^*$.

Initially we consider the general problem

$$u_{tt} - u_{xx} = g(t), \quad 0 < x < \frac{1}{2}, \quad t > 0, \tag{6.3a}$$

$$u(0,t) = 0, \quad u_x\Big(\frac{1}{2}, t\Big) = 0, \quad t > 0, \qquad u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < \frac{1}{2}, \tag{6.3b}$$

with $g(t) > 0$ and continuous. For $x \geq t$, the solution of (6.3) is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} g(s)\, d\xi\, ds = \frac{1}{2}\int_0^t g(t)\,[x + (t-s) - x + (t-s)]\, ds \\ &= \int_0^t (t-s) g(s)\, ds := G(t), \end{aligned} \tag{6.4}$$

since in that case the domain of dependence of the point (x,t) is the triangle $D = \{(\xi, s)|\; x - (t-s) < \xi < x + (t-s) \quad \text{and} \quad 0 < s < t\}$. On the other hand, for $x \leq t$ we have to subtract the contribution coming from $x < 0$ so we get

$$u(x,t) = G(t) - G(t-x) = \int_0^t (t-s) g(s)\, ds - \int_0^{t-x} (t-x-s) g(s)\, ds\,. \tag{6.5}$$

Note that in this case the domain of dependence is no longer a triangular but is instead split into two parts $D_1 = \{(\xi, s)|\; x - (t-s) < \xi < x + (t-s) \quad \text{and} \quad t - x < s < t\}$ and $D_2 = \{(\xi, s)|\; t - x - s < \xi < x + (t-s) \quad \text{and} \quad 0 < s < t - x\}$.

From the above analysis we bear in mind that the solution of (6.3) in the area $x \geq t$ is only a function of time, $u(x,t) = G(t)$. In addition for $t \leq 1/2$ by the form of the solution we can easily deduce that $u_x \geq 0$ ($= 0$ for $x \geq t$, $= \int_0^{t-x} g(s)\, ds > 0$ for $x \leq t$). Finally for every $0 < x < 1/2,\ 0 < t < 1/2$ we have $0 \leq u(x,t) \leq G(t) = \max_{x\in[0,1/2]} u(x,t)$.

In the following we consider the more general problem

$$u_{tt} - u_{xx} = h(x,t), \quad 0 < x < \frac{1}{2}, \quad t > 0, \tag{6.6a}$$

$$u(0,t) = 0, \quad u_x\Big(\frac{1}{2}, t\Big) = 0, \quad t > 0, \qquad u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < \frac{1}{2}, \tag{6.6b}$$

with $h(x,t) > 0$, $h_x(x,t) > 0$ and continuous.

Using the same reasoning as above we easily obtain that the solution of problem (6.6) is given by

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} h(y,s)\, d\xi\, ds, \tag{6.7}$$

for $x \geq t$, and

$$u(x,t) = \frac{1}{2}\int_{t-x}^t \int_{x-(t-s)}^{x+(t-s)} h(y,s)\, d\xi\, ds + \frac{1}{2}\int_0^{t-x} \int_{t-s-x}^{x+(t-s)} h(y,s)\, d\xi\, ds\,,$$

for $x \leq t$. Due to the fact that $h(x,t) > 0$ and $h_x(x,t) > 0$ we have $u_x(x,t) \geq 0$ and hence $\max_{x\in[0,1/2]} u(x,t)$ is given by (6.7).

In our case, going back to our original problem (6.2), the function h has the specific form $h(x,t) = f(u)g(t)$ where $f(u) = \frac{1}{(1-u)^2}$ and $g(t) = \frac{\lambda}{\left(1+2\alpha\int_0^{1/2} \frac{1}{1-u}dx\right)^2}$. Note also that the function f is such that $f(0) = 1$ and $f'(s) > 0$ for $0 < s < 1$.

We shall show that for $x \geq t$ the solution of problem (6.2) is purely a function of time, *i.e.* $u(x,t) = U(t) = \max_{x\in[0,1/2]} u(x,t)$, by applying a Picard iteration.

Initially we define u_0 to be the solution of the problem

$$u_{0tt} - u_{0xx} = g(t),\ 0 < x < \frac{1}{2},\ 0 < t < \frac{1}{2},$$

with the boundary and initial conditions as defined above. As we have already stated, by the analysis of the problem (6.3), $u_0(x,t) = U_0(t)$ for $x \geq t$, $u_{0x} \geq 0$ and $0 \leq u_0(x,t) \leq U_0(t)$ where $U_0(t) = \max_{x\in[0,1/2]} u_0(x,t)$ coincides with $G(t)$ given by (6.4).

We inductively define the function $u_n(x,t)$ to be the solution of the problem

$$u_{ntt} - u_{nxx} = f(u_{n-1})g(t),\ 0 < x < \frac{1}{2},\ 0 < t < \frac{1}{2},$$

with the standard initial and boundary conditions. By the analysis of problem (6.6) we have, since $h(x,t) = f(u_{n-1})g(t) > 0$ and $h_x(x,t) = f'(u_{n-1})u_{(n-1)_x}g(t) > 0$ ($f' > 0$ and $(u_{n-1})_x > 0$ by the induction hypothesis), that u_n has the required property $u_{nx} > 0$. By the induction hypothesis we also have that $u_{n-1}(x,t)$ is purely a function of time in the area $x \geq t$ hence $f(u_{n-1})g(t)$ is also, implying that $u_n(x,t) = U_n(t) = \max_{x\in[0,1/2]} u_n(x,t)$ for any $n \in \mathbb{N}$ and $x \geq t$ and $U_n(t)$ is given by

$$U_n(t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(U_{n-1}(s))g(s)\, d\xi\, ds\,, \tag{6.8}$$

which implies that the sequence $\{U_n(t)\}_{n=1}^{\infty}$ is increasing, recalling that $f(s) > 1$ and $f'(s) > 0$ for $0 < s < 1$. Moreover, we have that $0 < U_n(t) < 1$ for every $0 < t < t_0$ for some $t_0 \leq 1/2$ hence $\{U_n(t)\}_{n=1}^{\infty}$ converges as $n \to \infty$ to a function $U(t)$ which is actually the maximum value of the unique solution $u(x,t)$ of (6.2) achieved for $x \geq t$, *i.e.* $u(x,t) = U(t) = \max_{x\in[0,1/2]} u(x,t)$ for $x \geq t$.

The latter implies that the solution of problem (6.2) satisfies for $x \geq t$ the equation

$$U_{tt} = f(U)g(t),$$

and since

$$g(t) \geq \lambda / \left[1 + \alpha/(1-U)\right]^2,$$

$U(t)$ finally satisfies the differential inequality

$$U_{tt} \geq \lambda / \left[1 + \alpha - U\right]^2, \tag{6.9}$$

with $U(0) = 0$ and $U_t(0) = 0$.

Therefore we obtain that the following inequality is satisfied

$$\sqrt{2\lambda}t \leq (1+\alpha)^{\frac{3}{2}} \left[\sin^{-1}\left(\sqrt{\frac{U}{1+\alpha}}\right) + \frac{\sqrt{U(1+\alpha-U)}}{1+\alpha}\right],$$

and we deduce that $U(t)$ reaches 1 before time $t = 1/2$, provided that

$$\lambda \geq 2(1+\alpha)\left[(1+\alpha)\sin^{-1}\left(\frac{1}{\sqrt{1+\alpha}}\right) + \sqrt{\alpha}\right]^2 = \lambda^+(\alpha). \tag{6.10}$$

Finally we have proved the following result:

Theorem 6.1. *If $\lambda \geq \lambda^+(\alpha)$, where $\lambda^+(\alpha)$ is given by (6.10), then the solution $u(x,t)$ to problem (6.2) quenches in finite time $T \leq 1/2$, i.e. $||u(\cdot,t)||_\infty \to 1-$ as $t \to T \leq 1/2$.*

The least upper bound for global existence, λ_+^*, then satisfies $\lambda_+^* \leq \lambda^+(\alpha)$.

Remark 6.2. *Note that for $\alpha = 1$ we have that the above relation gives $\lambda \geq \lambda^+ = 26.43596$.*

Remark 6.3. *For the local problem, i.e. when $\alpha = 0$, differential inequality (6.9) implies that $||u_{tt}(\cdot,t)||_\infty$ blows up at the quenching time, i.e. $||u_{tt}(\cdot,t)||_\infty \to +\infty$ as $t \to T$, which is in agreement with the result in* [4].

A result similar to Theorem 6.1 can be deduced when the initial data are non-zero and this is the subject of the next subsection.

6.2. **Quenching depending on the initial data.** We now prove quenching for λ large enough, depending on the initial data u_0 and v_0, using some of the arguments developed in the previous subsection. We consider problem (1.1) and we assume that u_0 is bounded away from one.

Let w be the solution of the problem

$$w_{tt} - w_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{6.11a}$$

$$w(0,t) = w(1,t) = 0, \ t > 0, \quad w(x,0) = u_0(x), \quad w_t(x,0) = v_0(x), \ 0 < x < 1. \tag{6.11b}$$

We also define the function v as the difference $v = u - w$ hence $v(x,0) = 0$ and $v_t(x,0) = 0$; moreover v satisfies homogeneous Dirichlet boundary conditions $v(0,t) = 0 = v(1,t)$.

Now we assume that $u_0(x)$ and $v_0(x)$ are smooth enough so that $w_t(x,t)$ is bounded below for $x \in (1/4, 3/4)$ and $t \in (0, 1/4)$. Then for some $t_0 < 1/4$ there exists $\epsilon > 0$ such that $\sup_{(1/4,3/4)\times(0,t_0)} w(x,t) < 1 - 2\epsilon$. Note that t_0 and ϵ can be taken independent of λ, and, having fixed t_0, ϵ can be arbitrarily small. We define C_0, independent of λ and of ϵ, by $C_0 = \inf_{[0,1]\times[0,t_0]}\{w\} \le 0$.

Now take $t_1(\epsilon,\lambda) \le t_0$ such that $||v(.,t)||_\infty < \epsilon$ for $0 \le t < t_1$. Then

$$C_0 - \epsilon \le u = w + v \le 1 - 2\epsilon + \epsilon \quad \text{for } \ 0 \le t \le t_1 .$$

Hence, for $0 < t < t_1$,

$$\epsilon < 1 - u < 1 + C_0 - \epsilon\,, \quad 1 + \alpha/(1 + C_0 - \epsilon) < 1 + \alpha \int_0^1 (1-u)^{-1} dx < 1 + \alpha/\epsilon$$

$$\text{and} \quad \lambda/\left((1 + C_0 - \epsilon)(1 + \alpha/\epsilon)\right)^2 < v_{tt} - v_{xx} < \lambda \Bigg/ \left(\epsilon\left[1 + \frac{\alpha}{1 + C_0 - \epsilon}\right]\right)^2 .$$

In particular, following the previous subsection, since $t \le x \le 1 - t$ for $1/4 \le x \le 3/4$, $0 \le t \le 1/4$, for $0 < t < t_1 \le t_0 \le 1/4$,

$$v(x,t) = V(t) \text{ for } \tfrac{1}{4} \le x \le \tfrac{3}{4}\,, \quad \text{with} \quad 0 \le v(x,t) \le V(t) \text{ for } x \le \tfrac{1}{4} \text{ and for } x \ge \tfrac{3}{4}\,,$$

where

$$0 < \lambda/\left((1 + C_0 - \epsilon)(1 + \alpha/\epsilon)\right)^2 < d^2V/dt^2 < \lambda \Bigg/ \left(\epsilon\left[1 + \frac{\alpha}{1 + C_0 - \epsilon}\right]\right)^2 .$$

Then, at $t = t_1(\epsilon,\lambda)$,

$$v_t = \frac{dV}{dt} \ge \frac{\lambda t_1}{\left((1 + C_0 - \epsilon)(1 + \alpha/\epsilon)\right)^2} \quad \text{and}$$

$$\frac{\lambda t_1^2}{2\left((1 + C_0 - \epsilon)(1 + \alpha/\epsilon)\right)^2} < v = V < \frac{\lambda t_1^2}{2\left(\epsilon\left[1 + \frac{\alpha}{1 + C_0 - \epsilon}\right]\right)^2}\,,$$

for $1/4 < x < 3/4$.

Additionally, $v \leq V \leq \epsilon$ is guaranteed in $0 \leq t \leq t_1$ by taking $\lambda t_1^2 \leq 2\epsilon^3 \left[1 + \frac{\alpha}{1+C_0-\epsilon}\right]^2$, for example by

$$t_1 = \sqrt{\frac{2}{\lambda}} \left[1 + \frac{\alpha}{1+C_0-\epsilon}\right] \epsilon^{\frac{3}{2}}. \tag{6.12}$$

In $t_1 < t < t_0$, $1/4 \leq x \leq 3/4$, we still have $v(x,t) = V(t)$ with $d^2V/dt^2 > 0$, as long as the solutions u and v exist. It follows that

$$v\left(\frac{1}{2}, t\right) = V(t) > \frac{\lambda t_1^2}{2\left((1+C_0-\epsilon)\left(1+\alpha/\epsilon\right)\right)^2} + \frac{\lambda t_1(t-t_1)}{\left((1+C_0-\epsilon)\left(1+\alpha/\epsilon\right)\right)^2}.$$

Assuming the solution exists up to t_0, *i.e.* $T \geq t_0$,

$$u\left(\frac{1}{2}, t\right) = v\left(\frac{1}{2}, t\right) + w\left(\frac{1}{2}, t\right) > C_0 + \frac{\lambda t_1(t_0 - t_1/2)}{\left((1+C_0-\epsilon)\left(1+\alpha/\epsilon\right)\right)^2} \geq 1$$

if

$$\lambda t_1(t_0 - t_1/2) \geq \left((1+C_0-\epsilon)\left(1+\alpha/\epsilon\right)\right)^2. \tag{6.13}$$

Therefore, taking t_1 to be given by (6.12), quenching before $t = t_0$ will be guaranteed on choosing λ large enough to satisfy

$$\sqrt{2\lambda t_1}\left(t_0 - \left[1 + \frac{\alpha}{1+C_0-\epsilon}\right]\epsilon^{\frac{3}{2}}/\sqrt{2\lambda t_1}\right) \geq \left((1+C_0-\epsilon)\left(1+\alpha/\epsilon\right)\right)^2. \tag{6.14}$$

Note that, for restricted regions of smoothness for the initial data, different intervals of x and smaller values of t_0 could be used.

7. Numerical Solution

We now carry out a brief numerical study of problem (1.1), with a variety of initial conditions. A three-step Crank-Nicolson scheme (Richtmayer's method) is used, indicating unconditional stability.

Taking a partition of $M+1$ points in $[0,1]$, $0 = x_0, x_0 + \delta x = x_1, \cdots, x_M = 1$ and using a time step δt for a partition in the time interval $[0,T]$, with $t_i = i\delta t$, $i = 0, \ldots, N = \frac{T}{\delta t}$, we have for the approximation of the solution u, $u_j^i \simeq u(t_i, x_j)$ the following scheme

$$\begin{aligned} \frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\delta t^2} &= \frac{1}{2}\frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{\delta x^2} + \frac{1}{2}\frac{u_{j+1}^{i-1} - 2u_j^{i-1} + u_{j-1}^{i-1}}{\delta x^2} \\ &+ \lambda \frac{1}{(1-u_j^i)^2\left(1 + \alpha\int_0^1 \frac{1}{1-u^i}dx\right)^2}. \end{aligned} \tag{7.1}$$

The integral $\int_0^1 \frac{1}{1-u^i}dx$, where $u^i = (u_1^i, u_2^i, \cdots, u_M^i)^T$, is evaluated in each time step by Simpson's rule.

Taking into account the Dirichlet boundary conditions, the numerical scheme takes the form

$$\begin{aligned} -\frac{R^2}{2}u_{j-1}^{i+1} + (1+R^2)u_j^{i+1} - \frac{R^2}{2}u_{j+1}^{i+1} &= \frac{R^2}{2}u_{j-1}^{i-1} - (1+R^2)u_j^{i-1} + \frac{R^2}{2}u_{j+1}^{i-1} \\ &+ \delta t^2 F(u_j^i), \end{aligned} \tag{7.2}$$

for $j = 2, \ldots, M-2$,

$$(1+R^2)u_1^{i+1} - \frac{R^2}{2}u_2^{i+1} = -(1+R^2)u_1^{i-1} + \frac{R^2}{2}u_2^{i-1} + \delta t^2 F(u_1^i),$$

for $j = 2$,

$$-\frac{R^2}{2}u_{M-2}^{i+1} + (1+R^2)u_{M-1}^{i+1} = \frac{R^2}{2}u_{M-2}^{i-1} - (1+R^2)u_{M-1}^{i-1} + \delta t^2 F(u_{M-1}^i),$$

for $j = M-1$, where $R = \frac{\delta t}{\delta x}$, $u_0^i = 0$, $u_M^i = 0$ and

$$F(u_j^i) = \lambda \frac{1}{(1-u_j^i)^2\left(1+\alpha\int_0^1 \frac{1}{1-u^i}dx\right)^2}. \tag{7.3}$$

This results in the algebraic system

$$Au^{i+1} = Bu^{i-1} - bu^i$$

or

$$u^{i+1} = A^{-1}Bu^{i-1} - A^{-1}bu^i,$$

where A, B are $(M-2)\times(M-2)$ matrices and b is a $1\times(M-2)$ vector.

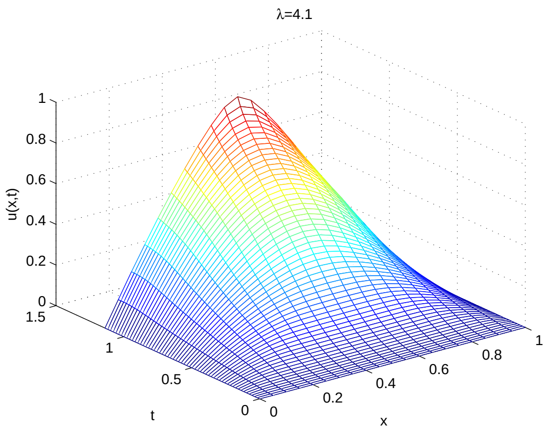

FIGURE 2. *The numerical solution of problem against space and time for $\lambda = 4.1$.*

In the figures of this section, the results of various numerical simulations are presented. The parameters that were used are $\alpha = 1$, $M = 21$, and $R = 0.005$. In the first two figures, $\lambda = 4.1$; this value is chosen to be slightly greater than the value 4 found, Theorem 5.1, to guarantee global existence.

In Figure 2, $u(x,t)$ is plotted against $x \in [0,1]$ and $t \in [0,T]$ for $T = 1.5$. In the next plot, Figure 3, $u(x,t)$ is plotted against x for various times. The uppermost line corresponds to the solution of the problem near quenching. These numerical results indicate that for zero initial data, $\lambda_-^* \le 4.1$ and suggest that Theorem 5.1 is close to best possible.

The remaining three figures show solutions for $\lambda = 1 < 4 \le \lambda_-^*$. In Figure 4, we consider a situation where the solution does not quench but instead oscillates. Additionally, in Figure 5, the solution at the mid point, $x = 1/2$, is plotted against time.

Finally, Figure 6 shows quenching of the solution for large enough initial data. In this example, $u_0 = 0$, $v_0 = 2$, still with $\lambda = 1$.

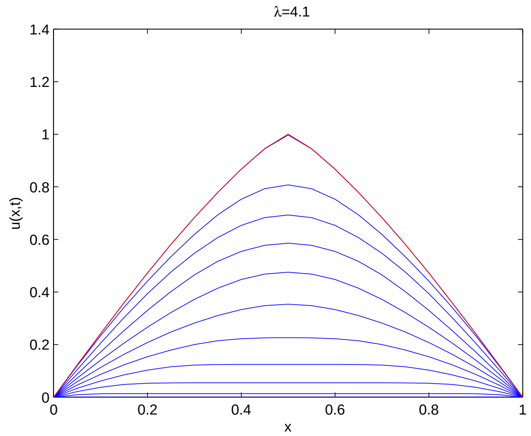

FIGURE 3. *Profile of the numerical solution of problem (6.1), for various times and for* $\lambda = 4.1$.

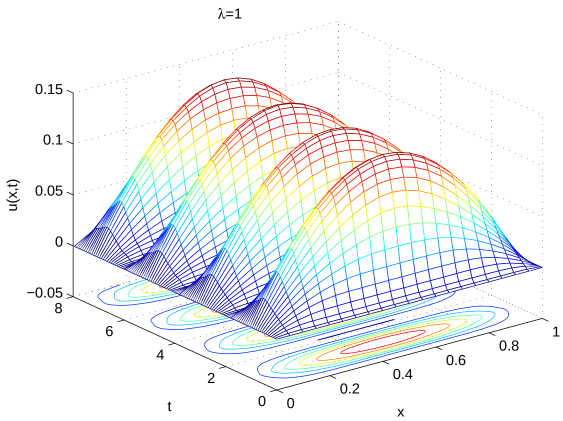

FIGURE 4. *The numerical solution of problem (6.1), for* $\lambda = 1$.

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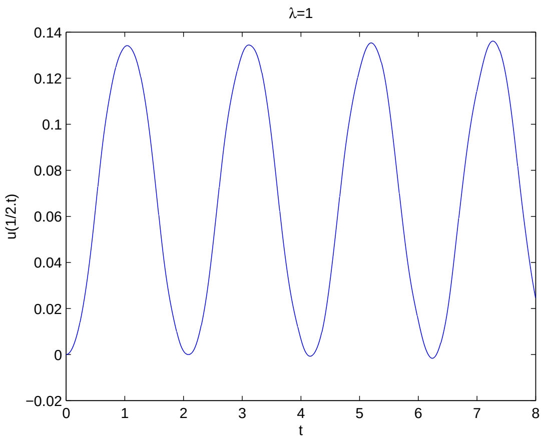

FIGURE 5. *The numerical solution of problem (6.1), $u(1/2, t)$ is plotted for $\lambda = 1$.*

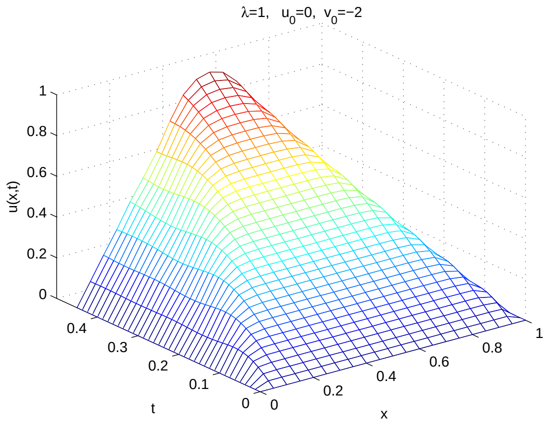

FIGURE 6. *The numerical solution of problem (6.1), $u(x, t)$ is plotted against space for $u_0 = 0$, $v_0 = 2$ and $\lambda = 1$.*

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