A Model for Housing Allocation of a Homeless Population due to a Natural Disaster

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Abstract

In the present work we derive and analyze a model considering housing allocation of homeless families due to a natural disaster; we use data from the earthquake of September 1999, in Athens, Greece. We derive a non-linear system of ordinary differential equations and analyze the stability of this system. Also we find an approximate solution of the model for a case study as well as and a numerical solution. Finally we consider possible extensions and improvements of the model making it more realistic.

Keywords: Mathematical Model, Population Dynamics, Asymptotic Analysis, Homeless Population.

1 Introduction

We are interested in investigating a problem concerning housing allocation of homeless families due to a natural disaster. Our purpose is to derive a mathematical model and to analyze it, taking into account data from the earthquake of September 1999, in Athens, Greece. The location of the epicenter of the earthquake, in 7/9/1999, was 18 km NE of Athens and its magnitude was 5.9 of the Richter scale degrees. About 700 people where wounded and 143 people lost their lives while about 70000 families became homeless.

In 1996 the European Study Group with Industry considered a similar problem related to the number of homeless people in the United Kingdom. The aim was to analyze a model, considering the mobility, of homeless and non-homeless

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people and in particular to see how these figures might be affected by different housing policies regarding various categories of people [3]; work [3] was extended further in [4], [5], [6].

This work is motivated, apart from [3], and also by [1] where a model about the evolution of population of tourists in Tenerife is demonstrated.

The analysis of such a model could be useful for welfare planning by the state. Prediction of at least qualititative characteristics of homeless and non-homeless population flows could lead to a better programming by the state to cope with the problems created by a natural disaster e.g. planning appropriate stock of tents or prefabricated houses etc. Also another aspect could be the estimation of the cost and time of resettlement for the victims of a natural disaster.

Here we derive a model considering a number of homeless people at some initial time after the natural disaster. We attempt to analyze the flow of this population to a temporary state (tents, prefabricated houses, relatives etc.) until their resettlement in a permanent residence.

In Section 2 a fairly simple model is derived. This model consists of a system of non-linear ordinary differential equations (ode's). The population is divided into three different categories regarding their allocation state. These are: a) homeless families, b) families in temporary accommodation and c) families that they have been resettled; modelling the flows between them leads to a system of ordinary differential equations.

We also study the possible steady - states of the system and analyze the stability of the model for the possible positive equilibrium points that can be found. For some specific cases the stability of the system cannot be deduced immediately and center manifold analysis is applied.

Moreover, we consider a case study concerning the earthquake of September 1999, in Athens, Greece, and by using data from census taken by The National Statistical Institute of Greece, we estimate the coefficients of the flows in the model. We solve numerically the system of ode's using a Runge-Kutta method and the results are analyzed. Also for certain choices of the coefficients of the flows we find an asymptotic solution of the system.

In Section 3 we apply similar analysis to a more sophisticated version of this model, considering the families in temporary accommodation in more than one category. More specifically, we have the families living in temporary accommodation divided among those living in a camp site organized by the state, those living in self-provided temporary accommodation, and those living with relatives and friends's house.

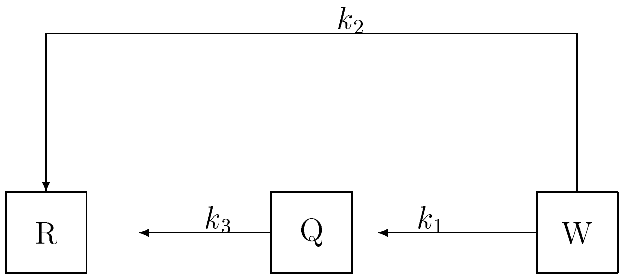

Fig. 1. Schematic representation of the model

In Section 4 we discuss the results of this work and the possible improvement of this model making it more realistic and conclude with the results of this work in Section 5.

2 Derivation of the Model

We consider three categories of population: a) number of families that become homeless after the natural disaster (having houses destroyed or badly damaged) denoted by $W = W(t)$ (families living in badly damaged houses will also be included in this category), b) number of families that are accommodated in a temporary state (families living in tents, prefabricated houses or hosted by relatives and friends) denoted by $Q = Q(t)$ and c) number of families that are resettled in a new home, denoted by $R = R(t)$. The variable t represents the time after the natural disaster.

We assume [3], that the rate of change of number of homeless families accommodated in temporary accommodation will be jointly proportional to W and to the number of available temporary accommodation $Q_a - Q$, where Q_a is the number of temporary accommodation available in stock by the state. Also the number of homeless families being resettled will be jointly proportional to W and to the availability of houses allowing families to resettle $R_a - R$, where R_a is the number of houses available.

Thus the rate of decrease of W will be given by the following equation

$$\frac{dW}{dt} = -k_1 W(Q_a - Q) - k_2 W(R_a - R), \tag{1}$$

where k_1 and k_2 are positive constants of proportionality.

The rate of change in the number of families living in temporary accommodation will be proportional to $W(Q_a - Q)$ and to the number of families that are resettled $Q(R_a - R)$. Therefore we have

$$\frac{dQ}{dt} = k_1 W(Q_a - Q) - k_3 Q(R_a - R), \tag{2}$$

where k_3 is a positive constant of proportionality, while for families resettled we have

$$\frac{dR}{dt} = (k_2 W + k_3 Q)(R_a - R). \tag{3}$$

We assume also that the population remains constant during this process therefore

$$W_0 = W(0) = W(t) + Q(t) + R(t). \tag{4}$$

Finally, using equation (4) we can eliminate equation (3) so that a system of two ordinary differential equations and an algebraic one is formed, for $t \geq 0$:

$$\frac{dW}{dt} = -k_1 W(Q_a - Q) - k_2 W(R_a - R), \tag{5}$$
$$\frac{dQ}{dt} = k_1 W(Q_a - Q) - k_3 Q(R_a - R), \tag{6}$$
$$W_0 = W(0) = W(t) + Q(t) + R(t), \tag{7}$$

where Q_a and R_a are positive constants of availability, being independent of time, so the system is autonomous. In order to derive this model we make the following assumptions. a) Birth or death rates can be neglected because we are interested in a time scale shorter than a generation. b) The number of families is large enough to be considered as continuum, so P, Q, R are considered to be non negative real functions of time no greater than W_0. c) Rates depend only on present time and delays are negligible i.e. there is an infinite fast rate of rehousing. d) Finally the conservation principle for the number of families is applied.

Estimation of the k's. Given the system of equations (5) and (6) we require that the values of $P(t)$, $Q(t)$, $R(t)$ are consistent to the data (P_s, Q_s, R_s) from the January 2000 census concerning earthquake victims at time $t_s = 3$ months after the earthquake. Therefore, at $t = t_s$ we must have $Q(t_s) = Q_s = 43,559$ families, $R(t_s) = R_s = 19,008$ families and $W(t_s) = W_s = 5,469$ families.

The results of the census are demonstrated in Figure (2). In our analysis we neglect for simplicity the category "Other cases".

We also have that at time $t = 0$, $W(0) = W_0 = 70,099$ families while $Q(0) = R(0) = 0$. Then we consider a function $G(k_1, k_2, k_3) = (W(t_s) - W_s,\ Q(t_s) - Q_s,\ R(t_s) - R_s)$. Finding the zeros of this function gives an estimate of the

Symbol	Categories	Number of families
P	Organized camp	5,528
T_1	Isolated tent	11,780
T_2	Trailer	906
T_3	Hotel	270
T_4	Ship	59
T_5	Stadium	47
F	Guest with friends or relatives	24,969
Q	Sum of families in temp. accommodation	43,559
R	Renting house after the earthquake or resettled	19,008
W	Living in a badly damaged house	5,469
N	Other cases	2,063
W_0	Summary of all categories	70,099

Fig. 2. Demonstration of the results by the census of January 2000 regarding victims of the earthquake in Athens.

values of k's. We solve this equation numerically, as a transcendental equation with an iteration scheme. We start with an initial guess of the $k's$ and in each step of the iteration scheme the system of equations, (5) and (6), is solved numerically, with a Runge-Kutta method, to give $W(t_s)$, $Q(t_s)$, $R(t_s)$. We make this initial guess so that k_1 and k_2 are much larger than k_3. This is due to the fact that we expect to have a very quick decrease of the number of homeless people settled in a temporary accommodation (supplied by the state) or permanent accommodation (for those families that they have the financial ability to do so). Also we expect to have a slow flow from people already in temporary accommodation to a permanent residence, something that it is expressed by the size of k_3. Moreover, we assume that there is enough availability in both temporary and permanent accommodation, that is $R_a = 100 \times 10^3$ units of permanent accommodation and $Q_a = 200 \times 10^3$. Proceeding in such a way we find that $k_1 \simeq 88 \times 10^{-5}$ per families per year, $k_2 \simeq 31 \times 10^{-5}$ per families per year and $k_3 \simeq .9 \times 10^{-5}$ per families per year.

Taking another choice of initial conditions, with $Q_a = 70 \times 10^3$ units of temporary accommodation and $R_a = 100 \times 10^3$ units of permanent accommodation, the same procedure gives $k_1 \simeq 119 \times 10^{-5}$ per families per year, $k_2 \simeq .5 \times 10^{-5}$ per families per year and $k_3 \simeq 1.3 \times 10^{-5}$ per families per year. This set of values correspond to a situation where initially the number of homeless people decay rapidly while, due to the fact that constants of availability (Q_a and R_a) are not very large, the majority of families are accumulated in temporary accommodation in the initial stage of the process.

Note that the first of the above considerations correspond to a situation where a respectable number of families move to permanent accommodation immediately while the second consideration corresponds to a situation where, initially, families can only settle to a temporary accommodation (e.g. due to financial inability). In general we will consider these two cases which are more close to reality. Other considerations could be possible as well as with a more in depth statistical research of the population motion in order to obtain more accurate and realistic estimates for the k's.

We scale quantities representing number of families W, Q, R with the initial number of homeless families W_0 so that $W = x_1 W_0$, $Q = x_2 W_0$, $R = x_3 W_0$ and as time the time that the system needs so that W becomes very small, t_0, i.e. $t = t_0 \tau$ where $t_0 = 1/k_1 W_0$. Therefore the system of equations, for $\tau \geq 0$, become:

$$\frac{dx_1}{d\tau} = -a_1 x_1 (c_2 - x_2) - a_2 x_1 (c_3 - x_3), \tag{8}$$

$$\frac{dx_2}{d\tau} = a_1 x_1 (c_2 - x_2) - a_3 x_2 (c_3 - x_3), \tag{9}$$

$$x_1(0) = 1 = x_1(\tau) + x_2(\tau) + x_3(\tau). \tag{10}$$

with $x_1(0) = 1$, $x_2(0) = x_3(0) = 0$ and $c_2 = Q_a/W_0$, $c_3 = R_a/W_0$. Also $a_1 = k_1/k_1 = 1$, $a_2 = k_2/k_1$, $a_3 = k_3/k_1$. In the rest of the analysis we keep the notation for a_1, even if this is scaled to be one, in order to have generality in our results.

In order to investigate the stability of the system we find the equilibrium (stationary, steady - state, fixed) points which are solutions of the following set of algebraic equations:

$$-a_1 x_1 (c_2 - x_2) - a_2 x_1 (c_3 - 1 + x_1 + x_2) = 0, \tag{11}$$

$$a_1 x_1 (c_2 - x_2) - a_3 x_2 (c_3 - 1 + x_1 + x_2) = 0. \tag{12}$$

We obtain the following sets of solutions $x_1 = 0,\ x_2 = 0$ or $x_1 = 0,\ x_2 = 1 - c_3$, or $x_1 = 1 - c_2 - c_3,\ x_2 = c_2$ and $x_1 = \frac{a_3[a_1 c_2 + a_2(c_3 - 1)]}{a_2(a_2 - a_1 - a_3)},\ x_2 = \frac{a_1 c_2 + a_2(c_3 - 1)}{a_1 - a_2 + a_3}$. We are especially interested in the state of the system expressed by the solution $x_1 = 0,\ x_2 = 0$ because we expect to have no homeless families or families in temporary accommodation after enough time has elapsed. The Jacobian JF of the system is

$$JF(x_1, x_2) =$$

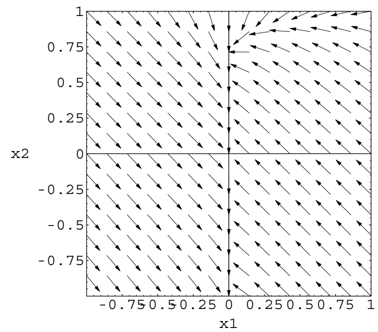

Fig. 3. Phase plane of the system of equations (8) and (9).

$$\begin{pmatrix} -a_1c_2 - a_2(c_3-1) - 2a_2x_1 + (a_1-a_2)x_2 & (a_1-a_2)x_1 \\ a_1c_2 - (a_1+a_3)x_2 & -a_3(c_3-1) - (1+a_3)x_1 - 2a_3x_2 \end{pmatrix},$$

where for $x_1 = x_2 = 0$, JF has eigenvalues $\lambda_1 = -a_1c_2 + a_2(1-c_3)$, $\lambda_2 = a_3(1-c_3)$.

Stability of the point $(0,0)$. The stability of this point depends crucially on the value of c_3 i.e. on the availability for resettlement. If $c_3 > 1$ then the two eigenvalues of the Jacobian of the system become negative and this point is asymptotically stable by linearization. In the case that $c_3 < 1$ we have $\lambda_2 > 0$ and this point becomes unstable. Finally it remains to examine the case $c_3 = 1$; then we have $\lambda_1 = -a_1c_2 < 0$ and $\lambda_2 = 0$. The presence of the zero eigenvalue indicates that we need further analysis in order to investigate the stability of this point for $c_3 = 1$.

Center manifold analysis for $c_3 = 1$. The matrix of the linear approximation of the system is

$$JF(0,0) = \begin{pmatrix} -a_1c_2 & 0 \\ a_1c_2 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -a_1c_2 < 0$ and $\lambda_2 = 0$. We now proceed by using the theory of center manifolds [2], (linearization and Liapunov functional do not help in this case). The eigenspace which corresponds to λ_1 is $E_{\lambda_1} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1\}$ and it is asymptotically stable, while $E_{\lambda_2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. For the stability at $(0,0)$ as well as of E_{λ_2} we find the center manifold: $W_c = \{(x_1 \oplus x_2) \in \mathbb{R}^2 : x_1 = h(x_2),\ h : E_{\lambda_2} \to E_{\lambda_1}\}$, where $\oplus$ denotes the direct sum. The function h cannot be defined uniquely but has a unique Taylor representation at the neighborhood of $(0,0)$, say $V = V(0,0)$, with $h(0) = h'(0) = 0$; therefore we seek a function h of the form $h(x_2) =$

$Ax_2^2 + Bx_2^3 + O(|x_2|^4)$, $x_2 \in V \cap E_{\lambda_2}$. We differentiate $x_1 = h(x_2)$ with respect to τ : $\dot{x}_1 = h'(x_2)\dot{x}_2$. The system of equations (8), (9) and the form of h give $x_1 = h(x_2) = 0$, recalling that $h(0) = h'(0) = 0$. The center manifold theorem predicts the existence of curves invariant under the flow and tangent to $x_1 = 0$. Here $x_1 = h(x_2) = 0$ for any $x_2 \in V \cap E_{\lambda_2}$, this means that $x_1 = 0$ is itself locally such a center manifold. Now the system on $x_1 = h(x_2) = 0$ becomes $\dot{x}_2 = -a_3 x_2^2$ which implies that $(0,0)$ is unstable on E_{λ_2}. Since we are interested in initial conditions of the form $(x_1(0), 0)$, $x_1(0) > 0$, the system starting from such a point is attracted finally by $(0,0)$, although $(0,0)$ is unstable (see the arrows in Figure (3)). Therefore in this sense this equilibrium point can be considered to be an attracting point. The phase portrait of the system is sketched in Figure (3), the trajectories are tangent to the vector field.

Other stationary points. We now consider the stationary points different from $(0,0)$.

As regards the point $(0, 1-c_3)$ we have that for $c_3 > 1$, this stationary point is not positive so it is not of interest to us. For $c_3 = 1$ we have that the eigenvalues of the Jacobian matrix of the system are $\lambda_1 = -c_2 < 0$ and $\lambda_2 = 0$, (taking $a_1 = 1$). In this case the equilibrium point is $(0,0)$ and it has been analyzed above.

When $c_3 < 1$ then the eigenvalues of Jacobian matrix of the system are $\lambda_1 = -c_2 + (1 - c_3) < 0$ and $\lambda_2 = a_3(c_3 - 1) < 0$, hence this point is asymptotically stable provided that $c_2 + c_3 > 1$. If $c_2 + c_3 < 1$ then $\lambda_1 > 0$ and this point becomes unstable. For $c_2 + c_3 = 1$ we have $\lambda_1 = 0$; again we apply a center manifold analysis. The matrix of the linear approximation of the system is:

$$JF(0, 1 - c_3) = \begin{pmatrix} 0 & 0 \\ a_3(c_3 - 1) & a_3(c_3 - 1) \end{pmatrix},$$

with $\lambda_2 = 0$ and $\lambda_1 = -a_3 c_2 < 0$. The eigenspace which corresponds to λ_1 is $E_{\lambda_1} = \{(0, x_2) : x_2 \in \mathbb{R}\}$ and the eigenspace corresponding to λ_2 is $E_{\lambda_2} = \{(x_1, -x_1) : x_1 \in \mathbb{R}\}$. We apply the transformation $x = x_1$ and $y = x_1 + x_2 - c_2$. In this way we transfer the equilibrium point, $(0, 1 - c_3) = (0, c_2)$, at the origin $(0,0)$ and linear manifold $\widehat{E}_{\lambda_2} = E_{\lambda_2} + c_2$ at the x axis. This transformation has been done since we want, for simplicity, to find a center manifold tangent for $\widehat{E}_{\lambda_2}$. Then the system (8)-(10) becomes:

$$\frac{dx}{d\tau} = (a_1 - a_2)xy - a_1 x(c_3 - x_3), \tag{13}$$

$$\frac{dy}{d\tau} = -a_3 c_2 y + (a_2 - a_3)xy - a_3 y^2. \tag{14}$$

We seek a function of the form $y = h(x)$, $h : \widehat{E}_{\lambda_2} \to E_{\lambda_1}$ and $h(x) = Ax^2 + Bx^3 + O(|x|^4)$, in a region of $(0,0)$ with $h(0) = h'(0) = 0$. By the relation $\dot{y} = h'(x)\dot{x}$ and the system (13) we deduce that $h(x) = 0$. Then the stability of the system (8)-(10) at the stationary point can be deduced by the equation $\dot{x} = -a_1 x^2$. Therefore the point $(0, 1-c_3)$is unstable but for initial conditions of the form $(x_1(0), 0)$, $x_1(0) > 0$, it is an attractor in the sense that we have a situation similar to the one demonstrated in Figure (3).

Looking now at the point $(1 - c_2 - c_3, c_2)$ we have that the eigenvalues of the Jacobian are $\lambda_1 = c_2 + c_3 - 1$ and $\lambda_2 = (c_2 + c_3 - 1)a_2 - c_2 a_3$. We assume $c_2 + c_3 < 1$ (otherwise this stationary point is not positive) and we get that both the eigenvalues are negative, so this point is asymptotically stable.

Finally for the point $\left(\frac{a_3[a_1 c_2 + a_2(c_3-1)]}{a_2(a_2-a_1-a_3)}, \frac{a_1 c_2 + a_2(c_3-1)}{a_1-a_2+a_3}\right)$, we have the first coordinate of this being $-\frac{a_3}{a_2}$ times the second coordinate. Thus this equilibrium point always has one negative coordinate and it is not of interest to us.

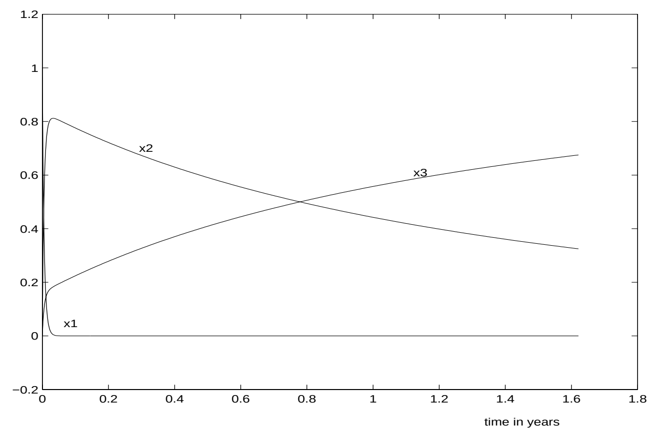

Fig. 4. Numerical solution of the simpler version of the model. The quantities x_i are plotted against time with the unit in the time axes representing a period of one month. Values that were used are $a_1 = 1$, $a_2 = 0.36$, $a_3 = 0.01$, $c_2 = 2.8531$, $c_3 = 1.4266$.

Using the first choice of k's, estimated in the beginning of this section, we have that $a_1 = 1$, $a_2 = 0.36$, $a_3 = 0.01$, $c_2 = 2.8531$, $c_3 = 1.4266$. We can solve the system numerically by using a Runge-Kutta method. Results of a numerical simulation are demonstrated in Figure (4). We see that x_2 attains its maximum at the time where $x_1 \simeq x_3$ and then decays exponentially. The time of the decay depends upon the constants c_3 and c_2 expressing availability in rehousing. For the second estimation for the k's the problem can be treated similarly.

Asymptotic analysis. Motivated by the form of the numerical solution and the relative sizes of the a's ($a_3 \ll a_1$ and $a_3 \ll a_2$) we seek an asymptotic solution. We see that initially we have a rapid change for time of order one. More precisely writing $a_3 = \epsilon a_1$ we have $\epsilon \simeq 0.01$ and the system (8)-(10) becomes:

$$\frac{dx_1}{d\tau} = -a_1 x_1 (c_2 - x_2) - a_2 x_1 (c_3 - x_3), \tag{15}$$

$$\frac{dx_2}{d\tau} = a_1 x_1 (c_2 - x_2) - \epsilon a_1 x_2 (c_3 - x_3), \tag{16}$$

$$x_1(0) = 1 = x_1(\tau) + x_2(\tau) + x_3(\tau). \tag{17}$$

Looking at terms of order one we have:

$$\frac{dx_1}{d\tau} \sim -a_1 x_1 (c_2 - x_2) - a_2 x_1 (c_3 - x_3), \tag{18}$$

$$\frac{dx_2}{d\tau} \sim a_1 x_1 (c_2 - x_2). \tag{19}$$

Also from equation (3) we obtain the equation for x_3:

$$\frac{dx_3}{d\tau} = a_2 x_1 (c_3 - x_3) + a_3 x_2 (c_3 - x_3). \tag{20}$$

Looking again at terms of order one we have

$$\frac{dx_3}{d\tau} \sim a_2 x_1 (c_3 - x_3). \tag{21}$$

Combining equations (19) and (21) we obtain the relation:

$$\frac{\dot{x}_2}{a_1(c_2 - x_2)} \sim \frac{\dot{x}_3}{a_2(c_3 - x_3)}, \qquad x_2(0) = x_3(0) = 0 \tag{22}$$

which implies that

$$x_2 \simeq c_2 \left[1 - c_3^{\frac{a_2}{a_1}} (c_3 - x_3)^{\frac{a_1}{a_2}} \right], \qquad x_2(0) = 0, \tag{23}$$

or equivalently $c_2 - x_2 = D(c_3 - x_3)^{\frac{a_1}{a_2}}$, with $D = c_2 / c_3^{\frac{a_1}{a_2}}$. Now setting $c_2 - x_2 = y$ the system of equations (18), (19) becomes

$$\frac{dx_1}{d\tau} \sim -a_1 x_1 y - a_2 D_1 x_1 y^{\frac{a_2}{a_1}}, \qquad D_1 = D^{-\frac{a_2}{a_1}}, \tag{24}$$

$$\frac{dy}{d\tau} \sim -a_1 x_1 y. \tag{25}$$

Combining these two equations we have

$$\frac{dx_1}{dy} \sim 1 + \frac{a_2}{a_1} D_1 y^{\frac{a_2}{a_1} - 1}, \tag{26}$$

and on integrating we get

$$x_1 \sim y + Ay^{\frac{a_2}{a_1}} + B, \tag{27}$$

where $A = (\frac{a_2}{a_1})^2 D_1$ and $B = 1 - c_2 - Ac_2^{\frac{a_2}{a_1}}$. Using this relation together with equation (25) we obtain

$$\frac{dy}{d\tau} \sim -y^2 - Ay^{\frac{a_2}{a_1}+1} - By, \tag{28}$$

and integrating we have

$$\tau \sim \int_{c_2}^{y(\tau)} \frac{1}{-s^2 - As^{\frac{a_2}{a_1}+1} - Bs} ds. \tag{29}$$

Therefore the initial system of equations is reduced to the single ode (28). In terms of x_2 this is written as

$$\frac{dx_2}{d\tau} \sim (c_2 - x_2)\left[c_2 + B - x_2 + A(c_2 - x_2)^{\frac{a_2}{a_1}}\right]. \tag{30}$$

For $c_2 > x_1(0)$, i.e. for enough availability in temporary accommodation, we have that x_2, solving equation (30), tends to a steady - state, which are the root of equation $c_2 + B - x_2 + A(c_2 - x_2)^{\frac{a_2}{a_1}} = 0$ (note that for $B < 0$ this root is the intersection of the straight line $c_2 + B - x_2 = 0$ and the curve $(c_2 - x_2)^{\frac{a_2}{a_1}} = 0$). We denote this steady - state by x_2^*.

From the following analysis we conclude that for $O(1)$ times the increase of x_2 is related to the increase of x_3 by equation (23). Also x_1 approaches zero when $y = c_2 - x_2$ attains its minimum (maximum of x_2), and this happens when $y = c_2 - x_2^*$. This time can be estimated by equation (29).

Looking now at later times, of order $O(\frac{1}{\epsilon})$ we need a change of the time scale by $O(\frac{1}{\epsilon})$, so $\epsilon\tau = s$. Then the system of equations becomes:

$$\frac{dx_1}{ds} \sim -\frac{a_1}{\epsilon} x_1(c_2 - x_2) - \frac{a_2}{\epsilon} x_1(c_3 - x_3), \tag{31}$$

$$\frac{dx_2}{ds} \sim \frac{a_1}{\epsilon} x_1(c_2 - x_2) - a_3 x_2(c_3 - x_3). \tag{32}$$

The right hand side of equation (31) and the first term of the right hand side of equation (32) are of the same size and of order $\frac{1}{\epsilon}$. This indicates that we should have $x_1 \sim 0$ and then the system becomes:

$$\frac{dx_2}{ds} \sim -a_3 x_2 (c_3 - x_3), \tag{33}$$
$$\frac{dx_3}{ds} \sim a_3 x_2 (c_3 - x_3), \tag{34}$$
$$\text{with} \qquad x_1 \sim 0 \quad \text{and} \quad x_2 + x_3 \sim 1. \tag{35}$$

The solution of this system is

$$x_2(s) = (c_3 - 1) \left[A_0 (c_3 - 1) e^{a_3 (c_3 - 1) s} - 1 \right]^{-1}, \tag{36}$$

with $x_3(s) = 1 - x_2(s)$. We see that x_2 decays exponentially to zero while x_3 is increasing. All quantities move to the steady - state $(x_1, x_2, x_3) = (0, 0, 1)$.

The parameter A_0 should be determined by a matching condition with the solution given by equation (30). We want for $s \to 0$ the solution of equation (36) to match with x_2^* so that we obtain

$$A_0 = \frac{1}{x_2^*} + \frac{1}{c_3 - 1}.$$

Then the full asymptotic solution can be written as

$$x_2(\tau) = x_2^{in} + x_2^{out} - x_2^*, \tag{37}$$

where x_2^{in} is given by equation (30) and x_2^{out} is given by equation (36).

Conclusion: The final result of the above analysis is that we have a respectable amount of families living in temporary accommodation for a long time, i.e. some years after the earthquake. This is in agreement with the real situation in Athens where still quite a lot of families are living in prefabricated houses three years after the earthquake of September 1999. It is estimated that in Athens about 10,000 families are still living in temporary accommodation (prefabricated houses) three years after the earthquake.

Now we consider the situation where $a_1 \gg a_2$ and $a_1 \gg a_3$. In such a case the pair of equations (8) and (9) becomes

$$\frac{dx_1}{d\tau} = -a_1 x_1 (c_2 - x_2) - \epsilon a_1 x_1 (c_3 - x_3), \tag{38}$$
$$\frac{dx_2}{d\tau} = a_1 x_1 (c_2 - x_2) - \epsilon \delta a_1 x_2 (c_3 - x_3), \tag{39}$$
$$x_1(0) = 1 = x_1(\tau) + x_2(\tau) + x_3(\tau), \tag{40}$$

where here $\epsilon a_1 = a_2$, $\delta = a_3/a_1$, and ϵ is small while δ is of order one.

For $\epsilon \ll 1$ the system becomes

$$\frac{dx_1}{d\tau} = -a_1 x_1(c_2 - x_2), \tag{41}$$
$$\frac{dx_2}{d\tau} = a_1 x_1(c_2 - x_2), \tag{42}$$
$$x_1(0) = 1 = x_1(\tau) + x_2(\tau) + x_3(\tau). \tag{43}$$

This means that $x_3 \sim O(\epsilon)$ and that $x_1 \sim 1 - x_2$. Then the equation for x_1 becomes

$$\frac{dx_1}{d\tau} = -a_1 x_1(c_2 - 1 + x_1), \quad x_2(0) = 1, \tag{44}$$

which has solution

$$x_1 = \frac{c_2 - 1}{c_2 e^{a_1(c_2-1)\tau} - 1}. \tag{45}$$

For τ large this gives $x_1 \sim O(\epsilon)$ while $x_2 \sim x_1(0) = 1$.

For larger times, i.e when $\tau \sim O(1/\epsilon)$ we need a change in the time scale so that $\epsilon\tau = s$. Applying this to equations (8) and (9) we obtain

$$\frac{dx_1}{ds} = -\frac{a_1}{\epsilon} x_1(c_2 - x_2) - a_1 x_1(c_3 - x_3), \tag{46}$$
$$\frac{dx_2}{ds} = \frac{a_1}{\epsilon} x_1(c_2 - x_2) - \delta a_1 x_2(c_3 - x_3), \tag{47}$$
$$x_1(0) = 1 = x_1(\tau) + x_2(\tau). \tag{48}$$

For $x_i's$ being bounded we need $x_1 \sim 0$ and this implies for the system

$$\frac{dx_2}{ds} = -\delta a_1 x_2(c_3 - x_3), \tag{49}$$
$$\frac{dx_3}{ds} = \delta a_1 x_2(c_3 - x_3), \tag{50}$$
$$x_1(0) = 1 = x_2(\tau) + x_3(\tau). \tag{51}$$

Therefore $x_1(0) = x_2(\tau) + x_3(\tau)$ and the system is reduced to the single equation

$$\frac{dx_2}{ds} = -\delta a_1 x_2(c_3 - 1 + x_2). \tag{52}$$

This has solution

$$x_2 = \frac{c_3 - 1}{(A_0(c_3 - 1) + 1)e^{\delta a_1(c_3-1)s} - 1}. \tag{53}$$

Now for $s \to 0$ we need this solution to match with the one given by (45). This implies that we must have $x_1(0) \sim \frac{1}{A_0}$. Then the full solution of the system for all times is

$$x_2 \sim -\frac{c_2 - 1}{c_2 e^{a_1(c_2-1)\tau} - 1} + \frac{c_3 - 1}{A_0(c_3 - x_1(0) + 1)e^{a_3(c_3 - x_1(0))\tau} - 1}. \tag{54}$$

In this case we initially observe a flow of x_1 only to temporary accommodation and only when x_2 takes its maximum value; for $x_1 \sim 0$ we observe people moving to permanent accommodation with a much slower rate. This could correspond to a situation where, initially, there is financially inability from the state or from the families on their own to resetlle.

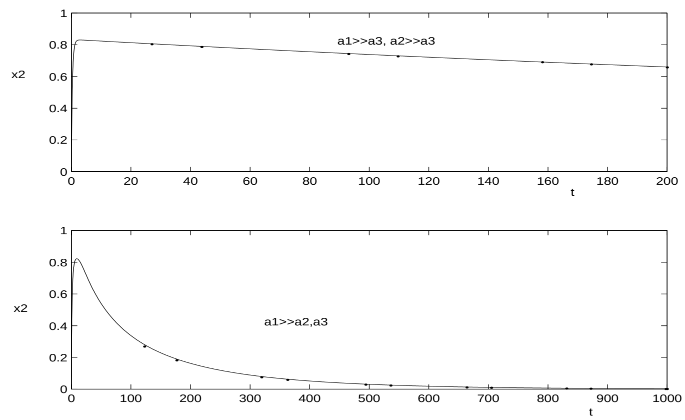

Fig. 5. Comparison of the numerical and asymptotic solutions for the model. In the first of these figures numerical solution of x_2 is plotted with the solid line while asymptotic solution of x_2 is plotted with dotted line against time (we see that the graphs of both solutions coincide). Values that were used are $a_1 = 1$, $a_2 = 0.36$, $a_3 = 0.01$, $c_2 = 2.8531$, $c_3 = 1.4266$. In the second, again numerical solution of x_2 is plotted with the solid line while asymptotic solution of x_2 is plotted with dotted line against time but with values $a_1 = 1$, $a_2 = 0.04$, $a_3 = 0.11$, $c_2 = 0.9986$, $c_3 = 1.4266$.

A more sophisticated model considering different categories of temporary accommodation is discussed in the next section.

3 Derivation of a more sophisticated model

We will consider once again three categories of population a) number of families that become homeless after the natural disaster (having houses destroyed or very badly damaged) denoted by W, b) number of families that are accommodated in a temporary state; we denote the number of families living in tents by T; the number of families living in prefabricated houses or in general, in house in camp organized by the state by P and the number of families living

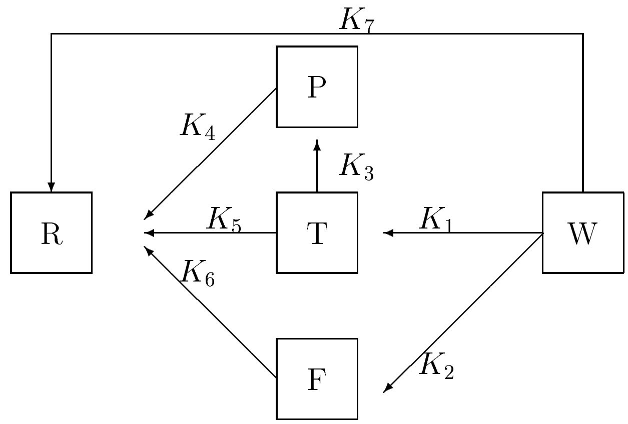

Fig. 6. Schematic representation of the model. The boxes represent the categories in which the population is divided and the arrows the flows between them.

with relatives or friends by F, and c) number of families that are resettled denoted by R. We assume that the number of homeless families accommodated in tents will be jointly proportional to W and to the number of available tents $T_a - T$ where T_a is the number of tents available in stock by the state. Similarly, the number of homeless families resettled will be jointly proportional to W and to $R_a - R$, where R_a is the number of houses available. The flow of homeless families to relatives and friends will be proportional to W. Thus the rate of decrease of W will be given by the following equation

$$\frac{dW}{dt} = -K_1 W(T_a - T) - K_2 W - K_7 W(R_a - R), \tag{55}$$

where K_1 and K_2 are positive constants of proportionality.

The rate of change of number of families living in tents will be proportional to $W(T_a - T)$, to the number of families that they are resettled $T(R_a - R)$ and to the number of families that are accommodated in prefabricated houses $T(P_a - P)$. Therefore we have

$$\frac{dT}{dt} = K_1 W(T_a - T) - K_5 T(R_a - R) - K_3 T(P_a - P), \tag{56}$$

where K_3 and K_5 are positive constants of proportionality. Similarly for the rate of change of number of families living in prefabricated houses we have

$$\frac{dP}{dt} = K_3 T(P_a - P) - K_4 P(R_a - R). \tag{57}$$

For the number of families accommodated in relative's or friend's house we have in a similar way

$$\frac{dF}{dt} = -K_6F(R_a - R) + K_2W. \tag{58}$$

Finally, for families resettled we have

$$\frac{dR}{dt} = (K_4P + K_5T + K_6F + K_7W)(R_a - R). \tag{59}$$

The symbols $K_4,\ K_6$ and K_7 are positive constant of proportionality. We assume also that the population remains constant during this process therefore,

$$W_0 = W(0) = W(t) + R(t) + P(t) + T(t) + F(t). \tag{60}$$

Using equation (60) we can eliminate equation (59) so that a system of four ode's and an algebraic one is formed:

$$\frac{dW}{dt} = -K_1W(T_a - T) - K_2W - K_7W(R_a - R), \tag{61}$$
$$\frac{dT}{dt} = K_1W(T_a - T) - K_5T(R_a - R) - K_3T(P_a - P), \tag{62}$$
$$\frac{dP}{dt} = K_3T(P_a - P) - K_4P(R_a - R), \tag{63}$$
$$\frac{dF}{dt} = -K_6F(R_a - R) + K_2W, \tag{64}$$
$$W_0 = W(0) = W(t) + R(t) + P(t) + T(t) + F(t), \tag{65}$$

with initial conditions $W(0) = W_0$, $R(0) = P(0) = T(0) = F(0) = 0$.

The derivation of this model is based on the same assumptions as in Section 2. Note also that the rate between F and P is taken to be negligible as well as the rate between W and P. The former, because of the data of the census about the earthquake, indicates that we have only a very small amount of families hosted by friends or relatives wanted to move to a camp organized by the state. The latter because people moving to organized accommodation had been living in a non-organized, temporary accommodation (in their own tents, hotels etc.).

We can apply the same method as in Section 2 to estimate the coefficients of the system. Note that in order to have agreement with the simpler model we must have $K_7 = k_2$ and the coefficients K_1 and K_2 can be calculated by considering the flow at time $t = 0$ from W to F and T should be

$$-K_1T_a - K_2 = -K_1Q_a,$$

while at time $t = t_s$ is

$$-K_1(T_a - T_s) - K_2 = -K_1(Q_a - Q_s).$$

Therefore by solving this linear system of equations, having $T_s = 38,031$ families and T_a taken to be $T_a = 150,000$ units of temporary accommodation, we have an estimate for these values: $K_1 = 294 \times 10^{-5}$ per families and per year and $K_2 = 29$ per families and per year. Now we consider a function $G(K_3, K_4, K_5, K_6) = (W(t_s) - W_s\,, P(t_s) - P_s,\ F(t_s) - F_s,\ T(t_s) - T_s)$. Finding the zeros of this function gives an estimate of the values of K's. We have $P_s = 5,528$, $T_s = 13,062$, $F_s = 19,008$ while we take $P_a = 50,000$ available units of temporary accommodation. Solving numerically this equation we find that $K_3 \simeq .6 \times 10^{-5}$ per families per year, $K_4 \simeq 2.77 \times 10^{-5}$ per families per year, $K_5 \simeq 4.22 \times 10^{-5}$ per families per year, and $K_6 \simeq 94.94 \times 10^{-5}$ per families per year.

Analysis of stability and numerical solution of the model. Now we scale quantities regarding number of families with the initial number of homeless families W_0 so that $W = x_1 W_0$, $T = x_2 W_0$, $P = x_3 W_0$, $F = x_4 W_0$, $R = x_5 W_0$. Also we scale as time the time that the system needs so that W becomes negligible, $t_0 = \frac{1}{k_1 W_0}$ and we have $t = \tau t_0$. Therefore the system of equations (61)-(65) becomes:

$$\frac{dx_1}{d\tau} = -a_1 x_1 (c_2 - x_2) - a_2 x_1 - a_7 x_1 (c_5 - x_5), \quad (66)$$

$$\frac{dx_2}{d\tau} = a_1 x_1 (c_2 - x_2) - a_5 x_2 (c_5 - x_5) - a_3 x_2 (c_3 - x_3), \quad (67)$$

$$\frac{dx_3}{d\tau} = a_3 x_2 (c_3 - x_3) - a_4 x_3 (c_5 - x_5), \quad (68)$$

$$\frac{dx_4}{d\tau} = -a_6 x_4 (c_5 - x_5) + a_2 x_1, \quad (69)$$

$$1 = x_1(0) = x_1(\tau) + x_2(\tau) + x_3(\tau) + x_4(\tau) + x_5(\tau). \quad (70)$$

Where now the a_i's are $a_i = \frac{k_i}{k_1}$, $i = 1, 3, 4, 5$, $a_2 = \frac{k_2}{k_1 W_0}$, $c_2 = \frac{T_a}{W_0}$, $c_3 = \frac{P_a}{W_0}$ and $c_5 = \frac{R_a}{W_0}$.

The system is solved numerically using a Runge-Kutta method and the results are demonstrated in Figure (7).

We can see here that x_1 tends to zero while at the same time x_4 and x_2 attain their maximum value. The number of families in prefabricated houses, represented by x_3, remains very small during this process. After a long period of time we find that the majority of the population is resettled while the rest of it is still accommodated in relatives houses (x_4 continuing to be significant).

The solution of the system of algebraic equations coming from equating the right-hand side of (66) - (69) to zero and (70) will give the steady - states of the system. One steady - state is $x_1 = x_2 = x_3 = x_4 = 0$ which is of interest in our analysis because it expresses the state when every family is resettled. Other

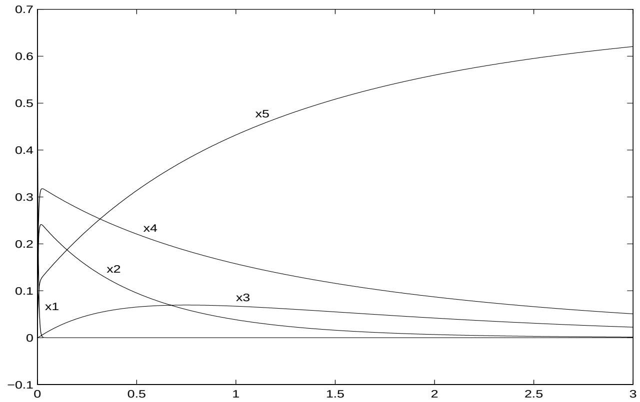

Fig. 7. Numerical solution of the model. The x's are plotted against time. Unit in the time axes represent a period of one month. Values that were used are $a_1 = 294.1$, $a_2 = 118$, $a_3 = 0.8056$, $a_4 = 0.8801$, $a_5 = 0.7179$, $a_6 = 0.6450$, $c_2 = 0.4531$, $c_3 = 1.4$, $c_5 = 1.4266$.

steady - states can be analyzed as regards their stability in a similar way. The parameter c_5 is crucial for the stability of the system at the origin. For $c_5 > 1$ all the eigenvalues of the Jacobian matrix of the system are negative and the point is asymptotically stable while for $c_5 < 1$ we obtain positive eigenvalues and the system is unstable.

If we take the Jacobian of the system we have for the zero solution, when $c_5 = 1$ that

$$JF(0,0) = \begin{pmatrix} -a_2 - a_7 - a_1 c_2 & 0 & 0 & 0 \\ a_1 c_2 & -a_3 c_3 & 0 & 0 \\ 0 & -a_3 c_3 & -a_4 & 0 \\ a_2 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the system are $\lambda_1 = -a_3 c_3$, $\lambda_2 = -a_4$, $\lambda_3 = -a_2 - a_1 c_2$ and $\lambda_4 = 0$. We see that the system attains the zero eigenvalue when $c_5 = 1$ with multiplicity 1 and the other three eigenvalues are negative, but starting with initial condition ($x_1 = 1$, $x_2 = x_3 = x_4 = 0$) the zero point should be an attracting point, therefore further analysis is needed.

Center manifold analysis. We proceed again by using the theory of center manifolds. The eigenspace which corresponds to the zero eigenvalue λ_4 is $R^c = \{(0,0,0,x_4) \in \mathbb{R}^4 : x_4 \in \mathbb{R}\}$ i.e. the x_4 axis in $\mathbb{R}^4$. For the stability of $(0,0,0,0)$ we find a center manifold $W_c = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_i = h_i(x_4),\ i = 1,2,3\}$ where $\tilde{h} = (h_1,h_2,h_3)$ with $\tilde{h} : R^c \to R^{\tilde{s}} = R^{s_1} \oplus R^{s_2} \oplus R^{s_3}$, $\tilde{s} = (s_1,s_2,s_3)$ and $R^c \oplus R^{\tilde{s}} = \mathbb{R}^4$. The form of h_i is $h_i(x_4) = A_i x_4^2 + B_i x_4^3 + O(|x_4|^4)$, $i = 1,2,3$, with

$\tilde{h}(0) = \tilde{h}'(0)$. We differentiate the relations $x_i = h_i(x_4)$ and by using equations (66) - (69) and the form of h_i we obtain that $x_i = h_i(x_4) = 0 + O(|x_4|^4)$. Therefore substituting in equation these expressions in (69) we get that $\dot{x}_4 = -a_6 x_4^2$. Hence the origin is unstable. Although for positive initial conditions, as it is in the case that we are interested in, we have that the origin is an attractor.

4 Possible improvements of the models

The models demonstrated in the previous sections are based upon the assumption that availability in accommodation (e.g. P_a, T_a) is a constant. This means that initially there is a sufficient number of tents or accommodation in an organized camp and prefabricated houses supplied by the welfare state according to demand. A more realistic approach would be to have these quantities i.e. P_a, T_a, R_a as functions of time and to assume for example that $P_a = P_o + \rho t$ for $t_1 < t < t_2$ and zero elsewhere. This would express the fact that prefabricated houses are supplied after some time of the disaster e.g. a couple of months with a constant rate only for some period $t_2 - t_1$. This modification would make the system of equations of the model non-autonomous and we should consider in such a case the time scales of the system more carefully as regards its stability and asymptotic behaviour.

Another modification would be to consider the time that families need to settle in a temporary accommodation or to resettle as far as such a possibility is available for a family. This would lead to a system of delay equations.

Finally in order to be more precise as regards the determination of the coefficients and the initial conditions of the model we could consider these as random variables following some distribution, estimated by available data in each case, and come out with a system of stochastic differential equations.

5 Discussion

Two models were derived regarding the movement of a homeless population after a natural disaster. In the second of these, the number of families in temporary accommodation is divided into three categories while in the first model it is taken to be one category. In both cases we analyze the stability of the system of derived ordinary differential equations. We find the steady - states of the system and we look at the zero solutions because we are interested in the state of the system where every family is resettled. With the initial conditions that we pose the origin is attracting but the presence of zero eigenvalue, when availability is equal to the demand for accommodation ($c_3 = 1$ for the simple version of the model or $c_5 = 1$ for the more sophisticated version of the model), the Jacobian of the system for this steady - state indicates that more analysis is needed in order to be able to see the behaviour of the system with positive initial conditions. Also an asymptotic solution of the system is given for the first model based on the dominant flows of it. The models are solved numerically using a Runge - Kutta scheme and the results are analyzed.

All the numerical estimates in our analysis are based on data taken from the census of the earthquake in Athens in September 1999. It would be interesting to test the results of this model with data taken from other cases of homeless populations of a natural disaster of the coefficients easier and more accurate and the role of the welfare state. This could also make the determination of the coefficients. Also possible improvements of the model are suggested in order to make it more realistic.

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References

[1] J.M. Gonzáles, *A system of logistic type equations which modellizes visitors demand in two areas of Tenerife Island.* Nonlinear Analysis: RWA, **35**, 111-123, 1999.

[2] J. Hale, H. Koçak, *Dynamics and Bifurcations.* Springer Verlag, New York, 1999.

[3] A.A Lacey, C. Nikolopoulos et all., *Report of the European study group with industry: Homeless Populations.* Oxford University Press, Oxford, April 1996.

[4] A.A. Lacey, A. Wough, *A Model of Counsil Housing Allocation.* MSc Dessertation , Heriot-Watt University 1997.

[5] A.A. Lacey, A. Wough, *Modelling Edinbourgh Consil Houses.* Preprint, 1999.

[6] A.A. Lacey, A. Wough, *Modelling Glasgow Consil Houses.* Preprint, 1999.