An estimate of blow-up time for a non-local problem modelling an Ohmic heating process

N. I. KAVALLARIS, C. V. NIKOLOPOULOS and D. E. TZANETIS

Department of Mathematics, Faculty of Applied Sciences,
National Technical University of Athens,
Zografou Campus, 15780 Athens, Greece

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We consider an initial boundary value problem for the non-local equation, $u_t = u_{xx} + \lambda f(u)/(\int_{-1}^{1} f(u)\,dx)^2$, with Robin boundary conditions. It is known that there exists a critical value of the parameter λ, say λ^*, such that for $\lambda > \lambda^*$ there is no stationary solution and the solution $u(x,t)$ blows up globally in finite time t^*, while for $\lambda < \lambda^*$ there exist stationary solutions. We find, for decreasing f and for $\lambda > \lambda^*$, upper and lower bounds for t^*, by using comparison methods. For the $f(u) = e^{-u}$, we give an asymptotic estimate : $t^* \sim t_u(\lambda - \lambda^*)^{-1/2}$ for $0 < (\lambda - \lambda^*) \ll 1$, where t_u is a constant. A numerical estimate is obtained using a Crank-Nicolson scheme.

Contents

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1 Introduction

We consider the non-local initial boundary value problem:

$$u_t(x,t) = u_{xx}(x,t) + \lambda \frac{f(u(x,t))}{\left(\int_{-1}^{1} f(u(x,t))\,dx\right)^2}, \qquad -1 < x < 1, \quad t > 0, \tag{1.1 a}$$

$$\mathcal{B}_{\pm}(u) := u_x(x,t) \pm a u(x,t) = 0\,, \;\; x = \pm 1\,, \;\; t > 0, \tag{1.1 b}$$

$$u(x,0) = u_0(x) \geqslant 0, \qquad -1 < x < 1, \tag{1.1 c}$$

where $\lambda > 0$, $a > 0$ and $\mathcal{B}_\pm$ are the Robin boundary operators, as defined above. The function f satisfies,

$$f(s) > 0, \quad f'(s) < 0, \quad s \geqslant 0\,, \tag{1.2 a}$$

$$f''(s) > 0 \quad \text{for } s \geqslant 0, \tag{1.2 b}$$

$$f(s) \leqslant \frac{c}{s^2}\,, \ c > 0 \ \text{for} \ s \gg 1, \tag{1.2 c}$$

for instance either $f(s) = e^{-s}$ or $f(s) = (1+s)^{-p}$, $p \geqslant 2$, satisfy (1.2).
For the initial data $u_0(x)$ we require $u_0(x)$, $u_0'(x)$ to be bounded and $u_0(x) \geqslant 0$ in $[-1,1]$ (the last requirement is a consequence of the fact that for any initial data the solution u becomes non-negative sometime [15]).
It is known that the solution $u = u(x,t) = u(x,t;\lambda)$ of (1.1), which represents temperature, blows up in finite time $t^* > 0$ under certain conditions (large enough values of λ or of initial data) [15, 16].

The key to the study of the behaviour of u is the knowledge of the corresponding steady problem to (1.1),

$$w'' + \mu f(w) = 0, \quad -1 < x < 1, \tag{1.3 a}$$

$$\mathcal{B}_\pm(w) = w'(x) \pm a w(x) = 0, \quad x = \pm 1, \tag{1.3 b}$$

where $w = w(x) = w(x;\lambda)$ (see [4, 11, 15, 16]). The parameter μ is referred to as a local parameter while the parameter λ as non-local, and the relation between them is

$$\mu = \frac{\lambda}{\left(\int_{-1}^{1} f(w)dx\right)^2}. \tag{1.4}$$

It is known that if

$$\int_0^\infty f(s)ds < \infty, \tag{1.5}$$

then there exists a critical value of the parameter λ, say $\lambda^* < \infty$, such that for $\lambda > \lambda^*$, $u(x,t;\lambda)$ blows up globally ($u \to \infty$ for all $x \in [-1,1]$ as $t \to t^*-$, actually the blow-up is uniform in x) in finite time t^* and the problem (1.3), (1.4), has no solutions (of any kind). For $0 < \lambda < \lambda^*$ there exist solutions $w(x;\lambda)$, and $u(x,t;\lambda)$ may either exist for all times or blow up globally depending upon the initial data (if u_0 is greater than the greatest steady solution $w(x;\lambda)$ and (1.5) holds) [15, 16]. We may take $\int_0^\infty f(s)ds = 1$, and in this case $\lambda^* < 8$, while for the Dirichlet problem $\lambda^* = 8$. The response (bifurcation) diagrams for problem (1.3), (1.4) are as in Figure 1.

Our purpose, in this work, is to find some estimates of the blow-up time t^* with respect to the parameter λ (more precisely, with respect to the difference $(\lambda - \lambda^*)$), when $\lambda > \lambda^*$.

In the physical problem modelled by (1.1), λ is equal to a constant times the square of the electrical potential difference driving an electric current through a conductor, see [15]. Works related to this model can be found in [2, 6, 5, 7, 8, 10]. Estimates of this type are very important since they answer to the question "when" the blow up takes place [3, 12].

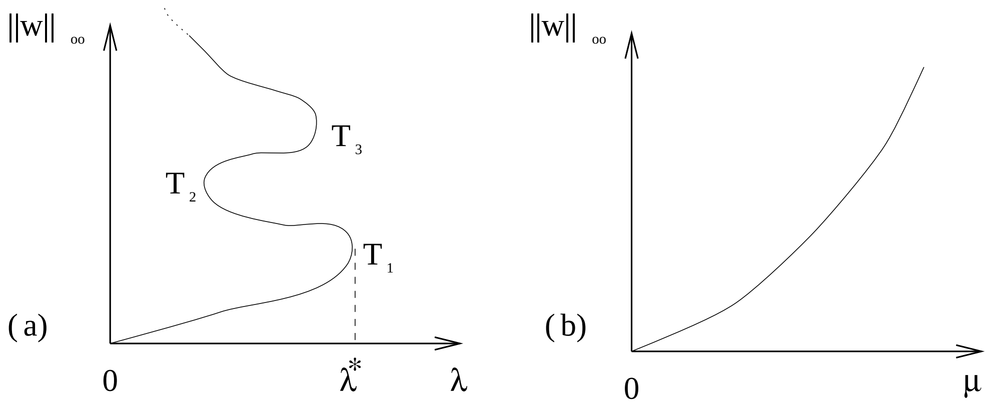

FIGURE 1. *The response diagram for problem (*1.3), (1.4*),*
(a) the non-local diagram if f satisfies (1.5), (b) the local diagram.

In Figure 1(a), there may be only one or more than one turning points $T_i, i = 1, 2, 3, ...$, depending upon f. One can find other forms of non-local diagrams in [15, 16]. Their shapes depend upon the boundary conditions and the function f (see (1.2 a), (1.5) or (1.2 a) together with $\int_0^\infty f(s)ds = \infty$).

Under the assumptions (1.2 a), (1.5), problem (1.3) has at least one classical (regular) steady solution $w^* = w(x;\lambda^*)$. (We may have more than one w^* when some of $T_1,\ T_2,\ T_3$ etc. have the same abscissa λ^*). In the following, we assume that w^* is unique, since in our proofs we require only the existence of at least one w^* and that the pair $(\underline{w}, \overline{w})$ at $\lambda < \lambda^*$ with $\underline{w} < \overline{w}$ for x in $(-1, 1)$, where $\overline{w}$ is the second smallest steady solution, has the property: $\underline{w}$ is stable while $\overline{w}$ is unstable, for $\lambda < \lambda^*$ and λ close to λ^*.

Also we emphasize that for $\lambda > \lambda^*$, $u(x, t; \lambda)$ blows up globally as $t \to t^*-$ which means:

$$F(u) = \frac{f(u)}{(\int_{-1}^1 f(u)dx)^2} \to \infty, \text{ as } t \to t^* - < \infty, \tag{1.6 a}$$

$$u(x, t; \lambda) \to \infty, \text{ for all } x \in [-1, 1] \text{ and } \lambda > \lambda^*, \text{ as } t \to t^* - < \infty, \tag{1.6 b}$$

we will see in Lemma 2.1 that this blow-up is actually uniform in x, see also [15, 16].

One can find similar situations, concerning the blow-up, in the study of the (local) reaction diffusion problem:

$$u_t = \triangle u + \lambda f(u), \quad x \in \Omega, \quad t > 0, \tag{1.7 a}$$

$$\mathcal{B}(u) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.7 b}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.7 c}$$

where $\mathcal{B}$ represents the boundary conditions (Dirichlet or Robin type), Ω is a bounded domain of $\mathbb{R}^n$, λ is a positive parameter and $f(u)$ behaves like e^u, i.e.

$$f(s) > 0, \;\; f'(s) > 0, \;\; f''(s) \geqslant 0, \;\; \text{for } s \geqslant 0, \;\; \text{and} \int_0^\infty ds/f(s) < \infty, \tag{1.8}$$

see [1, 13, 14]. Again, under certain conditions, the solution u of (1.7) blows up ($\limsup_{t\to t^*-} \| u(\cdot,t;\lambda) \|_\infty = \infty$ for $\lambda > \lambda^*$, $t^* < \infty$). It should be emphasized that the blow-up at (1.7) differs from that of the non-local problem in that, (1.7) does not normally blow-up globally. Moreover there exists a turning point $T^* = (\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty < \infty$ of the response diagram of the steady problem corresponding to (1.7). Then the following upper and lower bounds for t^* have been found:
$t_1 \leqslant t^* \leqslant t_2$ where $t_i = c_i(\lambda - \lambda^*)^{-1/2}$ and c_i some constants ($c_1 < c_2$), for f which satisfies (1.8).
Also, asymptotically $t^* \sim K(\lambda - \lambda^*)^{-1/2}$ as $\lambda \to \lambda^*+$, for $f(s) = e^s$; see [13].

In the present work, we find similar estimates for the non-local problem (1.1), for $f(s) = e^{-s}$, and for general $f(s)$ which satisfies (1.2).

In both problems the estimates of t^* can be found only if the spectrum of the steady problem is an interval closed on the right i.e. $(0, \lambda^*]$. It is still an open question, even for problem (1.7), to estimate t^* when the spectrum is an open interval, $(0, \lambda^*)$; see [13, 14].

We organize this work as follows: In Section 2 we use comparison techniques and find upper and lower bounds for t^*, when f satisfies (1.2). In the third section we use an asymptotic expansion and again obtain an estimate of t^* but for $f(s) = e^{-s}$. Also we compute numerically the blow-up time t^* and verify the previous estimate.

2 Comparison methods: upper and lower bounds for t^*, $\lambda > \lambda^*$

If the function f satisfies (1.2 a), one can prove that a maximum principle holds for (1.1) (here is where we need f to be decreasing). Then we may, in the usual way, define upper and lower solutions of (1.1): an upper (lower) solution $\bar{u}$ ($\underline{u}$) is defined as a function which satisfies (1.1) if we substitute $\geqslant$ ($\leqslant$) for $=$, see [15, 16, 18, 19].

An upper bound for t^*:
We wish now to find an upper bound for the blow-up time t^*. We assume for simplicity $0 \leqslant u_0 < w^*$. Firstly we write (1.3 a) in a different way, by using (1.4)

$$w'' + \frac{\lambda f(w)}{\left(\int_{-1}^1 f(w)\, dx\right)^2} = w'' + \lambda F(w) = 0, \quad -1 < x < 1, \tag{2.1}$$

where $F(\cdot) = f(\cdot)/\left(\int_{-1}^1 f(\cdot) dx\right)^2$ and λ is a positive parameter (eigenvalue). Then the linearized problem of (2.1) with boundary condition (1.3 b) is :

$$\phi'' + \lambda\, \delta F(w;\phi) = \rho\phi, \quad -1 < x < 1, \tag{2.2 a}$$

$$\mathcal{B}_{\pm}(\phi) = \phi'(x) \pm a\phi(x) = 0\,, \quad x = \pm 1\,, \tag{2.2 b}$$

where $\phi = \phi(x;\lambda)$, and $\delta F(w;\phi)$ is the first variation (or Gâteaux derivative) of F at w in the direction of ϕ, $(F(w;\phi) := F(w+\epsilon\phi) = J(\epsilon)$, and $\delta F(w;\phi) = J'(0) = \lim_{\epsilon\to 0}\frac{F(w+\epsilon\phi)-F(w)}{\epsilon})$.

As regards the first variation $\delta F(w;\phi)$ we have,

$$\delta F(w;\phi) = \frac{f'(w)\phi}{(\int_{-1}^{1} f(w)\,dx)^2} - \frac{2f(w)\int_{-1}^{1} f'(w)\phi\,dx}{(\int_{-1}^{1} f(w)\,dx)^3}.$$

In the following, in order to simplify the expressions, we use the notation:

$$I_{\nu\,k}(w,\phi) := \int_{-1}^{1} f^{(\nu)}(w)\,\phi^k\,dx,$$

and $I_\nu(w) := I_{\nu\,0}(w,\phi)$, $\nu,\, k = 0,1,2,3,\dots$, $f^{(\nu)}(w) = \frac{d^\nu}{dw^\nu}f(w)$, thus

$$\delta F(w;\phi) = \frac{f'(w)\phi}{I_0^2(w)} - \frac{2f(w)I_{11}(w,\phi)}{I_0^3(w)}. \tag{2.3}$$

Moreover, we know that the spectrum of problem (1.3), (1.4) is a closed interval from the right, and assume that there exists a unique turning point T^*, $((0,\lambda^*]$ and $T^* = (\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty < \infty$, i.e. $T_1 \equiv T^*$, see Figure (1a)). The lower branch of the response diagram is asymptotically stable with $\rho_1 = \rho_1(\lambda) > 0$ (ρ_1 is the first eigenvalue of (2.2) for $\lambda < \lambda^*$), while the upper branch is unstable with $\rho_1 = \rho_1(\lambda) < 0$ ([9, 15, 16, 17]). This continues to hold (with a suitable understanding of the "upper branch") even if there are more turning points T_i.

It is known [17], (see also [1, 9]), that $\rho_1(\lambda^*) = \rho_1^* = 0$. Hence problem (2.2) at $\lambda = \lambda^*$ gives

$$\phi^{*\prime\prime} + \lambda^*\,\delta F(w^*;\phi^*) = 0, \quad -1 < x < 1\,, \tag{2.4 a}$$

$$\mathcal{B}_{\pm}(\phi^*) = 0, \quad x = \pm 1\,, \tag{2.4 b}$$

where by ϕ^* we denote the first eigenfunction corresponding to ρ_1^* with $\phi^* > 0$, [17].

Now, in order to find an upper bound for t^*, we take the difference,

$$v = v(x,t) = v(x,t;\lambda) = u(x,t;\lambda) - w^*(x) = u - w^*. \tag{2.5}$$

Since w^* is bounded, v blows up at the same time as u does and in the same way. Hence $t^* = t^*(u) = t^*(v)$ and $v(x,t) \to \infty$ as $t \to t^*-$ for all $x \in [-1,1]$. In the following, we find an A-problem (see (2.19) below), where $A = A(t)$ is such that:

$$A(t) \leqslant const. \times \|v(\cdot,t)\|_\infty. \tag{2.6}$$

Now (2.5), (2.6) imply $t^*(u) = t^*(v) \leqslant t^*(A)$, thus we find an upper bound $t^*(A)$ for $t^*(u)$.

Therefore we obtain

$$\begin{aligned} v_t = u_t &= u_{xx} + \lambda F(u) = u_{xx} + \lambda F(u) - \lambda^* F(w^*) - w^{*\prime\prime} \\ &= v_{xx} + (\lambda - \lambda^*)F(u) + \lambda^*\left(F(u) - F(w^*)\right). \end{aligned} \tag{2.7}$$

By writing $J(\epsilon) = F(w^* + \epsilon v)$, whence $J(0) = F(w^*)$ and $J(1) = F(u)$, Taylor's formula gives,

$$F(u) - F(w^*) = J(1) - J(0) = J'(0) + \frac{J''(\xi)}{2!},$$

for some $\xi \in (0,1)$, where

$$J'(0) = \delta F(w^*; v) = \left[\frac{d}{d\epsilon} J(\epsilon)\right]_{\epsilon=0}.$$

Also

$$J''(\xi) = 2\delta^2 F(z;v) = \frac{f''(z)v^2}{I_0^2(z)} - \frac{4vf'(z)I_{1\,1}(z,v)}{I_0^3(z)} - \frac{2f(z)I_{2\,2}(z,v)}{I_0^3(z)} + \frac{6f(z)I_{1\,1}^2(z,v)}{I_0^4(z)},$$

where $z = w^* + \xi v$ and $\delta^2 F(z;v)$ is the second Gâteaux derivative). Thus from equation (2.7) we get the problem:

$$v_t = v_{xx} + (\lambda - \lambda^*)F(u) + \lambda^* \, \delta F(w^*; v) + \frac{\lambda^*}{2} J''(\xi), \quad -1 < x < 1, \quad t > t_1, \qquad (2.8\,a)$$

$$\mathcal{B}_{\pm}(v) = 0, \quad x = \pm 1, \quad t > t_1, \qquad (2.8\,b)$$

$$v(x, t_1) = u(x, t_1) - w^* \geqslant 0 \quad -1 < x < 1, \qquad (2.8\,c)$$

(it is obvious that there exists a $t_1 > 0$ so that $v(x,t) = u(x,t) - w^*(x) > 0$ for every $t > t_1$). Now we have the lemma:

Lemma 2.1 *The following limit holds:*

$$\lim_{t \to t^*-} |u(x_1, t) - u(x_2, t)| = 0, \quad -1 < x_1 < x_2 < 1,$$

i.e. the blow-up is uniform on compact subsets of (-1,1).

Proof: Following similar steps to those in [16], we have that the solution of ϑ-problem:

$$\vartheta_t = \vartheta_{xx} + g(t) f(M), \quad -1 < x < 1, \quad t > 0,$$

$$\mathcal{B}_{\pm}(\vartheta) = 0, \quad x = \pm 1, \quad t > 0,$$

$$\vartheta(x, 0) = 0, \quad -1 < x < 1,$$

where $g(t) = \frac{\lambda}{(\int_{-1}^{1} f(u)\,dx)^2}$. Then we have the integral representation:

$$\vartheta(x,t) = V(t) + \int_0^t \left[(\vartheta(y,\tau) - V(\tau)) G_y(x,y,t-\tau)\right]_{y=-1}^{y=1} d\tau, \qquad (2.9)$$

where $G(x,y,t)$ is the Green's function for the heat equation with Robin boundary conditions of the form $(1.1\,b)$ and V satisfies,

$$\frac{dV}{dt} = g(t) f(M(t)), \quad t > 0, \quad V(0) = 0, \quad M(t) = \max_x u(x,t).$$

For any given fixed x in $(-1,1)$ the second term on the right hand side of (2.9) is much smaller (on using maximum principle) than the first term, as $t \to t^*-$, so $\vartheta(x,t) \sim V(t)$ as $t \to t^*-$, for $-1 < x < 1$. The function ϑ is a lower solution to u-problem, hence

$u(x,t) \geqslant \vartheta(x,t) \sim V(t)$, as $t \to t^* -$. Moreover since

$$\frac{dM}{dt} \leqslant g(t) f(M(t)) = \frac{dV}{dt},$$

and $V(t) \lesssim u(x,t) \leqslant M(t)$, we get $M(t) \lesssim V(t) \lesssim M(t)$ as $t \to t^* -$. Hence $V(t) \sim M(t)$ and $u(x,t) \sim M(t)$ as $t \to t^* -$, so $|u(x_1,t) - u(x_2,t)| \leqslant (M(t) - u(x,t)) \to 0$ as $t \to t^* -$. □

As regards the last term of(2.8 a) we have:

$$J''(\xi) = v^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z) I_{1\,1}(z,v)}{v I_0^3(z)} - \frac{2f(z) I_{2\,2}(z,v)}{v^2 I_0^3(z)} + \frac{6f(z) I_{1\,1}^2(z,v)}{v^2 I_0^4} \right] =$$

$$= v^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z) I_1(z) v(\zeta_1,t)}{v I_0^3(z)} - \frac{2f(z) I_2(z) v^2(\zeta_2,t)}{v^2 I_0^3(z)} + \frac{6f(z) I_1^2(z) v^2(\zeta_3,t)}{v^2 I_0^4(z)} \right].$$

Now by lemma (2.1) we have $v(\zeta_i,t)$, $v(x,t) \sim M(t)$, as $t \to t^*-$, $i = 1,2,3$. since $v = u - w^*$ ($\zeta_i = \zeta_i(t), i = 1,2,3.$, are these values which come from applying the mean value theorem to the integrals $I_0(z)$, $I_{1\,1}(z)$ and $I_{2\,2}(z)$). Therefore

$$J''(\xi) = v^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z) I_1(z)}{I_0^3(z)} - \frac{2f(z) I_2(z)}{I_0^3(z)} + \frac{6f(z) I_1^2(z)}{I_0^4(z)} \right]$$
$$= v^2 \Gamma(x,t). \tag{2.10}$$

Since u blows up globally, see (1.6), we have that $F(u) - F(w^*) \to \infty$ as $t \to t^*-$. Also by lemma 2.1 and relation (1.2 c), $F(u) - F(w^*) \sim \frac{1}{4f(M)} > \frac{M^2}{4c}$ for $M \gg 1$(as $t \to t^*-$). Furthermore

$$F(u) - F(w^*) = \delta F(w^*; v) + \delta^2 F(z; v) = K_1(w^*) v + \Gamma(x,t) v^2 \tag{2.11}$$

By the fact that $v \sim M(t)$ as $t \to t^*-$ the second term of the right hand side of (2.11), $\Gamma(x,t)v^2$, dominates the first one ($|\delta^2 F(z;v)| \gg |\delta F(w^*;v)|$ as $v \to \infty$)., provided that $\Gamma(x,t) \nrightarrow 0$ for $t \to t^*-$. In this case

$$F(u) - F(w^*) \sim \frac{1}{4f(M)} \tag{2.12}$$

and

$$F(u) - F(w^*) \sim \Gamma(x,t) M^2 \tag{2.13}$$

By equations (2.12) and (2.13) we have

$$\Gamma(x,t) M^2 \sim \frac{1}{4f(M)} \geqslant \frac{M^2}{4c} \quad \text{which implies} \quad \Gamma(x,t) \geqslant \frac{1}{4c} > K > 0,$$

for some positive constant K. Allowing $\Gamma(x,t) \to 0$ as $t \to t^*-$, we would have that $\exists\, t_n \in [0,t^*]$ with $\Gamma(x,t_n) \to 0$. Then

$$F(u) - F(w^*) \sim \frac{1}{4f(M)} > \frac{M^2}{4c},\ M \gg 1 \tag{2.14}$$

and on the other hand

$$F(u) - F(w^*) \sim \delta F(w^*; v) \sim K_1 \sim M \quad \text{as} \quad t \to t^* -. \tag{2.15}$$

By (2.13) and (2.14) we would have $\frac{M^2}{4c} < K_1 M$ or $M < 4cK_1$ which implies a contradiction. Hence $\Gamma(x,t) \nrightarrow 0$ and finally $G(x,t) \geqslant K > 0$ as $t \to t^* -$. Thus relation (2.10) becomes

$$J''(\xi) \sim \Gamma(x,t) v^2 \geqslant K v^2 \quad \text{as} \quad t \to t^* - . \tag{2.16}$$

Also $F(u) \to \infty$ as $t \to t^*-$, (otherwise u, the solution of problem (1.1), does not blow up), then there exists a $t_2 \geqslant 0$ so that

$$F(u) \geqslant \beta\, \phi^*(x) > 0, \quad \phi^*(x) > 0, \quad t \in [t_2, t^*), \;\; t_2 > t_1, \tag{2.17}$$

where $\beta > 0$ is a constant.

Now we introduce the Ψ-problem:

$$\Psi_t \leqslant (\lambda - \lambda^*)\beta\phi^* + \Psi_{xx} + \lambda^*\, \delta F(w^*; \Psi) + \frac{\lambda^*}{2} K \Psi^2, \;\; -1 < x < 1, \;\; t > t_0 \geqslant t_2, \tag{2.18 a}$$

$$\mathcal{B}_{\pm}(\Psi) = 0\,, \;\; x = \pm 1\,, \;\; t > t_0, \tag{2.18 b}$$

$$\Psi(x, t_0) \leqslant v(x, t_0) = u(x, t_0) - w^*(x), \qquad -1 < x < 1. \tag{2.18 c}$$

Then $\Psi(x,t) = A(t)\phi^*(x)$ satisfies (2.18) provided that $A(t)$ is the solution of the equation:

$$\dot{A}(t) = (\lambda - \lambda^*)\beta + K^* A^2(t), \;\; t > t_0, \;\; A(t_0) = A_0, \tag{2.19}$$

where $K^* = \frac{\lambda^*}{2} K \inf_x \phi^*(x)$ and $A(t_0) = \inf_x \frac{|u_0(x) - w^*(x)|}{\phi^*(x)}$, $t_0 \geqslant t_2$. Moreover Ψ is a lower solution of problem (2.8).

Now the initial value problem (2.19) gives

$$A(t) \geqslant \left[(\lambda - \lambda^*) \frac{\beta}{K^*} \right]^{1/2} \tan\left\{ t\, [(\lambda - \lambda^*)\beta K^*]^{1/2} - \frac{\pi}{2} \right\}, \tag{2.20}$$

provided that $(\lambda - \lambda^*) < \pi/2t_0\, K^*$.

This relation implies that u ceases to exist at finite time t^* with

$$t^*\, [(\lambda - \lambda^*)\beta K^*]^{1/2} < \pi,$$

or

$$t^* < \frac{\pi}{(\beta K^*)^{1/2}} (\lambda - \lambda^*)^{-1/2} = t_u (\lambda - \lambda^*)^{-1/2},$$

where $t_u(\lambda - \lambda^*)^{-1/2}$ is an upper bound of t^*, with $t_u = \frac{\pi}{(\beta K^*)^{1/2}}$.

A lower bound for t^* :

We assume that $u_0(x) < w^*(x)$ for $-1 \leqslant x \leqslant 1$, with $\mathcal{B}_{\pm}(u_0) \leqslant \mathcal{B}_{\pm}(w^*)$ at $x = \pm 1$. Let $u^* = u^*(x,t) = u(x,t;\lambda^*)$ be the solution of (1.1).

In the following we use similar ideas to those in [13]. Therefore we write $u = u^* + u_1 \leqslant u^* + \psi_1 = w^* - \hat{u} + \psi_1 \leqslant w^* - \psi + \psi_1$, where $\hat{u}$ is given by $\hat{u} = w^* - u^* > 0$ and satisfies (2.21), u_1 solves (2.27), ψ_1 is an upper solution of the u_1-problem and ψ is a lower solution of the $\hat{u}$-problem. The $\hat{u}$-problem is defined by

$$\begin{aligned} \hat{u}_t &= -u^*_{xx} - \lambda^* F(u^*) + w^{*\prime\prime} + \lambda^* F(w^*) \\ &= \hat{u}_{xx} - \lambda^* \left(F(u^*) - F(w^*) \right), \;\; -1 < x < 1, \;\; 0 < t < T, \end{aligned} \tag{2.21 a}$$

$$\mathcal{B}_{\pm}(\hat{u}) = \mathcal{B}_{\pm}(w^*) - \mathcal{B}_{\pm}(u^*) = 0, \;\; x = \pm 1, \;\; 0 < t < T, \tag{2.21 b}$$

$$\hat{u}(x,0) = \hat{u}_0(x) = w^*(x) - u_0^*(x), \;\; -1 < x < 1, \tag{2.21 c}$$

with $\hat{u}_0 > 0$, hence $\hat{u} > 0, \;\; 0 < t < T < t^*$, for some $T > 0$.

We write $J(\epsilon) = F(w^* - \epsilon \hat{u})$ and examine the difference,

$$F(u^*) - F(w^*) = J(1) - J(0) = J'(0) + \frac{J''(\xi)}{2}, \quad 0 < \xi < 1,$$

where $J'(0) = \delta F(w^*; \hat{u})$ and

$$J''(\xi) = R(z, \hat{u}) = \frac{f''(z)\hat{u}^2}{I_0^2(z)} - \frac{4\hat{u} f'(z) I_{1\,1}(z,\hat{u})}{I_0^3(z)} - \frac{2 f(z) I_{2\,2}(z,\hat{u})}{I_0^3(z)} + \frac{6 f(z) I_{1\,1}^2(z,\hat{u})}{I_0^4(z)}, \tag{2.22}$$

with $0 < z = w^* - \xi \hat{u} < w^*$.

Equation (2.21 a) now becomes:

$$\hat{u}_t = \hat{u}_{xx} - \lambda^* \frac{f'(w^*)\hat{u}}{I_0^2(w^*)} + \frac{2\lambda^* f(w^*)\, I_{11}(w^*, \hat{u})}{I_0^3(w^*)} - \frac{\lambda^*}{2} J''(\xi). \tag{2.23}$$

Since $f'(z) I_{1\,1}(z,\hat{u})\,, \;\; f(z) I_{2\,2}(z,\hat{u}) > 0, \;\; \hat{u} > 0$ for $0 < t < T < t^*$, and (1.2 a) and (1.2 b) holds, equation (2.22) gives

$$J''(\xi) = R(z,\hat{u}) \leqslant \frac{1}{I_0^2(w^*)} \left[\hat{u}^2 f''(z) + \frac{6\, f(z) I_{11}^2(z,\hat{u})}{I_0^2(w^*)} \right] < K_0 \hat{u}^2 + K_1 \left(\int_{-1}^{1} \hat{u}\, dx \right)^2 := \Phi(\hat{u}),$$

where $K_0 = K_0(T) = \frac{\sup_{(x,t)} f''(z(x,t))}{I_0^2(w^*)}\,, \;\; K_1 = \frac{6 f(0)(f'(0))^2}{I_0^4(w^*)}, \;\; 0 < t < T < t^*$.

Then from (2.23) and since

$$\delta F(w^*; \hat{u}) = -\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} + \frac{2 f(w^*)\, I_{11}(w^*, \hat{u})}{I_0^3(w^*)},$$

we get

$$\hat{u}_t \geqslant \hat{u}_{xx} + \lambda^* \;\delta F(w^*; \hat{u}) - \frac{\lambda^*}{2} \Phi(\hat{u}).$$

Now we introduce the function $\psi = \frac{K_2 \phi^*}{t + t_0}$, where $K_2, \;\; t_0$ are to be determined. The function ψ satisfies

$$\psi_t \leqslant \psi_{xx} + \lambda^* \;\delta F(w^*; \psi) - \frac{\lambda^*}{2} \Phi(\psi),$$

or

$$\psi_t = -\frac{K_2 \phi^*}{(t+t_0)^2} \leqslant \frac{K_2}{(t+t_0)} \left[\phi^{*\prime\prime} + \lambda^* \,\delta\, F(w^*; \phi^*) \right] - \frac{\lambda^*}{2} \frac{K_2^2}{(t+t_0)^2} \left[K_0\, (\phi^*(x))^2 + K_1 \left(\int_{-1}^{1} \phi^*(x)\, dx \right)^2 \right],$$

provided that we choose $K_2 = 1/\frac{\lambda^*}{2} \sup_x [K_0\, \phi^*(x) + \frac{K_1}{\phi^*(x)}]$. Also we take $t_0 = K_2 \sup_x \frac{\phi^*(x)}{w^*(x) - u_0^*(x)}$, so we have $\psi(x,0) = \frac{K_2 \phi^*}{t_0} < \hat{u}(x,0) = \hat{u}_0(x) = w^*(x) - u_0^*(x)$.

Therefore ψ is a lower solution of $\hat{u}$-problem.

We now write $u = u^* + u_1 \leqslant w^*$ and find an upper solution of u_1-problem. The equation for u_1 is

$$\begin{aligned} u_{1t} =& u_{1xx} + (\lambda - \lambda^*)F(w^*) + \lambda\,(F(u) - F(w^*)) \\ &- \ \lambda^*\,(F(u^*) - F(w^*))\,, \quad -1 < x < 1, \quad t > 0. \end{aligned} \tag{2.24}$$

We again examine the difference $\lambda\,(F(u) - F(w^*))$ and, writing $v = u - w^*$, $(-w^* < v < 0)$, $J_1(\epsilon) = F(w^* + \epsilon v)$, $0 \leqslant \epsilon \leqslant 1$, we have:

$$\begin{aligned} \lambda\,(F(u) - F(w^*)) &= \lambda(J_1(1) - J_1(0)) = \lambda\left(\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*;v)}{I_0^3(w^*)}\right) + \frac{\lambda}{2}J_1''(\xi_1) \\ &= \lambda^*\delta\ F(w^*;v) + (\lambda - \lambda^*)\delta\ F(w^*;v) + \frac{\lambda}{2}R_1(z,v) = \\ &= \lambda^*\delta\ F(w^*;v) + Q(w^*,z,v), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} J_1''(\xi_1) = R_1(z,v) = \frac{1}{I_0^4(z)}\big[&I_0^2(z)\ v^2\ f''(z) - 4v\ f'(z)I_{11}(z,v)\ I_0(z) - 2f(z)I_0(z)\ I_{22}(z,v) \\ &+ 6f(z)\,I_0(z)\ I_{11}^2(z,v)\big]\,, \quad z = w^* + \xi_1 v. \end{aligned}$$

Also by writing $u^* = w^* - \hat{u}$ and $J_2(\epsilon) = F(w^* - \epsilon\hat{u}) = F(\zeta)$, the quantity $\lambda^*\,(F(u^*) - F(w^*))$ is written:

$$\begin{aligned} -\lambda^*\,(F(u^*) - F(w^*)) &= -\lambda^*(J_2(1) - J_2(0)) = \lambda^*\left(\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,\hat{u})}{I_0^3(w^*)}\right) \\ &- \frac{\lambda^*}{2}J_2''(\xi_2) = \lambda^*\ \delta\ F(w^*;\hat{u}) - \frac{\lambda^*}{2}R_2(\zeta,\hat{u}), \end{aligned} \tag{2.26}$$

with

$$\begin{aligned} J_2''(\xi_2) = R_2(\zeta,\hat{u}) = \frac{1}{I_0^4(\zeta)}\big[&I_0^2(\zeta)\ \hat{u}^2\ f''(\zeta) - 4\hat{u}\ f'(\zeta)I_{11}(\zeta,\hat{u})\ I_0(\zeta) - 2f(\zeta)I_0(\zeta)\ I_{22}(\zeta,\hat{u}) \\ &+ 6f(\zeta)\,I_0(\zeta)\ I_{11}^2(\zeta,\hat{u})\big]\,. \end{aligned}$$

Thus the u_1-problem now becomes

$$\begin{aligned} u_{1t} =& u_{1xx} + (\lambda - \lambda^*)F(w^*) + \lambda^*\delta\ F(w^*;v) + Q(w^*,z,v) \\ &- \ \lambda^*\delta\ F(w^*;\hat{u}) - \frac{\lambda^*}{2}R_2(\zeta,\hat{u}), \quad -1 < x < 1, \quad t > 0, \end{aligned} \tag{2.27 a}$$

$$\mathcal{B}_\pm(u_1) = 0, \quad x = \pm 1\,, \ \ t > 0, \tag{2.27 b}$$

$$u_1(x,0) = u_0(x) - u^*(x), \quad -1 < x < 1, \tag{2.27 c}$$

where $0 < z,\ \zeta < w^*$, $0 < \hat{u} < w^*$, $u < u_1 < w^*$ as far as $u < w^*$, so that $Q(w^*,z,v)$, $J_2''(\xi_2)$ are bounded from above and below.
Hence we can always find B_1, B_2 such that

$$Q(w^*,z,v) < B_1, \qquad -\frac{\lambda^*}{2}J_2''(\xi_2) < B_2.$$

From (2.24) - (2.27) we obtain :

$$u_{1t} \leqslant u_{1xx} + (\lambda-\lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,v)}{I_0^3(w^*)}\right]$$

$$-\lambda^* \left[\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,\hat{u})}{I_0^3(w^*)}\right] + B_1 + B_2.$$

Due to the fact that $u_1 = u - u^* = u - w^* + w^* - u^* = v + \hat{u}$, the previous relation becomes:

$$u_{1t} \leqslant u_{1xx} + (\lambda-\lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)u_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,u_1)}{I_0^3(w^*)}\right] + B_1 + B_2.$$

Then we introduce $\psi_1 = [(\lambda-\lambda^*)\Lambda t]\phi^*$. By substituting ψ_1 for u_1 in the right hand side of the above relation, we get

$$\psi_{1xx} + (\lambda-\lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)\psi_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,\psi_1)}{I_0^3(w^*)}\right] + B_1 + B_2 \leqslant (\lambda-\lambda^*)\Lambda\phi^* = \frac{\partial\psi_1}{\partial t},$$

or

$$F(w^*) + \Gamma \leqslant \Lambda\phi^*,$$

where $\Gamma = (B_1+B_2)/(\lambda-\lambda^*)$; thus it is sufficient to choose $\Lambda = \sup_x \frac{F(w^*)+\Gamma}{\phi^*}$.

If $\psi - \psi_1 \geqslant 0$ then we have:

$$u \leqslant w^* - \psi + \psi_1 = w^* - \frac{K_2\phi^*}{t+t_0} + (\lambda-\lambda^*)\Lambda\, t\, \phi^*.$$

The right-hand side of the above relation is no greater than w^*, as long as

$$\frac{K_2\phi^*}{t+t_0} \geqslant (\lambda-\lambda^*)\,\Lambda\,\phi^*\, t, \quad x \in [-1,1],$$

so that

$$K_2 \geqslant (\lambda-\lambda^*)(t+t_0)\Lambda t.$$

Hence we require

$$(\lambda-\lambda^*)\,\Lambda\, t^2 + \Lambda\, t_0(\lambda-\lambda^*)\, t - K_2 \leqslant 0,$$

which for λ sufficiently close to λ^* $(\lambda > \lambda^*)$ gives

$$t \leqslant t_l(\lambda-\lambda^*)^{-1/2},$$

with $t_l = \frac{1}{2}(\frac{K_2}{\Lambda})^{1/2}$. Hence, as long as $u = u(x,t) < w^*$ at $t = t_l(\lambda-\lambda^*)^{-1/2}$, we deduce that $t^* > t_l(\lambda-\lambda^*)^{-1/2}$ and $t_l(\lambda-\lambda^*)^{-1/2}$ is a lower bound for t^*.

3 Asymptotic estimate of t^* for small $(\lambda - \lambda^*)$

We now examine the special case $f(s) = e^{-s}$. Motivated by Section 2 we wish to find an estimate for the blow-up time t^* of problem (1.1) as an asymptotic series in $0 < \eta = (\lambda-\lambda^*)^{1/2} \ll 1$. We again assume that $u_0(x) < w^*(x)$ for $-1 < x < 1$, with $\mathcal{B}_\pm(u_0) \leqslant \mathcal{B}_\pm(w^*)$ at $x = \pm 1$.

Following similar ideas to [13] we consider three intervals of time, say I, II, and III. In I and III t varies by $O(1)$ and we expand $u \sim u^* + \eta^2 v_2 + ...$ as $\eta \to 0$. More precisely, in I $u^* < w^*$ since $u_0 = u_0^* < w^*$ in $[-1,1]$, while in III $u^* > w^*$. Moreover in I

$$u^* \sim w^* - \frac{\phi^*}{\lambda^* I_2 t} \text{ as } t \to \infty,$$

with $I_2 = \int_{-1}^{1} R_2(z,\phi^*)dx$, $z = w^* + \frac{1}{\sigma}\phi^*$ with $\sigma > t$ and R_2 is the second order residual of the Taylor expansion (see previous sections or below). This can be obtained by assuming that $u^* \sim w^* + \frac{K}{t}\phi^*$ as $t \to \infty$ and substituting this expansion in the equation for u^*, we find that $K = -\frac{1}{\lambda^* I_2}$.

In III, again $u^* \sim w^* - \frac{\phi^*}{\lambda^* I_2 \hat{t}}$, $t = t_* + \hat{t}$ for some large t_* ($t_* \sim t^*$ for $\eta \to 0$).

In the interval II, we expand $u \sim w^* + \eta v_0 + \eta^2 v_1 + ...$ as $\eta \to 0$, and on making a change in time scale $t = \tau/\eta$, equation (1.1) gives:

$$\eta^2 v_{0\tau} + \eta^3 v_{1\tau} + ... \sim w^*_{xx} + \eta v_{0xx} + \eta^2 v_{1xx} + ... + \lambda R(\eta), \text{ as } \eta \to 0, \tag{3.1}$$

where

$$F(u) \sim \hat{R}(x,t;\eta) := R(\eta) = \frac{e^{-(w^*+\eta v_0+\eta^2 v_1+...)}}{\left(\int_{-1}^{1} e^{-(w^*+\eta v_0+\eta^2 v_1+...)}dx\right)^2}, \text{ as } \eta \to 0.$$

We require an expansion for $R(\eta)$ as follows

$$R(\eta) = R(0) + \eta R'(0) + \frac{\eta^2}{2} R''(0) + ... \,. \tag{3.2}$$

From (3.1), (3.2) we obtain

$$\begin{aligned} \eta^2 v_{0\tau} + \eta^3 v_{1\tau} + ... &\sim w^*_{xx} + \eta v_{0xx} + \eta^2 v_{1xx} + ... \\ &+ (\lambda^* + \eta^2)\left(R(0) + \eta R'(0) + \frac{\eta^2}{2} R''(0) + ...\right). \end{aligned} \tag{3.3}$$

As regards the boundary condition $\mathcal{B}_\pm(u) = 0$ at $x = \pm 1$, we have

$$\begin{aligned} & w^{*\prime}(\pm 1) + \eta v_{0x}(\pm 1,\tau) + \eta^2 v_{1x}(\pm 1,\tau) + ... \\ \sim & \mp \alpha w^*(\pm 1) \mp \eta \alpha v_0(\pm 1,\tau) \mp \eta^2 \alpha v_1(\pm 1,\tau) + ... \,. \end{aligned} \tag{3.4}$$

We equate the terms of zero order ($O(1)$ or $O(\eta^0)$) and we get

$$w^*{}_{xx} + \lambda^* R(0) = 0, \quad -1 < x < 1, \tag{3.5 a}$$

$$\mathcal{B}_\pm(w^*) = 0, \quad x = \pm 1, \tag{3.5 b}$$

$R(0) = e^{-w^*} / \left(\int_{-1}^{1} e^{w^*}\, dx\right)^2$, where problem (3.5) is actually problem (1.3). By looking now at the terms of $O(\eta)$ we have

$$v_{0xx} + \lambda^* R'(0) = 0, \quad -1 < x < 1, \tag{3.6 a}$$

$$\mathcal{B}_\pm(v_0) = 0, \quad x = \pm 1, \tag{3.6 b}$$

where $R'(0) = \delta\ F(w^*; v_0) = -\frac{e^{-w^*}\, v_0}{\left(\int_{-1}^{1} e^{-w^*}\, dx\right)^2} + \frac{2e^{-w^*}\int_{-1}^{1} e^{-w^*}\, v_0\, dx}{\left(\int_{-1}^{1} e^{-w^*}\, dx\right)^3}$. Problem (3.6) has the form of problem (2.4), thus we can write

$$v_0 = A\phi^* \tag{3.7}$$

where now we normalize ϕ^* according to $\int_{-1}^{1} \phi^{*2} dx = 1$. Looking next at the $O(\eta^2)$ terms we have

$$v_{0\tau} = v_{1\,xx} + R(0) + \frac{\lambda^*}{2} R''(0),$$

which becomes

$$\begin{aligned} v_{0\tau} = v_{1\,xx} &+ \frac{e^{-w^*}}{S_0^2(\phi^*)} + \frac{\lambda^* v_0^2 e^{-w^*}}{S_0^2(\phi^*)} - \frac{\lambda^* v_1 e^{-w^*}}{S_0^2(\phi^*)} - \frac{2\lambda^* e^{-w^*} v_0 S_1(v_0)}{S_0^3(\phi^*)} \\ &+ \frac{2\lambda^* e^{-w^*} S_1(v_1)}{S_0^3(\phi^*)} - \frac{\lambda^* e^{-w^*} S_2(v_0)}{S_0^3(\phi^*)} + \frac{3\lambda^* e^{-w^*} S_1^2(v_0)}{S_0^4(\phi^*)}, \end{aligned} \tag{3.8}$$

where now we denote by $I_{\nu\,k}(w^*, v) = (-1)^\nu S_k(v)$ with $S_k(v) = \int_{-1}^{1} e^{-w^*} v^k dx, \quad k = 0, 1, 2, 3$ and $S_0(\phi^*) = S_0$.

Multiplying (3.8) by ϕ^*, integrating over $[-1, 1]$, applying Green's identity, using (3.7), and the normalization of ϕ^*, we obtain

$$\begin{aligned} \dot{A}(\tau) = &\int_{-1}^{1} \left(\phi^{*\prime\prime} - \frac{\lambda^* e^{-w^*} \phi^*}{S_0^2} + \frac{2\lambda^* e^{-w^*} S_1(\phi^*)}{S_0^3} \right) v_1 dx + \frac{S_1(\phi^*)}{S_0^2} \\ &+ \lambda^* \frac{A^2(\tau)}{2S_0^4} \left(S_3(\phi^*) S_0^2 - 6S_1(\phi^*) S_2(\phi^*) S_0 + 6S_1^3(\phi^*) \right). \end{aligned} \tag{3.9}$$

The solution w^* of problem (3.5) is of the form $w^* = 2\ln(\beta\cos(\gamma x))$ with β, γ to be determined. This will give that $\lambda = 8\sin^2(\gamma)\exp(-(2\gamma\tan(\gamma))/\alpha)$ and for $\lambda = \lambda^*$ we should have $\alpha = \tan(\gamma^*)\left(\tan(\gamma^*) + \gamma^* \sec^2(\gamma^*)\right)$, thus we get $\lambda^* = 8\sin^2(\gamma^*)\exp(-2\gamma^*/(\tan(\gamma^*) + \gamma^* \sec^2(\gamma^*))$ and

$$w^* = \frac{2\gamma^* \tan(\gamma^*)}{\alpha} + 2\ln\left(\frac{\cos(\gamma^* x)}{\cos(\gamma^*)}\right).$$

Also the solution ϕ^* of the linearized problem (2.4) is equal to $\phi^* = \varphi^*/(\int_{-1}^{1} \varphi^{*2} dx)^{1/2}$, where φ^* is the solution (non normalized) of problem (2.4). Moreover

$$\varphi^* = \cot(\gamma^*) + \tan(\gamma^*) - x\tan(\gamma^* x).$$

Having the explicit forms of w^* and ϕ^* we can now calculate the quantity

$$S = S_3(\phi^*)\, S_0^2 + 6S_1^3(\phi^*) - 6S_1(\phi^*)\, S_2(\phi^*)\, S_0.$$

Thus we obtain

$$S = \left[\exp(-\frac{2\gamma^* \tan(\gamma^*)}{\alpha}) \cos^3(\gamma^*) \right]^3 \frac{\sec^4(\gamma^*)}{\gamma^*} \left(4\gamma^* - 2\gamma^* \cos(2\gamma^*) + 3\sin(2\gamma^*)\right),$$

which is always positive for $0 < \gamma^* < \frac{\pi}{2}$. Therefore the equation (3.9) can be written as

$$\dot{A}(\tau) = \frac{S_1}{S_0^2} + \lambda^* \frac{S}{2S_0^4} A^2(\tau), \quad A(\tau) \to -\infty \ \text{ as } \ \tau \to 0, \tag{3.10}$$

which has solution

$$A(\tau) = (\frac{B}{K})^{\frac{1}{2}} \tan\left[\tau (BK)^{\frac{1}{2}} - \frac{\pi}{2} \right],$$

for $K = \lambda^* S/2S_0^4$, $B = S_1/S_0^2$. Returning to the original time variable this expression becomes

$$A(t) = (\frac{B}{K})^{\frac{1}{2}} \tan\left[t(\lambda-\lambda^*)^{1/2}(BK)^{\frac{1}{2}} - \frac{\pi}{2}\right].$$

Because of $u = w^* + \eta v_0 + ...$ and $v_0 = A(t)\phi^*(x)$, it is obvious that u ceases to exist at time

$$t^* \sim t_b = t_u(\lambda-\lambda^*)^{-\frac{1}{2}}$$

where $t_u = \pi/(BK)^{1/2}$ and t_b is the blow-up time of $A(\tau) = A(t(\lambda-\lambda^*))$.

Numerical Solution:

We solve problem (1.1) by using a Crank - Nicolson scheme. For the linear terms we apply the usual form of the scheme i.e.

$$-\frac{r}{2}u_{j-1}^{n+1} + (r+1)u_j^{n+1} - \frac{r}{2}u_{j+1}^{n+1} = \frac{r}{2}u_{j-1}^n + (1-r)u_j^n + \frac{r}{2}u_{j-1}^n + \delta t\,\lambda F(u_j^n)$$

where u_j^n is the temperature at the nth time level and at the jth space grid, $r = \frac{\delta t}{(\delta x)^2}$ and the non-local term $F(u_j^n)$ is evaluated at the nth time step when we are solving a system of equations to evaluate temperature at the $(n+1)$th time step. For this term we have

$$F(u_j^n) = \frac{f(u_j^n)}{\left(\int_{-1}^1 f(u_j^n)dx\right)^2}.$$

The integral in the denominator is evaluated by Simpson's rule.

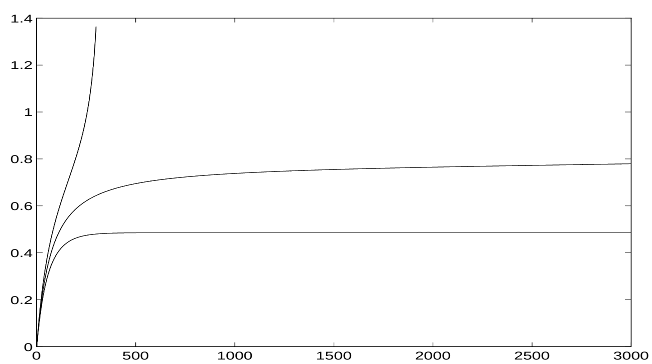

FIGURE 2. *Numerical solution of problem (1.1). We plot the* $\max_x(u(x,t))$, *for* x *in* $[-1,1]$ *against time (the upper curve corresponds to* $\lambda > \lambda^*$, *the intermediate to* $\lambda = \lambda^*$ *and the lower one to* $\lambda < \lambda^*$ *).*

In Figure 2 we use this scheme to solve the problem numerically for $f(u) = e^{-u}$ and taking $u(x,0) = 0$, $\alpha = 1$. We see that for $\lambda < \lambda^*$ the solution u tends to a steady state, for $\lambda = \lambda^*$ the behaviour is similar, and for $\lambda > \lambda^*$ the solution blows up (the decay is faster for $\lambda < \lambda^*$ than it is for $\lambda = \lambda^*$). More precisely, in Figure 2 the maximum of solutions are plotted against time.

In Figure 3, we plot the numerical estimate of the blow-up time together with the asymptotic estimate of it.

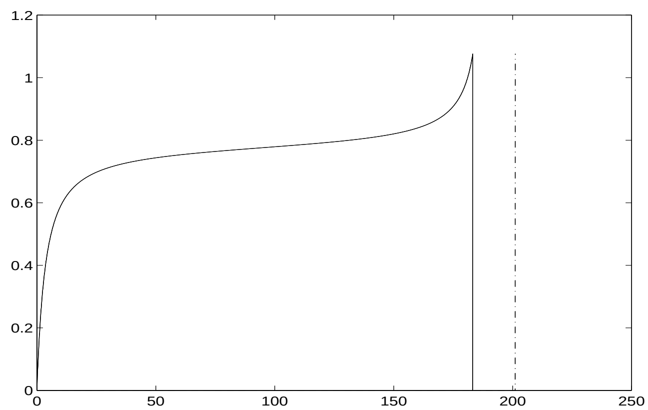

FIGURE 3. *Numerical solution of problem* (1.1) *for* $\lambda > \lambda^*$. *Again we plot the* $\max_x(u(x,t))$, *for* x *in* $[-1,1]$ *against time. The solid vertical line* $t_1^* \simeq 181.25$ *indicates the numerical estimate of the blow-up time, while the dotted vertical line* $t_2^* \simeq 203$ *indicates the asymptotic estimate of it.*

4 Conclusion

In this work we have dealt with the estimate of blow-up time t^*. It is interesting from the point of view of applications to know when the temperature u becomes infinity, which means, in many cases, depending on how the model arises, the blow-up might correspond to a short-circuit or a circuit breaking. Similar estimates are also known for the reaction diffusion problem [13]. In both cases the results are obtained when $0 < \lambda - \lambda^* \ll 1$ and for problems in which there exists a steady solution $w^* = w(x; \lambda^*)$ of the time dependent u-problem at $\lambda = \lambda^*$. Here the methods that are applied are comparison techniques and asymptotic expansion, as well as numerical computations. Our main result is the estimate $t^* = t_u(\lambda - \lambda^*)^{-1/2}$ as $0 < \lambda - \lambda^* \ll 1$, where t_u is a constant and λ, λ^* are given. It remains an open question how to estimate t^* when there is no regular solution w^* at $\lambda = \lambda^*$.

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References

[1] H. Amann, (1976), Fixed point equations and nonlinear eigenvalue problems in ordered Banach Spaces, *SIAM Rev.*, **18**, 620–709.

[2] W. Allegretto and H. Xie,(1992), Existence of solutions for the time–dependent thermistor equations, *IMA J. Appl. Math.*,**48** , 271–281.

[3] J. W. Bebernes and D. Eberly,(1989) Mathematical Problems from Combustion Theory, Springer–Verlag, New York.

[4] J.A. Carrillo, (1998), On a non-local elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction. *Nonlinear Analysis TMA.,* **32**, 97-115.

[5] N. Chafee, (1981), The electric ballast resistor: homogeneous and nonhomogeneous equilibria, In Nonlinear differential equations: invariance stability and bifurcations, P. de Mottoni & L. Salvatori Eds., Academic Press, New York., 97–127.

[6] X. Chem and A. Friedman,(1993), The thermistor problem for conductivity which vanishes at large temperature, *Quart. Appl. Math.*, **LI** , 101–115.

[7] G. Cimatti, (1989),Remark on existence and uniqueness for the thermistor problem under mixed boundary conditions, *Quart. J. Appl. Mayh.*, **47** , 117–121.

[8] G. Cimatti, (1990), The stationary thermistor problem with a current limiting device, *Proc. Roy. Soc. Edinburgh* **116A** , 79–84.

[9] M. G. Crandall & P.H. Rabinowitz, (1973), Bifurcation, pertubation of simple eigenvalues, and linearized stability. *Arch.Rational Mech. Anal.*, **582**, 161-180.

[10] A. C. Fowler, I. Frigaard and S. D. Howison,(1992), Temperature surges in current–limiting circuit devices, **SIAM J. Appl. Math. 52** , 998–1011.

[11] P. Freitas & M. Grinfeld, (1994), Stationary solutions of an equation modelling Ohmic heating, *Appl. Math. Lett.*, Vol. **7**, No **3**, 1-6.

[12] Galaktionov V.A., Vazquez J.L. (2000), The Problem of Blow-up in Nonlinear Parabolic Equations, *Proc. Summer Course on PDEs*, Chile (Temuco, 1999), Pitman.

[13] A. A. Lacey, (1983), Mathematical analysis of thermal runaway for spartially inhomogeneous reactions, *SIAM J. Appl. Math.*, **43**, 1350–1366.

[14] A. A. Lacey and D. Tzanetis, (1986), Global existence and convergence to a singular steady state for a semilinear heat equation, Proc. Royal Soc. Edin. 105A, 289-305.

[15] A.A. Lacey, (1995a), Thermal runaway in a non-local problem modelling Ohmic heating. Part I : Model derivation and some special cases, *Euro. Jl. Appl. Maths.*, **6**, 127-144.

[16] A.A. Lacey, (1995b), Thermal runaway in a non-local problem modelling Ohmic heating. Part II : General proof of blow-up and asymptotics of runaway, *Euro. Jl. Appl. Maths.*, **6**, 201-224.

[17] J. Lopez-Gomez, (1998), On the structure and stability of the set of solutions of a non-local problem modelling Ohmic heating, *J. Dynam. Diff. Eqns* **10**, **4**, 537-559.

[18] D. H. Sattinger, (1972), Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. **21**, 979–1000.

[19] D.E. Tzanetis, Blow-up of radially symmetric solutions of a non-local problem modelling Ohmic heating, preprint.