Proceedings of the Edinburgh Mathematical Society

Submitted Paper
Paper 22 December 2004

ESTIMATES OF BLOW-UP TIME FOR A NON-LOCAL REACTIVE-CONVECTIVE PROBLEM MODELLING OHMIC HEATING OF FOODS

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(Received)

Abstract In this work, we estimate the blow-up time for the non-local hyperbolic equation of Ohmic type, $u_t + u_x = \lambda f(u)/(\int_0^1 f(u)\,dx)^2$, together with initial and boundary conditions. It is known, that for $f(s),\ -f'(s)$ positive and $\int_0^\infty f(s)ds < \infty$, there exists a critical value of the parameter $\lambda > 0$, say λ^*, such that for $\lambda > \lambda^*$ there is no stationary solution and the solution $u(x,t)$ blows up globally in finite time t^*, while for $\lambda \le \lambda^*$ there exist stationary solutions. Moreover the solution $u(x,t)$ also blows up for large enough initial data and $\lambda \le \lambda^*$. Thus, estimates for t^* were found either for λ greater than the critical value λ^* and fixed initial data $u_0(x) \ge 0$, or for $u_0(x)$ greater than the greatest steady-state solution (denote $w_2 \ge w^*$) and fixed $\lambda \le \lambda^*$. The estimates are obtained by comparison, by asymptotical and by numerical methods. Finally, amongst the others, for given λ, λ^* and $0 < \lambda - \lambda^* \ll 1$, estimates of the form were found: upper bound $\epsilon + c_1 \ln\left[c_2\,(\lambda - \lambda^*)^{-1}\right]$; lower bound $c_3(\lambda - \lambda^*)^{-1/2}$; asymptotic estimate $t^* \sim c_4(\lambda - \lambda^*)^{-1/2}$ for $f(s) = e^{-s}$. Moreover, for $0 < \lambda \le \lambda^*$ and given initial data $u_0(x)$ greater than the greatest steady-state solution $w_2(x)$, we have upper estimates: either $c_5 \ln(c_6\, A_0^{-1} + 1)$ or $\epsilon + c_7 \ln(c_8 \zeta^{-1})$, where $A_0,\ \zeta$ measure, in some sense, the difference $u_0 - w_2$, (if $u_0 \to w_2+$ then $A_0,\ \zeta\ \to 0+$). $c_i > 0$ are some constants and $0 < \epsilon \ll 1,\ \ 0 < A_0, \zeta$. Some numerical results are also given.

Keywords:
Non-local hyperbolic equations; blow-up; estimates of blow-up time; asymptotic and numerical estimates.

1. Introduction

We consider the non-local initial boundary value problem,

$$u_t(x,t) + u_x(x,t) = \lambda \frac{f(u(x,t))}{\left(\int_0^1 f(u(x,t))\,dx\right)^2}, \qquad 0 < x < 1, \quad t > 0, \tag{1.1a}$$

$$u(0,t) = 0\,,\ \ t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x) \ge 0, \qquad 0 < x < 1, \tag{1.1c}$$

where $\lambda > 0$. The function $u(x,t)$ represents the dimensionless temperature when an electric current flows through a conductor (e.g. food) with temperature dependent on electrical resistivity $f(u) > 0$, subject to a fixed potential difference $V > 0$. The (dimensionless) resistivity $f(u)$ may be either an increasing or a decreasing function of temperature depending strongly on the type of the material (food). Problem (1.1) models one of the main methods for sterilizing food. The sterilization can take place by electrically heating the food rapidly. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in Ohmic heating of the food which quickly gets hot. The problem was considered by Please *et al.* [23] who looked at the stability of models allowing for different types of flow. More background on this type of process can be found in Biss *et al.* [5], De Alwiss & Fryer [3], Fryer *et al.* [14], Skudder & Biss [25], Stirling [26], and Zhang & Fryer [28].
Lacey, Tzanetis and Vlamos [21] have also studied problem (1.1): For f decreasing with $\int_0^\infty f(s)ds < \infty$ then blow-up occurs if λ is too large for a steady state to exist or if the initial condition is too big. If f is increasing with $\int_0^\infty ds/f(s) < \infty$ blow-up is also possible. If f is increasing with $\int_0^\infty ds/f(s) = \infty$ or decreasing with $\int_0^\infty f(s)ds = \infty$ the solution is global in time; some special cases with particular forms of f are discussed to illustrate what the solution can do, for details see [21] and the references therein.

Here λ is a dimensionless parameter and can be identified, amongst other things, with the square (is actually proportional) of the applied potential difference V. In the case where $f(u)$ is a sufficiently rapidly decreasing function of temperature, there exists a critical value of the potential difference V, say V^*, such that for $V > V^*$ (equivalently $\lambda > \lambda^*$) a thermal runaway (blow-up of the temperature u or burning of the food) takes place, see [19, 20, 21].
In the following we assume f to satisfy

$$f(s) > 0, \quad f'(s) < 0, \quad s \geq 0, \tag{1.2a}$$

$$\int_0^\infty f(s)\, ds < \infty \tag{1.2b}$$

for instance either $f(s) = e^{-s}$ or $f(s) = (1+s)^{-p}$, $p > 1$, satisfy (1.2).
For the initial data it is required that $u_0(x)$, $u_0'(x)$ to be bounded, $u_0(x) \geq 0$ in $[0,1]$ (the last requirement is a consequence of the fact that for any initial data the solution u becomes non-negative over $(0,1]$ for some time t and so, with an appropriate redefinition of t, we can always make this assumption [19, 21]).

The solution $u(x,t)$ also blows up for large enough initial data even if $0 < \lambda \leq \lambda^*$, [19, 20, 21]; we give (see Section 5), in this case, an analogous estimate of blow up time, as for the case of $\lambda > \lambda^*$.

The corresponding steady problem to (1.1) is

$$w' = \mu f(w) = \lambda \frac{f(w)}{\left(\int_0^1 f(w)\, dx\right)^2}, \quad 0 < x < 1, \quad w(0) = 0, \tag{1.3}$$

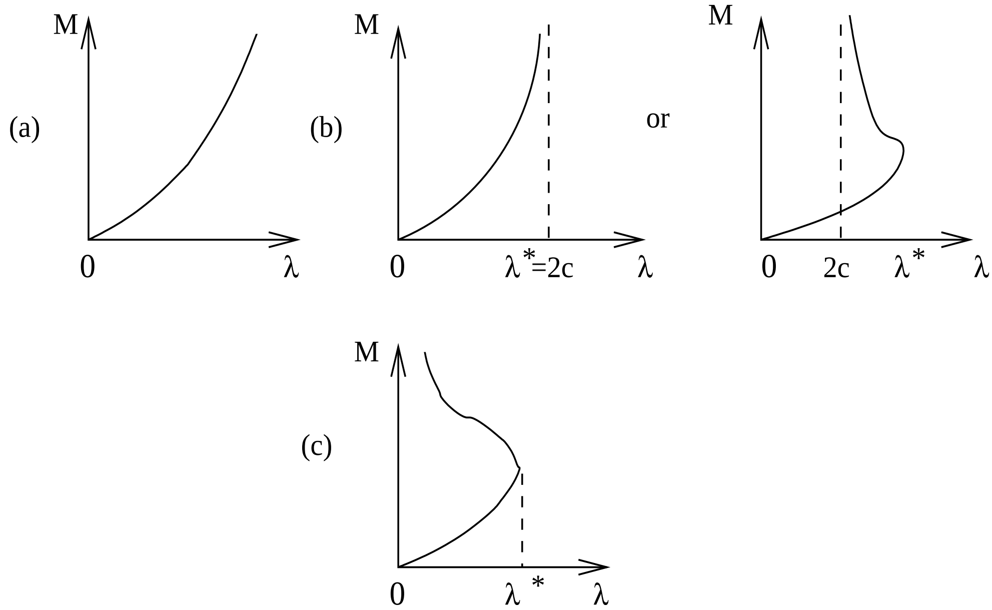

Figure 1. Possible response diagrams for equilibrium solutions: (a) $Mf(M)$ unbounded; (b) $Mf(M) \to c,\ 0 < c < \infty$, as $M \to \infty$; (c) $Mf(M) \to 0$, as $M \to \infty$. In each diagram it is possible for there to be more turning points than shown (so that for some λ there are more solutions).

where $w = w(x) = w(x;\lambda)$ (see [6, 13, 19, 20, 21]). The parameter μ is referred to as a local parameter while λ as a non-local one and the relation between them is $\mu = \lambda/\left(\int_0^1 f(w)dx\right)^2$.

On integrating the ordinary differential equation (1.3) we see that $\lambda = \frac{M^2}{\mu}$, where $M = M(\mu) = \max\limits_{[0,1]} w = w(1;\mu)$ and $\mu = \int_0^M ds/f(s)$.

It is clear that $M(\mu) \to 0+$ as $\mu \to 0+$ and, with $f(0) > 0$, $\lambda \to 0+$ as $\mu \to 0+$.

It is also known that if (1.2b) holds, then there exists a critical value of the parameter λ, say $\lambda^* < \infty$, such that for $\lambda > \lambda^*$, the solution $u(x,t;\lambda)$ to problem (1.1), blows up globally in finite time t^* ($u \to \infty$ for all $x \in (0,1]$ as $t \to t^*-$, actually the blow-up is uniform in x on compact subsets of $(0,1]$, in fact, for two points $0 < x_1 < x_2 \leq 1$, $|u(x_1,t) - u(x_2,t)| \to 0$ as $t \to t^*$, [21]) and problem (1.3) has no solutions (of any kind). For a fixed $\lambda \in (0,\lambda^*)$ there exist at least two solutions $w(x;\lambda)$ and a unique $u(x,t;\lambda)$; $u(x,t;\lambda)$ may either exist for all time or blow up globally depending on the initial data (for the blow-up, u_0 must be greater than the greatest steady solution $w(x;\lambda)$ and (1.2) holds) [19, 20, 21]. The response (bifurcation) diagrams for problem (1.3) are as in Figure 1, see also [21].

Our purpose, in this work, is to find some estimates of the blow-up time t^*, either with respect to the parameter λ (more precisely, with respect to the difference $\lambda - \lambda^*$, when $\lambda > \lambda^*$ and fixed initial data $u_0(x)$, or with respect to initial data $u_0(x)$ and fixed $\lambda \leq \lambda^*$.

Works related to this model (thermistor problem) can be found in [2, 7, 8, 9, 10, 12]. Estimates of blow-up time are very important since they answer the question "when" the blow up takes place (for problem (1.1) this is the time when the food is burnt) [4, 15, 16].

In Figure 1, (c) or (b) with $2c < \lambda^*$, there may be either only one or more than one turning point $(\lambda^*,\, M^*)$ depending on f. One can find other forms of non-local diagrams in [19, 20, 21]; their shapes depend strongly on the boundary conditions and the function f.

Under the assumptions (1.2), problem (1.3) has at least one classical (regular) steady solution $w^* = w(x;\lambda^*)$, (more than one w^* may exist). In the following, we assume that w^* is unique, i.e. Figure 1(c), and that the pair $(\underline{w}, \overline{w})$ at $\lambda < \lambda^*$ (λ close to λ^*) has the property: $\underline{w} = w_1$ is stable while $\overline{w} = w_2$ is unstable, (since in our proofs we require only the existence of at least one w^* at λ^* and that $\underline{w}(x) < \overline{w}(x)$ for x in $(0,1]$ where $\overline{w}$ is the next steady solution greater than $\underline{w}(x)$ at $\lambda < \lambda^*$).

Also we emphasize that for $\lambda > \lambda^*$ and for all $x \in (0,1]$ we have:

$$F(u) = \frac{f(u)}{(\int_0^1 f(u)dx)^2} \to \infty \ \text{ as } \ t \to t^*- < \infty, \tag{1.4a}$$

$$u(x,t;\lambda) \to \infty \ \text{ as } \ t \to t^*- < \infty, \tag{1.4b}$$

the latter means that $u(x,t;\lambda)$ blows up globally, see [19, 20, 21].

Similar situations, concerning the blow-up, can be found in the study of the (local) reaction diffusion problem:

$$u_t = \triangle u + \lambda f(u), \quad x \in \Omega, \quad t > 0, \tag{1.5a}$$

$$B(u) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.5b}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.5c}$$

where B represents Dirichlet or Robin boundary conditions, Ω is a bounded domain of $\mathbb{R}^n$, λ is a positive parameter and $f(u)$ behaves like e^u:

$$f(s) > 0,\ \ f'(s) > 0,\ \ f''(s) \geq 0, \text{ for } s \geq 0, \text{ and } \int_0^\infty ds/f(s) < \infty, \tag{1.6}$$

[1, 17, 18]. Again, under certain conditions, the solution u to (1.5) blows up ($\limsup_{t\to t^*-} \| u(\cdot,t;\lambda) \|_\infty = \infty$ for $\lambda > \lambda^*$, $t^* < \infty$). It should be emphasized that the blow-up for problem (1.5) differs from that of non-local problem (1.1), in that, (1.5) does not normally blow up globally. Moreover, there exists a turning point $P^* = (\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty < \infty$ of the response diagram of the steady problem corresponding to (1.5). For f which satisfies (1.6), the following upper and lower bounds for t^* hold: $t_1 \leq t^* \leq t_2$ where $t_i = c_i(\lambda - \lambda^*)^{-1/2}$ and c_i are some constants ($c_1 < c_2$); also for $f(s) = e^s$, $t^* \sim K(\lambda - \lambda^*)^{-1/2}$ as $\lambda \to \lambda^*+$ asymptotically, where K is a constant, [17].
Estimates of blow-up time also can be found in [16], (upper and lower bounds for t^*,

similar to those for the local problem (1.5)), by using comparison methods, for the non-local equation $u_t = u_{xx} + \lambda f(u)/(\int_{-1}^{1} f(u)\,dx)^2$, with initial conditions $u_0(x)$, Robin type boundary conditions, decreasing f and $\lambda > \lambda^*$. Also for $f(u) = e^{-u}$, the following asymptotic estimate holds: $t^* \sim t_u(\lambda - \lambda^*)^{-1/2}$ for $0 < \lambda - \lambda^* \ll 1$, where t_u is a constant. A numerical estimate is also obtained by using a Crank-Nicolson scheme.

In the present work, we find estimates for the non-local problem (1.1), for $f(s) = e^{-s}$ and for general $f(s)$ which satisfies (1.2).

In both problems, estimates of t^* have been found only if the spectrum of the steady problem is an interval closed on the right i.e. $(0, \lambda^*]$. It is still an open question, even for problem (1.5), how to estimate t^* when the spectrum is an open interval $(0, \lambda^*)$; see [16, 17, 18].

We organize this work as follows: in Section 2 we use comparison techniques and find upper and lower bounds for t^*, when f satisfies (1.2). In Section 3, we use an asymptotic expansion and again obtain an estimate of t^* but for $f(s) = e^{-s}$. Also in Section 4, we compute numerically the blow-up time t^* using an up-wind scheme and verifying the previous estimate. Finally, in Section 5 we give an estimate for t^* with respect to the initial data $u_0(x)$.

2. Comparison methods: upper and lower bounds of t^* for $\lambda > \lambda^*$

If the function f satisfies (1.2a), one can prove (see appendix in [21]) that a maximum principle holds for (1.1) (here is where we need f to be decreasing). Then we may, in the usual way, define upper and lower solutions of (1.1): an upper (lower) solution $\overline{u}$ ($\underline{u}$) is defined as a function which satisfies (1.1) if we substitute $\geq$ ($\leq$) for $=$, see [19, 20, 21, 24, 27]. The same comparison properties also hold for the equation $u_t + u_x = 0$ together with the data (1.1b), (1.1c). In the following work, we use ideas and techniques similar to [16].

(*a*) *An upper bound for* t^*

We now wish to find an upper bound for the blow-up time t^*. For simplicity, we assume that $0 \leq u_0 < w^*$. Firstly, we write (1.3) in a slightly different way,

$$w' = \mu f(w) = \frac{\lambda f(w)}{\left(\int_0^1 f(w)\,dx\right)^2} = \lambda\, F(w), \quad 0 < x < 1, \quad w(0) = 0, \tag{2.1}$$

where $F(\cdot) = f(\cdot)/\left(\int_0^1 f(\cdot)dx\right)^2$ and λ is a positive parameter (eigenvalue). Then, the related linearized eigenvalue problem of (2.1) for a function $\phi = \phi(x;\lambda) \in \mathbb{R}$ (actually ϕ is assumed to be a real valued function) is:

$$\phi' - \lambda\, \delta F(w;\phi) = -\rho\, \phi, \quad 0 < x < 1, \quad \phi(0) = \phi_0 = 0, \tag{2.2}$$

where $\phi = \phi(x;\lambda)$, $\rho = \rho(w,\lambda)$, (ρ,ϕ) is the eigenpair and $\delta F(w;\phi)$ is the first variation (or Gâteaux derivative) of F at w in the direction of ϕ, $(F(w;\phi) := F(w+\epsilon\phi) = J(\epsilon)$ and $\delta F(w;\phi) = J'(0) = \lim_{\epsilon\to 0}\frac{F(w+\epsilon\phi)-F(w)}{\epsilon})$.

As regards the first variation $\delta F(w;\phi)$ we have,

$$\delta F(w;\phi) = \frac{f'(w)\phi}{(\int_0^1 f(w)\,dx)^2} - \frac{2f(w)\int_0^1 f'(w)\phi\,dx}{(\int_0^1 f(w)\,dx)^3}.$$

In the following, in order to simplify the expressions, we use the shorthand notation:

$$I_{\nu\,k}(w,\phi) := \int_0^1 f^{(\nu)}(w(x))\,\phi^k(x)\,dx,$$

and $I_\nu(w) := I_{\nu\,0}(w,\phi)$, $\nu,\,k = 0,1,2,3,\ldots$, $f^{(\nu)}(w) = \frac{d^\nu}{dw^\nu}f(w)$, thus

$$\delta F(w;\phi) = \frac{f'(w)\phi}{I_0^2(w)} - \frac{2f(w)I_{11}(w,\phi)}{I_0^3(w)}.$$

Now we can have the following lemma concerning the eigenpair of problem (2.2).

Lemma 1. Problem (2.2) has the eigenpair (ρ,ϕ) where $\phi(x) > 0$, $x \in (0,1]$ and its spectrum is a continuum of eigenvalues in $\mathbb{R}$, generated by $\rho = \rho(w,\lambda)$ for each $\lambda \in (0,\lambda^*]$. The eigenvalue $\rho = \rho(w,\lambda)$ is continuous with respect to λ.

Proof: Firstly, if $\phi(x)$ is a real valued function then equation (2.2) implies that $\rho \in \mathbb{R}$. Now, we write equation (2.2) in a different way

$$\phi' + (g(x)+\rho)\phi = qh(x),$$

where $q = -I_{11}(w,\phi)$ is a number, $h(x) = h(x;w(x)) = \lambda\,\frac{2f(w(x))}{I_0^3(w)} > 0$ and $g(x) = g(x;w(x)) = -\lambda\frac{f'(w(x))}{I_0^2(w)}$. Problem (2.2) can be written under the equivalent integral formulation,

$$\phi(x) = q\left[\exp\left(-\int_0^x g(z)\,dz - \rho\,x\right)\right]\int_0^x h(s)\left[\exp\left(\int_0^s g(z)\,dz + \rho\,s\right)\right]ds.$$

The above form of ϕ implies that if a non trivial ϕ satisfying (2.2) exists, then this is positive (ϕ actually does not change sign in $(0,1)$ and can be taken as positive).

Now we can normalize ϕ so that $q = 1$. Therefore there exists a real valued function ϕ satisfying (2.2) if we can find $\rho \neq 0$ satisfying the following equation

$$1 = -I_{11}(w,\phi) = \int_0^1 -f'(w(x))\phi(x)dx\,, \quad \text{where} \quad \phi(x) = \phi(x;\rho)\,.$$

This can be written as

$$1 = \int_0^1 -f'(w(x))\left[\int_0^x h(x-s)\exp\left[-\rho s - G(x) + G(x-s)\right]ds\right]dx$$

for $G(s) = \int_0^s g(z)\,dz$. Therefore

$$1 = \int_0^1 \exp(-\rho s) \left[\int_s^1 (-f'(w(x))h(x-s) \exp\left[-G(x) + G(x-s)\right])dx \right] ds \quad \text{or}$$

$$1 = \int_0^1 Y(s) \exp(-\rho s)\, ds, \tag{2.3}$$

where $Y(s) = Y(s;w) > 0$ for $-f'(w)h(x-s)\exp\left[-G(x)+G(x-s)\right] > 0$ and $0 < s < x < 1$ (the eigenvalue ρ is a real number). For a fixed λ, w's are also fixed. Now for each $Y(s) > 1$ we have only one real solution $\rho > 0$, which satisfies equation (2.3). Also for each $Y(s) < 1$ we have only one real solution $\rho < 0$ and for $Y(s) = 1$ we have only the trivial solution $\rho = 0$. The eigenvalues $\rho = \rho(w,\lambda) \in C((0,\lambda^*];\mathbb{R})$ since the function $w(x;\lambda)$, $g(x) = g(x;\lambda)$ and $Y(s) = Y(s;w)$ are continuous functions with respect to $\lambda \in (0,\lambda^*]$. This proves the lemma. □

It is known that the spectrum to problem (2.1) can be either a closed interval from the right or an open one. Here we consider the case where the spectrum to problem (2.1) is a closed interval from the right and that there exists a unique turning point $(\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty = w^*(1) = M^* < \infty$, see Figure 1(c); at $\lambda < \lambda^*$ two steady states correspond w_1, w_2 with $w_1 < w_2$, while at $\lambda = \lambda^*$ correspond $w_1 = w_2 = w^*$. We need the following:

Lemma 2. Let w_1, w_2 with $w_1 < w_2$ be the solutions of (2.1) at $\lambda < \lambda^*$, then $\rho_1 = \rho(w_1,\lambda) \le 0$, $\rho_2 = \rho(w_2,\lambda) \ge 0$ and $\rho^* = \rho(w^*,\lambda^*) = 0$ where ρ represents the eigenvalues of problem (2.2).

Proof: We assume $u(x,t) = w(x) + \epsilon\,\phi(x)\,e^{\rho t} + O(\epsilon^2)$, with $\phi(x) > 0$, for $x \in (0,1]$ and for some $\epsilon \in \mathbb{R}$. Then we find, to the first order of ϵ, that ϕ and ρ must satisfy problem (2.2). We also know that w_1 is asymptotically stable, w_2 is unstable and w^* is stable from below and unstable from above, see [21]. These imply that $\rho_1 \le 0$, $\rho_2 \ge 0$ and $\rho^* = 0$. □

Thus, the lower branch of the response diagram (Figure 1(c)) is asymptotically stable with $\rho_1 = \rho(w_1,\lambda) \le 0$ (ρ is the eigenvalue of (2.2) for $\lambda < \lambda^*$ at w), while the upper branch is unstable with $\rho_2 = \rho(w_2,\lambda) \ge 0$, see also [11, 19, 20, 22]. This continues to hold (with a suitable understanding of the "upper branch") even if there are more turning points P^*. Moreover, from Lemma 2 we have $\rho^* = \rho(w^*,\lambda^*) = 0$, where ρ is the eigenvalue of linearized problem (2.2); for related results, see also [1, 11, 22]. Because of Lemmas 1, 2, problem (2.2) at $\lambda = \lambda^*$, with $\phi^*(x) > 0$, becomes

$$\phi^{*\prime} - \lambda^*\,\delta F(w^*;\phi^*) = 0, \quad 0 < x < 1\,, \quad \phi^*(0) = \phi_0^* = 0. \tag{2.4}$$

Now, in order to find an upper bound for t^*, we take the difference,

$$v = v(x,t) = v(x,t;\lambda) = u(x,t;\lambda) - w^*(x) = u - w^*. \tag{2.5}$$

Since w^* is bounded, v blows up at the same time as u does and in the same manner i.e. globally. Hence $t^* = t^*(u) = t^*(v)$ ($t^*(u)$ is the blow-up time for u) and $v(x,t) \to \infty$

as $t \to t^*-$ for all $x \in (0,1]$. In the following, we find an A-problem (see below (2.13)), where $A = A(t)$ blows up and is such that: $A(t) \leq c\,\|v(\cdot,t)\|_\infty$ where $c = 1/\sup_x \phi^*(x)$ as long as $v(x,t) \geq \psi(x,t) = A(t)\,\phi^*(x)$. The latter relation and (2.5) imply $t^*(u) = t^*(v) \leq T^* = T^*(A)$, for some T^*, thus we find an upper bound T^* for $t^*(u)$.

Therefore we obtain

$$\begin{aligned} v_t = u_t &= -u_x + \lambda F(u) = -u_x + \lambda F(u) - \lambda^* F(w^*) + {w^*}' \\ &= -v_x + (\lambda - \lambda^*)F(u) + \lambda^*\left(F(u) - F(w^*)\right). \end{aligned} \tag{2.6}$$

By writing $J(\epsilon) = F(w^* + \epsilon v)$, $0 \leq \epsilon \leq 1$, whence $J(0) = F(w^*)$ and $J(1) = F(u)$, Taylor's formula gives,

$$F(u) - F(w^*) = J(1) - J(0) = J'(0) + \frac{J''(\xi)}{2},$$

for some $\xi = \xi(t) \in (0,1)$, where $J'(0) = \delta F(w^*; v) = \left[\frac{d}{d\epsilon} J(\epsilon)\right]_{\epsilon=0}$. Also

$$\delta^2 F(z; v) = \frac{f''(z)v^2}{I_0^2(z)} - \frac{4vf'(z)I_{1\,1}(z,v)}{I_0^3(z)} - \frac{2f(z)I_{2\,2}(z,v)}{I_0^3(z)} + \frac{6f(z)I_{1\,1}^2(z,v)}{I_0^4(z)},$$

where $z = w^* + \xi v$ and $\delta^2 F(z; v) = J''(\xi)$ is the second Gâteaux derivative. Thus, from equation (2.6) we get the problem:

$$v_t + v_x = (\lambda - \lambda^*)F(u) + \lambda^*\,\delta F(w^*; v) + \frac{\lambda^*}{2} J''(\xi), \;\; 0 < x < 1, \;\; t > t_1, \tag{2.7a}$$

$$v(0,t) = 0, \quad t > t_1, \tag{2.7b}$$

$$v(x,t_1) = u(x,t_1) - w^*(x) \geq 0, \quad 0 \leq x \leq 1, \tag{2.7c}$$

(it is easily seen, due to (1.4), that there exists a $t_1 \in (0,t^*)$ so that $v(x,t) = u(x,t) - w^*(x) > 0$ in $(0,1]$ for every $t > t_1$).

Now we set $v = u - w^* = \theta\hat{v}$, for $0 < \theta = \lambda - \lambda^* \ll 1$. Thus, problem (2.7) becomes

$$\theta\hat{v}_t + \theta\hat{v}_x = \theta F(u) + \lambda^*\theta\,\delta F(w^*; \hat{v}) + \frac{\lambda^*}{2} J''(\xi), \;\; 0 < x < 1, \;\; t > t_1, \tag{2.8a}$$

$$\hat{v}(0,t) = 0, \quad t > t_1, \tag{2.8b}$$

$$\theta\hat{v}(x,t_1) = u(x,t_1) - w^*(x) \geq 0, \quad 0 < x < 1. \tag{2.8c}$$

This is simplified to

$$\hat{v}_t + \hat{v}_x = F(u) + \lambda^*\,\delta F(w^*; \hat{v}) + \frac{\lambda^*}{2}\theta \hat{J}''(\xi), \quad 0 < x < 1, \quad t > t_1,$$

where $J(\xi) = \theta^2\,\hat{J}(\xi) = \theta^2\,\delta^2 F(z; \hat{v})$. Now we find a lower solution ψ for $\hat{v}$-problem (2.8). Therefore we require $\psi = \psi(x,t)$ to satisfy

$$\psi_t + \psi_x \leq F(u) + \lambda^*\,\delta F(w^*; \psi) + \frac{\lambda^*}{2}\theta\,\delta^2 F(z; \psi). \tag{2.9}$$

Setting $\psi(x,t) = A(t)\phi^*(x)$ and $\dot{A}(t) = \frac{d}{dt}A(t)$ we obtain

$$\dot{A}(t)\phi^* + A(t)\phi^{*\prime} - \lambda^* A(t)\, \delta F(w^*; \phi^*) \leq F(u) + \frac{\lambda^*}{2}\theta\, \delta^2 F(z;\psi). \tag{2.10}$$

Using ϕ^*-problem (2.4), equation (2.10) becomes

$$\dot{A}(t)\phi^* \leq F(u) + \frac{\lambda^*}{2}\theta\, \delta^2 F(z;\psi). \tag{2.11}$$

Due to the fact that u blows up as $t \to t^*-$ and relation (1.4), we have that there exists β such that $F(u) > \beta A(t)\phi^*(x) > 0$, for some $t \in [t^* - \varepsilon, T^*(A))$ where $\varepsilon > 0$ and $T^* = T^*(A) \geq t^*$ is the maximum time of existence (the blow-up time) of $A(t)$. Therefore it is enough to consider, for $t > \tau_1 = \max\{t_1,\ t^* - \varepsilon\}$,

$$\dot{A}(t)\phi^*(x) \leq \beta A(t)\phi^*(x) + \frac{\lambda^*}{2}\,\theta\, A^2(t)\, \delta^2 F(z;\phi^*). \tag{2.12}$$

For $0 < \theta \ll 1$ i.e. λ close to λ^*, we can find $\beta_1 > 0$, so that we get

$$\dot{A}(t)\phi^* \leq \beta_1 A(t)\phi^* \leq \beta A(t)\phi^* + \frac{\lambda^*}{2}\,\theta A^2(t)\, \delta^2 F(z;\phi^*), \quad t > \tau_1\,. \tag{2.13}$$

Taking c small enough so that $\theta \leq \frac{c}{A(t)}$, (for some fixed θ we choose c, and c_1, see below, so that $\theta \leq \frac{c}{A(t)}$ where c is about the time that u is smaller than order one i.e. $u(x,t)$ is bounded, $A(\tau)\phi^*\theta + w^* \leq u(x,t)$ and $0 < t^* - \tau \ll 1$) we have that $A(t) \leq c_1\, e^{\beta_1 t} \leq \frac{c}{\theta}$ with $c_1 = A(\tau_1)\, e^{-\beta_1 \tau_1}$ and this holds for time $t = \tau = \frac{1}{\beta_1}\ln(\frac{c}{\theta\, c_1})$.
Now we can obtain an upper estimate T^*_u for $t^*(u)$ which is $T^*_u = \tau + t^*_1 > t^*(u) = t^*$, where t^*_1 is the blow-up time of the problem:

$$u_t(x,t) + u_x(x,t) = \lambda\, \frac{f(u(x,t))}{\left(\int_0^1 f(u(x,t))\, dx\right)^2}, \qquad 0 < x < 1, \quad t > \tau,$$

$$u(0,t) = 0\,, \quad t > \tau, \qquad u(x,\tau) = w^* + c\phi^* \geq 0, \qquad 0 < x < 1,$$

and $t^*_1 \ll \tau$.

(b) A lower bound for t^*

We take $u_0(x)$ such that $u_0(x) < w^*(x)$ for $0 < x < 1$ and $u_0(0) = w^*(0) = 0$. Let $u^* = u^*(x,t) = u(x,t;\lambda^*)$ be the solution to (1.1) with $u^*_0 = u_0$.

In the following we use a similar concept to those in [16, 17]. Therefore we set $u = u^* + u_1$ and we shall prove that $u \leq u^* + \psi_1 = w^* - \hat{u} + \psi_1 \leq w^* - \psi + \psi_1$, where $\hat{u}$ is given by $\hat{u} = w^* - u^* > 0$ and satisfies (2.15), u_1 solves (2.24) (see below), ψ_1 is an upper solution to the u_1-problem and ψ is lower solution to $\hat{u}-$problem i.e. $\psi_1 \geq u_1$ and $\psi \leq \hat{u}$. The $\hat{u}$-problem is defined by

$$\begin{aligned} \hat{u}_t &= -u^*_t = u^*_x - \lambda^* F(u^*) - w^{*\prime} + \lambda^* F(w^*) \\ &= -\hat{u}_x - \lambda^*\left(F(u^*) - F(w^*)\right), \ \ 0 < x < 1,\ \ 0 < t < T, \end{aligned} \tag{2.15a}$$

$$\hat{u}(0,t) = w^*(0) - u^*(0,t) = 0, \;\; 0 < t < T, \tag{2.15b}$$

$$\hat{u}(x,0) = \hat{u}_0(x) = w^*(x) - u_0^*(x), \;\; 0 < x < 1, \tag{2.15c}$$

with $\hat{u}_0 > 0$, hence $\hat{u} > 0$, for $0 < x < 1, \;\; 0 < t < T < t^*$ and for some $T > 0$. We write $J(\epsilon) = F(w^* - \epsilon\hat{u}), \; 0 \le \epsilon \le 1$ and examine the difference,

$$F(u^*) - F(w^*) = J(1) - J(0) = J'(0) + \frac{J''(\xi)}{2}, \quad 0 < \xi < 1,$$

where $J'(0) = \delta F(w^*; -\hat{u}) = -\delta F(w^*; \hat{u}) = \lambda^* \frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2\lambda^* f(w^*)\, I_{11}(w^*,\hat{u})}{I_0^3(w^*)}$ and

$$J''(\xi) = \delta^2\, F(z; -\hat{u}) = \delta^2\, F(z; \hat{u}) = \frac{f''(z)\hat{u}^2}{I_0^2(z)} - \frac{4\hat{u}f'(z)I_{1\,1}(z,\hat{u})}{I_0^3(z)} - \frac{2f(z)I_{2\,2}(z,\hat{u})}{I_0^3(z)} + \frac{6f(z)I_{1\,1}^2(z,\hat{u})}{I_0^4(z)}, \tag{2.16}$$

with $0 < z = w^* - \xi\hat{u} < w^*$, for some $\xi = \xi(t) \in (0,1)$.

Thus equation (2.15a) and (2.16) give:

$$\mathrm{L}(\hat{u}) := \hat{u}_t + \hat{u}_x - \lambda^*\delta F(w^*; \hat{u}) + \frac{\lambda^*}{2}\delta^2\, F(z; \hat{u}) = 0. \tag{2.17}$$

Now we introduce the function $\psi = \psi(x,t) = \frac{c\phi^*(x)}{t+t_0} + \frac{u_2(x)}{(t+t_0)^2}$, where c, t_0 (positive constants), $u_2 = u_2(x) \ge 0$ are to be determined and $\phi^* = \phi^*(x)$ satisfies problem (2.4). For $u^* = w^* - \psi - r = w^* - \frac{c\phi^*(x)}{t+t_0} - \frac{u_2(x)}{(t+t_0)^2} - r \ge 0$, where $r = r(x,t) = \frac{u_3(x)}{(t+t_0)^3} + \frac{u_4(x)}{(t+t_0)^4} + ...$, ($r \in \mathbb{R}$), since $u^* \to w^*-$ or $(\psi + r) \to 0+$ as $t \to \infty$ $\;(u_0^*(x) < w^*(x))$, actually $u^* < w^*$ and $\psi + r > 0$ for all $t \ge 0$, [21]. Thus we have

$$u_t^* + u_x^* = \lambda^* F(u^*) \quad \text{or}$$

$$\frac{c\phi^*}{(t+t_0)^2} + w^{*\prime} - \frac{c\phi^{*\prime}}{t+t_0} - \frac{u_2'}{(t+t_0)^2} + ... = \lambda^* F(w^* - \frac{c\phi^*}{t+t_0} - \frac{u_2}{(t+t_0)^2} + ...) =$$

$$\lambda^* F(w^*) - \lambda^* \frac{c}{t+t_0}\, \delta F(w^*;\, \phi^*) - \frac{\lambda^*}{(t+t_0)^2}\left[\delta F(w^*;\, u_2) - \frac{c^2}{2}\, \delta^2 F(w^*;\, \phi^*)\right] + \ldots .$$

Equating terms of the same order with respect to powers of $\frac{1}{t+t_0}$ and taking into the account ϕ^*-problem, we have

$$(c\phi^* - u_2') = -\frac{\lambda^*}{2}c^2\delta^2 F(w^*;\, \phi^*) - \lambda^*\delta F(w^*;\, u_2). \tag{2.18}$$

We choose c so that

$$c\int_0^1 \phi^* dx = c^2\, \frac{\lambda^*}{2} | \int_0^1 \delta^2 F(w^*;\, \phi^*)\, dx\, |,$$

which implies that $c = 2/\lambda^*|\int_0^1 \delta^2 F(w^*;\, \phi^*)\, dx\,| > 0$, since $c = 0$ is rejected and provided that $\int_0^1 \phi^*\, dx = 1$, (it should be $c > 0$ otherwise we would have $u^* > w^*$ for $t \gg 1$).
Then we need to estimate u_2 by the inequality

$$u_2' \le \lambda^* \delta F(w^*;\, u_2) + c\phi^* + \frac{\lambda^*}{2} c^2 \delta^2 F(z;\, \phi^*), \tag{2.19}$$

where $0 < z = w^* - \xi \hat{u} < w^*$, for some $\xi = \xi(t) \in (0, 1)$.
Therefore, by taking $m_0 = \frac{\lambda^*}{2} c^2 \inf_x\{\delta^2 F(z;\, \phi^*)\}$ we have

$$u_2' \le \lambda^* \left[\frac{f'(w^*) u_2}{I_0^2(w^*)} - \frac{2 f(w^*) I_{11}(w^*, u_2)}{I_0^3(w^*)} \right] + c\phi^* + m_0. \tag{2.20}$$

Taking now $c_1 = \inf_x \left\{ \lambda^* \frac{f'(w^*)}{I_0^2(w^*)} \right\}$, $c_2 = \sup_x \left\{ -\frac{2f(w^*)}{I_0^3(w^*)} \right\}$ and $q = \int_0^1 f'(w^*) u_2\, dx = I_{11}(w^*, u_2) < 0$, we only need to estimate u_2 by the expression

$$u_2'(x) \le c_1 u_2(x) + c_2 q + c\phi^*(x) + m_0.$$

This implies that we need

$$u_2'(x) \le c_1 u_2(x) + c_2 q + m_1,$$

where $m_1 = c \inf_x \phi^*(x) + m_0 = m_0$, hence

$$u_2 \le \frac{c_2 q + m_1}{c_1} (e^{c_1 x} - 1)$$

for $u_2(0) = 0$, which is satisfied if

$$q = \frac{c_2 q + m_1}{c_1} \int_0^1 f'(w^*)(e^{c_1 x} - 1) dx.$$

Therefore $q = \frac{m_1 m_2}{c_1 - c_2 m_2}$ for $m_2 = \int_0^1 f'(w^*)(e^{c_1 x} - 1)\, dx$, which can be estimated.

Substituting now ψ, which is knowing, for $\hat{u}$ in (2.17) and taking into the account ϕ^*-problem and (2.19), we obtain (for the operator L see (2.17)),

$$\begin{aligned} \mathrm{L}(\psi) = \quad & \tfrac{c}{t+t_0} \quad (\phi^{*\prime} - \lambda^* \delta\, F(w^*; \phi^*)) + \frac{1}{(t+t_0)^2} (u_2' - \lambda^* \delta F(w^*;\, u_2) - \\ c\,\phi^* \quad & + \frac{\lambda^*}{2} c^2\, \delta^2 F(z;\, \phi^*) + O\left(\frac{1}{(t+t_0)^3} \right) \le 0. \end{aligned}$$

The last inequality holds since the term of order $\frac{1}{(t+t_0)^2}$ dominates the term of order $\frac{1}{(t+t_0)^3}$, this due to the u^*-problem, actually due to the fact that $u^* \to w^*-$ or $(\psi + r) \to 0+$ as $t \to \infty$ and that $\psi + r > 0$ for all $t > 0$.
Requiring $\psi(x, 0) \le \hat{u}_0$ and now knowing u_2, since $\hat{u}_0 = w^* - u_0^* = w^* - u_0$, we choose $t_0 = \sup_x \frac{c\,\phi^* + \sqrt{c^2\, \phi^{*2} + 4 \hat{u}_0\, u_2}}{2\hat{u}_0} = \sup_x \frac{E(x)}{2\hat{u}_0} < \infty$, in $[0, 1]$, provided that $u_0^*(x) = u_0(x) =$

$w^*(x) - \hat{u}_0(x)$ is chosen so that $E(x) = O(\hat{u}_0(x))$ as $x \to 0+$, without lost of generality we may choose $u_0 \to 0+$ properly as $x \to 0+$. Then we have that ψ is a lower solution to $\hat{u}$-problem and thus $\psi \le \hat{u}$.

We now write $u = u^* + u_1 \le w^*$ and find an upper solution to u_1-problem. The equation for u_1 is

$$\begin{aligned} u_{1t} &= -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda\left(F(u) - F(w^*)\right) - \\ &\quad \lambda^*\left(F(u^*) - F(w^*)\right), \quad 0 < x < 1, \quad t > 0. \end{aligned} \tag{2.21}$$

We again examine the difference $\lambda\left(F(u) - F(w^*)\right)$ and write $v = u - w^*$, $(-w^* < v < 0)$, $J_1(\epsilon) = F(w^* + \epsilon v)$, $0 \le \epsilon \le 1$, we have:

$$\begin{aligned} \lambda(F(u) - F(w^*)) &= \lambda(J_1(1) - J_1(0)) \\ &= \lambda\left(\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*;v)}{I_0^3(w^*)}\right) + \frac{\lambda}{2}J_1''(\xi_1) \\ &= \lambda^*\delta\, F(w^*;v) + (\lambda - \lambda^*)\,\delta\, F(w^*;v) + \frac{\lambda}{2}\,\delta^2\, F(z;v) \\ &= \lambda^*\,\delta\, F(w^*;v) + Q(w^*,z,v), \end{aligned} \tag{2.22}$$

where $Q(w^*,z,v) = (\lambda - \lambda^*)\delta\, F(w^*;v) + \frac{\lambda}{2}\,\delta^2\, F(z;v)$ and

$$\begin{aligned} J_1''(\xi_1) &= \delta^2\, F(z;v) = \frac{1}{I_0^4(z)}\left[I_0^2(z)\, v^2\, f''(z) - 4v\, f'(z)I_{11}(z,v)\, I_0(z)\right. \\ &- \left. 2f(z)I_0(z)\, I_{22}(z,v) + 6f(z)\, I_0(z)\, I_{11}^2(z,v)\right], \end{aligned}$$

$z = w^* + \xi_1 v$, $\xi_1 = \xi_1(t) \in (0,1)$.
Also by setting $u^* = w^* - \hat{u}$ and $J_2(\epsilon) = F(w^* - \epsilon\hat{u})$, $0 \le \epsilon \le 1$ the quantity $\lambda^*\left(F(u^*) - F(w^*)\right)$ is writen:

$$\begin{aligned} -\lambda^*\left(F(u^*) - F(w^*)\right) &= -\lambda^*(J_2(1) - J_2(0)) \\ &= \lambda^*\left(\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,\hat{u})}{I_0^3(w^*)}\right) - \frac{\lambda^*}{2}J_2''(\xi_2) \\ &= \lambda^*\,\delta\, F(w^*;\hat{u}) - \frac{\lambda^*}{2}\delta^2\, F(\zeta;\hat{u}), \end{aligned} \tag{2.23}$$

$$\begin{aligned} \text{with } J_2''(\xi_2) &= \delta^2\, F(\zeta;\hat{u}) = \frac{1}{I_0^4(\zeta)}\left[I_0^2(\zeta)\, \hat{u}^2\, f''(\zeta) - 4\hat{u}\, f'(\zeta)I_{11}(\zeta,\hat{u})\, I_0(\zeta)\right. \\ &- \left. 2f(\zeta)I_0(\zeta)\, I_{22}(\zeta,\hat{u}) + 6f(\zeta)\, I_0(\zeta)\, I_{11}^2(\zeta,\hat{u})\right], \end{aligned}$$

where $\zeta = w^* - \xi_2\hat{u}$, $\xi_2 = \xi_2(t) \in (0,1)$.
The u_1-problem (2.21), with relations (2.22), (2.23) now becomes

$$\begin{aligned} u_{1t} = \quad -u_{1x} &\quad +(\lambda - \lambda^*)F(w^*) + \lambda^*\delta\, F(w^*;v) + Q(w^*,z,v) \\ &+ \quad \lambda^*\delta\, F(w^*;\hat{u}) - \frac{\lambda^*}{2}\,\delta^2\, F(\zeta;\hat{u}), \quad 0 < x < 1, \quad t > 0, \end{aligned} \tag{2.24a}$$

$$u_1(0,t) = 0, \quad t > 0, \tag{2.24b}$$

$$u_1(x,0) = u_0(x) - u_0^*(x) = 0, \quad 0 < x < 1, \tag{2.24c}$$

where $0 < z < w^*$, $0 < \zeta < w^*$, $0 < \hat{u} < w^*$, $u < u_1 < w^*$ as far as $u < w^*$, so that $Q(w^*, z, v)$, $J_2''(\xi_2)$ are bounded from above and below.
Hence, for a fixed $\lambda > \lambda^*$, there exist some constants B_1, B_2 and $B > 0$, such that

$$Q(w^*, z, v) < B_1 < \frac{(\lambda - \lambda^*)}{2} B, \qquad -\frac{\lambda^*}{2} J_2''(\xi_2) < B_2 < \frac{(\lambda - \lambda^*)}{2} B$$

and $Q - \frac{\lambda^*}{2} J_2'' < (\lambda - \lambda^*)B$. From (2.21) - (2.24) we obtain :

$$\begin{aligned} u_{1t} \leq -u_{1x} &+ (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, v)}{I_0^3(w^*)} \right] \\ &+ \lambda^* \left[\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, \hat{u})}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B. \end{aligned} \tag{2.25}$$

Due to the fact that $u_1 = u - u^* = u - w^* + w^* - u^* = v + \hat{u}$, the previous relation becomes:

$$u_{1t} \leq -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)u_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, u_1)}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B.$$

Now we introduce $\psi_1(x,t) = [(\lambda - \lambda^*)\Lambda(t + t_0)]\phi^*(x)$, where Λ is a constant which is determined, so that ψ_1 to be an upper solution to u_1-problem. Here ϕ^* again satisfies problem (2.4).
By substituting $-\psi_1$ for u_1 in the right hand side of the above relation, we get

$$-\psi_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)\psi_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, \psi_1)}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B$$

$$= (\lambda - \lambda^*)F(w^*) + (\lambda - \lambda^*)B \leq (\lambda - \lambda^*)\, \Lambda\, \phi^* = \frac{\partial \psi_1}{\partial t},$$

$$F(w^*(x)) + B \leq \Lambda \phi^*(x), \quad \text{or} \quad \sup_x F(w^*(x)) + B \leq \Lambda \phi^*(x).$$

Since $\sup_x F(w^*(x)) = \frac{f(0)}{f^2(w^*(1))}$, it is enough to take,

$$\frac{f(0)}{f^2(w^*(1))} + B = \Gamma \leq \Lambda\, \phi^*(x) \quad \text{or} \quad 0 < \Gamma \leq \Lambda \inf_{x \in [\gamma,\, 1]} \phi^*(x) = \Lambda\, \Theta, \tag{2.26}$$

for some $\gamma \in (0,1)$. Also $\psi_1(x,0) = [(\lambda - \lambda^*)\Lambda t_0]\phi^*(x) > u_1(x,0) = 0$, and

$$\psi_1(\gamma,t) = [(\lambda - \lambda^*)\Lambda(t + t_0)]\phi^*(\gamma) > u_1(\gamma,t) \geq 0\,. \tag{2.27}$$

Relations (2.26), (2.27) leads us to choose $\Lambda = max\{\frac{w^*(\gamma)}{(\lambda - \lambda^*)\, t_0\, \phi^*(\gamma)}, \frac{\Gamma}{\Theta}\}$, for some $\gamma > 0$ (such a γ exists since $\phi^*(0) = 0$ and $\phi^*(x) > 0$ in $(0,1]$). Then, ψ_1 is a “restricted” upper solution for u_1-problem in the interval $[\gamma, 1]$, [18]. Here we have to notice that u blows up

globally, see relation (1.4b), this means that $u(x,t)$ is bounded for $(x,t) \in [0,\gamma] \times [0,T]$ for some $T < t^*$. In other words u is bounded in $[\gamma,1] \times [0,T]$, (actually here we require $u < w^*$, see below) and hence a lower bound for t^* can be found working in the restricted interval $[\gamma, 1]$.
Hence, $u \le w^*$ as far as $\psi - \psi_1 \ge 0$, thus we have

$$u = u_1 + u^* \le w^* - \frac{c\phi^*}{t+t_0} - \frac{u_2}{(t+t_0)^2} + (\lambda - \lambda^*)\Lambda\,(t+t_0)\,\phi^*.$$

The right-hand side of the above relation is no greater than w^*, as long as $\psi \ge \psi_1$ or

$$\frac{c\phi^*}{(t+t_0)} + \frac{u_2}{(t+t_0)^2} \ge (\lambda - \lambda^*)\,\Lambda\,\phi^*\,(t+t_0), \quad x \in [\gamma, 1].$$

Hence, $u \le w^*$ as far as $\psi - \psi_1 \ge 0$; it is enough $u \le w^*$ for any $x \in [\gamma, 1]$, for some $\gamma > 0$, due to the fact that u blows up globally.
Therefore, for simplicity, since $0 < \lambda - \lambda^* \ll 1$, it is enough to choose

$$\frac{c\phi^*}{t+t_0} \ge (\lambda - \lambda^*)\,\Lambda\,\phi^*\,(t+t_0), \quad x \in [\gamma, 1],$$

$$\text{so that} \qquad c \ge (\lambda - \lambda^*)\Lambda(t+t_0)^2.$$

Thus we get

$$t \le c^{1/2}\,\Lambda^{-1/2}\,(\lambda - \lambda^*)^{-1/2} - t_0,$$

which for λ sufficiently close to λ^* ($\lambda > \lambda^*$) gives

$$t \lesssim c^{1/2}\,\Lambda^{-1/2}\,(\lambda - \lambda^*)^{-1/2} = t_l(\lambda - \lambda^*)^{-1/2}.$$

Hence, as long as $u = u(x,t) < w^*(x)$, $x \in [\gamma, 1]$ at $t = t_l(\lambda - \lambda^*)^{-1/2}$, we deduce that $t^* > t_l(\lambda - \lambda^*)^{-1/2}$ and $t_l(\lambda - \lambda^*)^{-1/2}$ is a lower bound for t^* with $t_l = c^{1/2}\Lambda^{-1/2}$.

3. Asymptotic estimate of t^* for small $\lambda - \lambda^* > 0$

We now examine the special case $f(s) = e^{-s}$. Motivated by Section 2 we wish to find an estimate for the blow-up time t^* to problem (1.1) as an asymptotic series of $\eta = \theta^{1/2} = (\lambda - \lambda^*)^{1/2} \ll 1$, $\eta > 0$. We again assume that $u_0(x) < w^*(x)$ for $0 < x \le 1$, with $u_0(0) = w^*(0) = 0$.

Following similar concepts to [16, 17], as well as motivated from numerical calculations, we consider three intervals of time, say I, II and III.
In I and III t varies by $O(1)$ as $\theta = (\lambda - \lambda^*) \to 0$ and we expand $u \sim u^* + \theta v_1 + \theta^2 v_2 + ...$ as $\theta \to 0+$, where u^* satisfies problem (1.1) at $\lambda = \lambda^*$. More precisely, in I we have $u^* < w^*$ since $u_0 = u_0^* < w^*$ in $(0,1]$, while in III we have $u^* > w^*$. Moreover in I

$$u^*(x,t) \sim w^*(x) - \frac{2\phi^*(x)}{\lambda^*\,J_2\,t} \quad \text{as} \quad t \to \infty,$$

with $J_2 = \int_0^1 R_2(w^*, \phi^*)\, dx$ and $R_2(w^*, \phi^*) = \delta^2 F(w^*; \phi^*)$, where R_2 follows from the second term of Taylor's expansion, see in Section 2 or below relation (3.1). This can be obtained by assuming that $u^* \sim w^* + \frac{K}{t}\phi^*$ as $t \to \infty$, $u^* < w^*$ and substituting this expansion in the equation for u^*, we find that $K = -\frac{2}{\lambda^* J_2}$, provided that $\int_0^1 \phi^*(x)dx = 1$.

In III ($u^* > w^*$), we again expand u in a similar manner as in I, but now $u^*(x,t) \sim w^*(x) - \frac{2\phi^*(x)}{\lambda^* J_2 \hat{t}}$ as $\hat{t} \to -\infty$ and $t = t_* + \hat{t} > 0$ for some large t_* ($t_* \sim t^* < \infty$ for $\theta \to 0$) u^* becomes infinity at some finite time $\hat{t}_*$ as $\theta \to 0$.

In interval II, we expand $u \sim w^* + \eta v_1 + \eta^2 v_2 + ...$ as $\eta \to 0$, and on making a change in time scale $t = \tau/\eta$, equation (1.1) gives:

$$\eta^2 v_{1\tau} + \eta^3 v_{2\tau} + ... + w_x^* + \eta v_{1x} + \eta^2 v_{2x} + ... = \lambda R(\eta) \text{ as } \eta \to 0, \tag{3.1}$$

where

$$F(u) \sim \hat{R}(x,t;\eta) := R(\eta) = \frac{e^{-(w^*+\eta v_1+\eta^2 v_2+...)}}{\left(\int_0^1 e^{-(w^*+\eta v_1+\eta^2 v_2+...)}\right)^2 dx} \quad \text{as } \eta \to 0.$$

We require an expansion for $R(\eta)$ as follows

$$R(\eta) = R(0) + \eta R'(0) + \frac{\eta^2}{2} R''(0) + ... \,. \tag{3.2}$$

From (3.1), (3.2) we obtain

$$\begin{aligned} \eta^2 v_{1\tau} \quad &+ \quad \eta^3 v_{2\tau} + ... + w_x^* + \eta v_{1x} + \eta^2 v_{2x} + ... \\ &= \quad (\lambda^* + \eta^2)\left(R(0) + \eta R'(0) + \frac{\eta^2}{2} R''(0) + ...\right). \end{aligned}$$

As regards the boundary condition $u(0,t) = 0$ at $x = 0$, we have

$$w^*(0) + \eta v_1(0,\tau) + \eta^2 v_2(0,\tau) + ... = 0.$$

We equate the terms of zero order ($O(1)$ or $O(\eta^0)$) and get

$$w^{*\prime} = \lambda^* R(0), \quad 0 < x < 1, \;\; w^*(0) = 0, \tag{3.3}$$

where $R(0) = e^{-w^*} / \left(\int_0^1 e^{-w^*}\, dx\right)^2$. Problem (3.3) is actually problem (1.3). By looking now at the terms of $O(\eta)$ we have

$$v_{1x}(x,\tau) = \lambda^* R'(0), \quad 0 < x < 1, \;\; \tau > 0, \qquad v_1(0,\tau) = 0, \quad \tau > 0\,, \tag{3.4}$$

where $R'(0) = \delta\, F(w^*; v_1) = -\frac{e^{-w^*} v_1}{(\int_0^1 e^{-w^*}\, dx)^2} + \frac{2e^{-w^*} \int_0^1 e^{-w^*} v_1\, dx}{(\int_0^1 e^{-w^*}\, dx)^3}$.

Problem (3.4) has the form of problem (2.4), thus we can write

$$v_1(x,\tau) = a(\tau)\phi^*(x), \tag{3.5}$$

where now we normalize ϕ^* according to $\int_0^1 \phi^{*2}(x)\,dx = 1$ and we denote the normalized ϕ^* again by ϕ^*. Looking next at the $O(\eta^2)$ terms we have

$$v_{1\tau} + v_{2x} = R(0) + \frac{\lambda^*}{2} R''(0),$$

which becomes

$$\begin{aligned} v_{1\tau} + v_{2x} &= \frac{e^{-w^*}}{S_0^2(\phi^*)} + \frac{2\lambda^* e^{-w^*} v_1\, S_1(v_2)}{S_0^3(\phi^*)} + \frac{2\lambda^* e^{-w^*} S_1(v_2)}{S_0^3(\phi^*)} \\ &- \frac{\lambda^* e^{-w^*} S_2(v_2)}{S_0^3(\phi^*)} + \frac{3\lambda^* e^{-w^*} S_1^2(v_2)}{S_0^4(\phi^*)}, \end{aligned} \tag{3.6}$$

where now we denote by $I_{\nu\, k}(w^*, v) = (-1)^\nu S_k(v)$ with $S_k(v) = \int_0^1 e^{-w^*} v^k dx, \quad k = 0, 1, 2, 3$ and $S_k(\phi^*) = S_k$.

Multiplying (3.6) by ϕ^*, integrating over $[0, 1]$, using (3.5) and the normalization of ϕ^*, we obtain

$$\begin{aligned} \dot{a}(\tau) = \int_0^1 \left(-\phi^{*\prime} - \frac{\lambda^* e^{-w^*} \phi^*}{S_0^2} + \frac{2\lambda^* e^{-w^*} S_1}{S_0^3} \right) v_2\, dx + \frac{S_1}{S_0^2} + \\ \lambda^* \frac{a^2(\tau)}{2S_0^4} \left(S_3\, S_0^2 - 6\, S_1\, S_2\, S_0 + 6\, S_1^3 \right). \end{aligned} \tag{3.7}$$

Now we look at the equation for ϕ^* which is

$$\phi^{*\prime} = -\frac{\lambda^* e^{-w^*} \phi^*}{S_0^2} + \frac{2\lambda^* e^{-w^*} S_1}{S_0^3}, \;\; 0 < x < 1, \;\; \phi^*(0) = 0, \tag{3.8}$$

and integrate over $[0, 1]$, to obtain $\phi^*(1) = \frac{\lambda^* S_1}{S_0^2}$. Moreover, multiplying (3.6) by ϕ^* and then integrating over $[0, 1]$, we obtain

$$\frac{\phi^{*2}(1) S_0^3}{2\lambda^*} = -S_2 S_0 + 2S_1^2. \tag{3.9}$$

In the same way, but multiplying with ϕ^{*2}, we get

$$\frac{\phi^{*3}(1) S_0^3}{3\lambda^*} = -S_3 S_0 + 2S_2 S_1. \tag{3.10}$$

Now for the quantity $S = S_3 S_0^2 - 6 S_2 S_1 S_0 + 6 S_1^3$, after using $\phi^*(1)$ and relations (3.9), (3.10) we obtain

$$S = S_1^3 \left(-2 + 2\frac{\lambda^*}{S_0} - \frac{\lambda^{*2}}{3 S_0^2} \right). \tag{3.11}$$

The quantity S is positive as long as the quantity $S_q = -6 S_0^2 + 6\lambda^* S_0 - \lambda^{*2}$ is positive. We know the relations $\lambda^* = \frac{M^2}{\mu}$, where $M = max\{w^*(x)\} = w^*(1)$, for $0 \le x \le 1$ and $\mu = \frac{\lambda^*}{S_0^2}$. Using these relations we get that S_q is positive provided

$$S_q M^2 = {\lambda^*}^2(-6+6M-M^2)$$

is positive. We have (see [21]) that $M \sim 1.5936$ and $\lambda^* \sim 0.6476$. The above quantity, for these values, is positive ($S_q M^2 \sim 0.4286$).

Therefore the equation (3.7) can be written as

$$\dot{a}(\tau) = \frac{S_1}{S_0^2} + \lambda^* \frac{S}{2S_0^4}\, a^2(\tau), \quad a(\tau) \to -\infty \text{ as } \tau \to 0, \tag{3.12}$$

which has as a solution

$$a(\tau) = (\frac{B}{K})^{\frac{1}{2}} \tan\left[\tau (BK)^{\frac{1}{2}} - \frac{\pi}{2}\right],$$

for $K = \lambda^* S/2S_0^4$, $B = S_1/S_0^2$ (this choice of initial condition as $\tau \to 0+$ gives constant of integration $-\pi/2$). Returning to the original time variable this expression becomes

$$A(t) = (\frac{B}{K})^{\frac{1}{2}} \tan\left[t(\lambda-\lambda^*)^{1/2}(BK)^{\frac{1}{2}} - \frac{\pi}{2}\right].$$

Because $u = w^* + \eta v_1 + \ldots$ and $v_1(x,t) = A(t)\phi^*(x)$, it is obvious that u ceases to exist at time

$$t^* \sim t_b = t_u(\lambda-\lambda^*)^{-\frac{1}{2}}$$

where $t_u = \pi(4BK)^{-1/2}$ and t_b is the blow-up time of $a(\tau) = a(t(\lambda-\lambda^*)^{-1/2}) = A(t)$.

4. Numerical Solutions

We solve problem (1.1) by using a two-step up-wind scheme. For the linear terms we apply the usual form of the scheme:

$$v_j^{n+1} = u_j^n - r\left(u_j^n - u_{j-1}^n\right) + \lambda F(u_j^n),$$

where u_j^n is the temperature at the nth time level and at the jth space grid, $r = \frac{\delta t}{\delta x}$ and the non-local term $F(u_j^n)$ is evaluated at the nth time step. For this term we have

$$F(u_j^n) = \frac{f(u_j^n)}{\left(\int_0^1 f(u_j^n)\, dx\right)^2}.$$

The integral in the denominator is evaluated by Simpson's rule. In the next step we evaluate w

$$w_j^{n+1} = u_j^n - r\left(v_j^{n+1} - v_{j-1}^{n+1}\right) + \lambda F(v_j^{n+1}).$$

Finally u at the $(n+1)$th time step is approximated by

$$u_j^{n+1} = \frac{1}{2}(v_j^{n+1} + w_j^{n+1}).$$

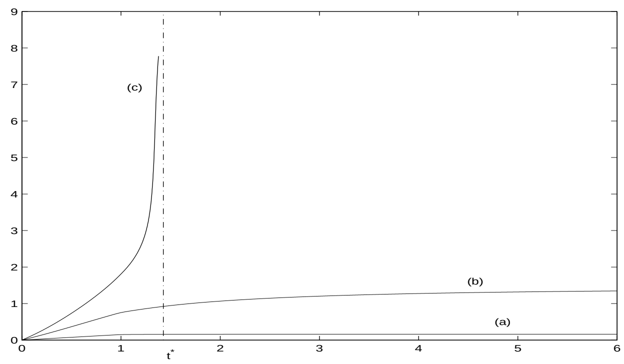

Figure 2. *Numerical solution to problem* (1.1)*. We plot the* $\max_x(u(x,t)) = u(1,t) = M(t)$*, for* x *in* $[0,1]$ *against time for* $\delta x = 0.033$*,* $\delta t = 0.002$ *(the upper curve, (c), corresponds to* $\lambda = 1.1476 > \lambda^* = 0.6476$*, the intermediate, (b), to* $\lambda = \lambda^*$ *and the lower one, (a), to* $\lambda = 0.1476 < \lambda^*$*). Also the dash-dotted axis corresponds to the asymptotic estimate of the blow up time* $t^* \sim 1.3367$ *for* $\lambda = 1.1476$*. To obtain this estimate we calculate numerically* w^* *by an iteration scheme and then we solve the equation for* ϕ^* *using the appropriate normalization.*

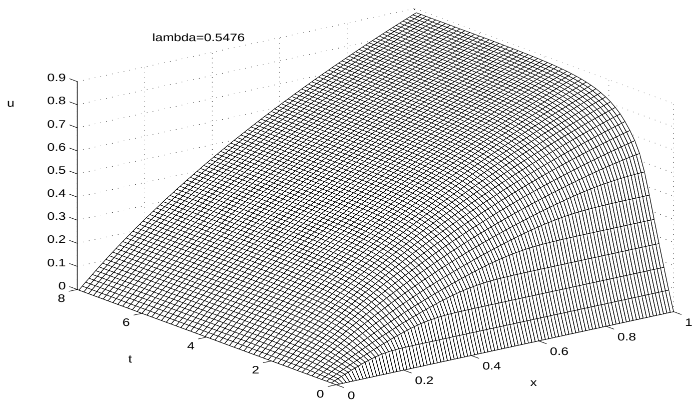

Figure 3. *Numerical solution to problem* (1.1) *for* $\lambda = 0.5476 < \lambda^*$.

In Figure 2 we use this scheme to solve the problem numerically for $f(u) = e^{-u}$ and taking $u(x,0) = 0$. We see that for $\lambda < \lambda^*$ the solution u tends to a steady state, for $\lambda = \lambda^*$ the behaviour is similar and for $\lambda > \lambda^*$ the solution blows up (the decay is faster for $\lambda < \lambda^*$ than it is for $\lambda = \lambda^*$). More precisely, in Figure 2 the maximum of solutions are plotted against time.

In Figure 3, we plot the numerical solution of u for $\lambda = 0.5476$.

5. Upper bounds of t^* for sufficiently large $u_0(x)$ and $\lambda \le \lambda^*$

In this Section we study the case of $\lambda \le \lambda^*$, where the steady-state problem (1.3) has two (bounded) solutions $w_1(x)$, $w_2(x)$ with $w_2(x) > w_1(x)$ ($w_1 = w_2 = w^*$ at $\lambda = \lambda^*$) for $x \in (0,1]$, see Figure 1(c). Following similar concepts as in the case of the upper bound for t^* when $\lambda > \lambda^*$, we may proceed by two ways: either (i) as in [16] or (ii) as in Section 2.

(i) First method: Again we set $v(x,t) = u(x,t) - w_2(x)$, $0 \le x \le 1$, $t > 0$ with $v_0(x) = u_0(x) - w_2(x) > 0$, $0 < x \le 1$ and $v_0(0) = u_0(0) - w_2(0) = 0$. The function $w_2(x)$ is the greatest steady-state solution to problem (1.3) while $u(x,t)$ is the unique solution to problem (1.1) which blows-up globally in finite time and (1.4) holds. Therefore $v(x,t) \to \infty$ and $F(u) = F(v + w_2) \to \infty$ as $t \to t^*-$.
Also $F(u) - F(w_2) = F(v + w_2) - F(w_2) = \delta F(w_2; v) + \frac{1}{2}\,\delta^2 F(s; v)$.
Then v satisfies the problem

$$v_t + v_x = \lambda\,\delta F(w_2; v) + \frac{\lambda}{2}\,\delta^2 F(s; v), \;\; 0 < x < 1, \;\; t > 0, \tag{5.1a}$$

$$v(0,t) = 0, \quad t > 0, \tag{5.1b}$$

$$v_0 = v(x,0) = u(x,0) - w_2(x) \ge 0, \quad 0 \le x \le 1, \tag{5.1c}$$

where $v_0 = 0$ only at $x = 0$, $s = s(x,t) = w_2 + \xi v$, $\xi = \xi(t) \in (0,1)$, $F(w_2 + \xi v) := L(\xi)$ and $\delta^2\, F(s, v) := L''(\xi)$. Moreover $\delta^2 F(s; v) \to \infty$ as $t \to t^*-$, otherwise $v(x,t)$ would not blow-up (in finite time).
Now we introduce the function Ψ for which we require to satisfy,

$$\Psi_t + \Psi_x \le \lambda\,\delta F(w_2; \Psi) + \frac{\lambda}{2}\,\delta^2 F(z; \Psi) = \lambda(F(w_2 + \Psi) - F(w_2)), \tag{5.2}$$

where $\Psi(x,t) = A(t)\,\phi(x)$, $\phi(x) = \phi(x;\rho)$, $z = w_2 + \epsilon\Psi$, $\epsilon = \epsilon(t) \in (0,1)$, $J(\epsilon) = F(w_2 + \epsilon\Psi)$, $J''(\epsilon) = \delta^2 F(z; \Psi)$, $\phi(x)$ satisfies problem (2.2) and $A(t)$ satisfies problem (5.7) (see below). We require the function Ψ to be a lower solution to problem (5.1) which also blows up, due to the blow-up of $A(t)$ at T^*; since $\Psi \le v$, we have $T^* = T^*(A) \ge t^*(v) = t^*(u) = t^*$, actually this blow-up is global because of the form of Ψ.
Setting $\Psi(x,t) = A(t)\,\phi(x)$ in relation (5.2) we obtain

$$\dot{A}(t)\phi + A(t)\phi' - \lambda\,\delta F(w_2; \phi) A(t) \le \frac{\lambda}{2}\,\delta^2 F(z; \Psi). \tag{5.3}$$

Also, by the same argument as for v, we have that $J''(\epsilon) = \delta^2 F(z; \Psi) \to \infty$, $\Psi \to \infty$ as $t \to T^*-$, otherwise $\Psi(x,t)$ would not blow-up. Then for $J''(\epsilon)$ we have,

$$\begin{aligned} J''(\epsilon) &= \Psi^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z) I_{1\,1}(z,\Psi)}{\Psi\, I_0^3(z)} - \frac{2f(z) I_{2\,2}(z,\Psi)}{\Psi^2\, I_0^3(z)} + \frac{6f(z) I_{1\,1}^2(z,\Psi)}{\Psi^2\, I_0^4(z)} \right] \\ &= \Psi^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z) I_{1\,1}(z,\phi)\, A}{A\,\phi\, I_0^3(z)} - \frac{2f(z) I_{2\,2}(z,\phi)\, A^2}{A^2\,\phi^2\, I_0^3(z)} + \right. \\ &\qquad \left. \frac{6f(z) I_{1\,1}^2(z,\phi)\, A^2}{A^2\,\phi^2\, I_0^4(z)} \right] = \Psi^2 \Gamma(x,t) = A^2(t)\,\phi(x)\,\Gamma_1(x,t), \end{aligned} \tag{5.4}$$

where $\Gamma_1(x,t) = \Gamma(x,t)\,\phi(x)$. Since Ψ blows up globally, we have that $F(w_2+\Psi) - F(w_2) \sim F(w_2+\Psi) \sim F(\Psi) \to \infty$ as $t \to T^*-$ for every $x \in (0,1]$.
Now we impose an extra condition on the function f,

$$f(s) \le \frac{c}{s^2}, \quad c>0, \quad \text{for} \quad s \gg 1, \tag{5.5}$$

and have the following lemma.

Lemma 3. Let f satisfy (1.2) and (5.5), then $J''(\epsilon) = \delta^2 F(z;\Psi) \ge \Lambda\, A^2(t)\phi(x) > 0$ for $\Psi \gg 1$, or $A \gg 1$, where Λ is a positive constant.

Proof: Since Ψ blows up globally, $\delta^2\, F(z;\Psi) \to \infty$ as $\Psi \to \infty$. Let that the conclusion of the lemma do not hold, then due to (5.3) we would have $\delta^2 F(z;\Psi) \lesssim K_0\, A^p(t)\,\phi(x)$ as $A(t) \to \infty$ or as $t \to T^*-$, for some $1 < p < 2$ and $K_0 > 0$, otherwise Ψ does not blow-up. The later, the fact that $F(w_2+\Psi) - F(w_2) \sim F(\Psi)$, for $\Psi \gg 1$ and (5.4) give,

$$\begin{aligned} F(w_2+\Psi) - F(w_2) \quad &= \quad \delta F(w_2;\Psi) + \frac{1}{2}\delta^2 F(z;\Psi) = \delta F(w_2,\phi)A(t) + \\ &\frac{1}{2}\Gamma(x,t)\,\Psi^2 \sim F(\Psi) \lesssim K_0\, A^p\, \phi, \quad \text{for} \quad A \gg 1 \text{ and } \ 1<p<2\,. \end{aligned}$$

Also by using the mean value theorem we get, $F(\Psi) = \frac{f(\Psi(x,t))}{(\int_0^1 f(\Psi(x,t)\,dx)^2} = \frac{f(A(t)\phi(x))}{f^2(A(t)\phi(\xi))}$, where $\xi = \xi(t) \in (0,1)$. Then, due to the continuity of $\phi(x)$, we can always find $x_0 = x_0(t) \in (0,1)$ such that $0 < \phi(x_0(t)) < \min\{\phi(\xi(t)),\ \phi^2(\xi(t))\}$. This implies $\frac{f(A(t)\phi(x_0(t)))}{f(A(t)\phi(\xi(t)))} > 1$, and as $t \to T^*-$ we have,

$$F(\Psi(x_0,t)) = \frac{f(A(t)\phi(x_0))}{f^2(A(t)\phi(\xi))} > \frac{1}{f(A(t)\phi(\xi))}, \qquad \text{and}$$

$$\frac{1}{f(A(t)\phi(\xi))} < \frac{f(A(t)\phi(x_0))}{f^2(A(t)\phi(\xi))} = F(\Psi(x_0,t)) \lesssim K_0\, A^p(t)\,\phi(x_0).$$

Then, by using and condition (5.5), we get

$$K_0\, A^p(t)\,\phi(x_0(t)) \gtrsim \frac{1}{f(A(t)\phi(\xi(t)))} \ge \frac{A^2(t)\phi^2(\xi(t))}{c} \quad \text{as} \quad t \to T^*.$$

The latter implies $K_0\, c \gtrsim A^{2-p}(t)\,\frac{\phi^2(\xi(t))}{\phi(x_0(t))} > A^{2-p}(t)$ for t close to T^*, but $A^{2-p}(t) \to \infty$ as $t \to T^*$ for $1<p<2$ while $K_0\, c < \infty$, which is a contradiction. □

Due to Lemma 3, relation (5.4) now becomes,

$$J''(\epsilon) = \Gamma(x,t)\Psi^2 \ge \Lambda\, A^2(t)\,\phi(x) \;\text{ as } \quad t \to T^*-, \quad x \in (0,\ 1]. \tag{5.6}$$

From relation (5.6), it is enough to take

$$\dot{A}(t)\phi + A(t)\,[\phi' - \lambda\delta F(w_2;\phi)] \le K_1 A^2(t)\phi, \;\; t>0, \;\; A(0) = A_0,$$

where $K_1 = \frac{\lambda}{2}\,\Lambda$. Now, by using problem (2.2) and Lemmas 2, 3 we obtain:

$$\dot{A}(t) - A(t)\rho_2 \le K_1\, A^2(t), \quad t>0, \;\; A(0) = A_0, \;\; \rho_2 \ge 0.$$

Thus, we can take $A(t)$ to satisfy,

$$\dot{A}(t) = \rho A(t) + KA^2(t), \quad t \in [0, T^*), \;\; A(0) = A_0, \tag{5.7}$$

with $K \leq K_1$ and $\rho = \rho_2 \geq 0$.

Also we take $\psi(x, 0) = A(0)\phi(x) \leq v(x, 0) = u_0(x) - w_2(x)$ or $A_0 = A(0) \leq \frac{u_0(x) - w_2(x)}{\phi(x)}$, hence we choose $A_0 = \inf_x \frac{u_0(x) - w_2(x)}{\phi(x)} = \inf_x \frac{E(x)}{\phi(x)} > 0$, (this infimum exists provided that $u_0(x)$, $u_0'(x)$ are bounded in $[0, 1]$). Moreover $\Psi(0,\, t) = v(0,\, t)$. Finally Ψ is a lower solution to v-problem (5.1).

Now problem (5.7) for $A(t)$ and $\rho > 0$ gives:

$$A(t) = \left[-\frac{K}{\rho} + (A_0^{-1} + \frac{K}{\rho}) e^{-\rho t} \right]^{-1},$$

where K is a constant depending on c and λ, while A_0 depends on the infimum of the difference $(u_0(x) - w_2(x))/\phi(x)$.

Then the blow-up time T^* for $A(t)$ is

$$T^* = \frac{1}{\rho} \ln \left(\frac{\rho}{K} A_0^{-1} + 1 \right), \quad \text{for} \;\; \rho > 0.$$

While for $\rho = 0$, problem (5.7) gives, $T^* = (KA_0)^{-1}$. Since Ψ is a lower solution of v-problem (5.1), we get

$$t^* = t^*(u) = t^*(v) \leq t^*(\Psi) = T^*,$$

hence T^* is an upper bound for $t^*(u)$.

(ii) Second method: This method works only if $\rho = \rho_2 > 0$ or if $\rho = \rho_2 = 0$ and $\delta^2 F(z; \phi) > 0$. Let now $u(x, t) = u(x, t; u_0)$ be the solution to problem (1.1) for $\lambda < \lambda^*$ and assume, for simplicity, initial data of the form $u_0(x) = \sigma w_2(x)$, with $\sigma > 1$, then $u_0(x) - w_2(x) = (\sigma - 1) w_2(x) = \zeta w_2(x)$, with $\zeta = \sigma - 1 > 0$. Also we have that $u \to w_2+$ as $\zeta \to 0+$. Again we set $v(x, t) = u(x, t) - w_2(x)$, $0 \leq x \leq 1$, $t > 0$ with $v_0(x) = u_0(x) - w_2(x) > 0$, $0 < x \leq 1$ and $v_0(0) = u_0(0) - w_2(0) = 0$. We call $v = u - w_2 = \zeta \hat{v}$, thus problem (5.1) becomes

$$\zeta \hat{v}_t + \zeta \hat{v}_x = \lambda \, \zeta \, \delta F(w_2; \hat{v}) + \frac{\lambda}{2} J''(\xi), \;\; 0 < x < 1, \;\; t > 0, \tag{5.8a}$$

$$\hat{v}(0, t) = 0, \quad t > 0, \tag{5.8b}$$

$$\hat{v}_0(x) = \hat{v}(x, 0) = \frac{u_0(x) - w_2(x)}{\zeta} > 0, \;\; 0 < x < 1. \tag{5.8c}$$

This is simplified to

$$\hat{v}_t + \hat{v}_x = \lambda \, \delta F(w_2; \hat{v}) + \frac{\lambda}{2} \, \zeta \, \hat{J}''(\xi), \;\; 0 < x < 1, \;\; t > 0,$$

where $J(\xi)=\zeta^2\,\hat{J}(\xi)=\zeta^2\,\delta^2F(z;\hat{v})$. Now we find a lower solution ψ for the $\hat{v}$-problem. Therefore we require $\psi=\psi(x,t)$ to satisfy

$$\psi_t+\psi_x\le\lambda\,\delta F(w_2;\psi)+\frac{\lambda}{2}\,\zeta\,\delta^2F(z;\psi). \tag{5.9}$$

Setting $\psi=A(t)\phi(x)$ and $\dot{A}(t)=\frac{d}{dt}A(t)$ we obtain

$$\dot{A}(t)\phi+A(t)(\phi'-\lambda\,\delta F(w_2;\phi))\le\frac{\lambda}{2}\,\zeta\,\delta^2F(z;\psi). \tag{5.10}$$

Using ϕ-problem (2.2) and Lemma 2, equation (5.10) becomes

$$\dot{A}(t)\phi-\rho_2\,\phi\,A(t)\le\frac{\lambda}{2}\,\zeta\,\delta^2F(z;\psi),\quad \rho_2\ge 0. \tag{5.11}$$

Thus we have, on setting $\rho_2=\rho$,

$$\dot{A}(t)\phi(x)\le\rho A(t)\phi(x)+\frac{\lambda}{2}\,\zeta\,A^2(t)\,\delta^2F(z;\phi),\quad 0<x<1\quad t>0. \tag{5.12}$$

For $0<\zeta\ll 1$, in both cases, either $\rho=\rho_2>0$, or $\rho=\rho_2=0$ and $\delta^2F(z;\phi)>0$, we can find $\beta>0$, so that we get (for some τ_1 close to T^* and $\tau_1<T^*$)

$$\dot{A}(t)\phi\le\beta A(t)\phi\le\rho A(t)\phi+\frac{\lambda}{2}\,\zeta A^2(t)\,\delta^2F(z;\phi),\quad t>\tau_1>0\,.$$

Taking now c small enough so that $\zeta\le\frac{c}{A(t)}$, (c is about the time that u is smaller than order one) we have that $A(t)\le c_1\,e^{\beta t}\le\frac{c}{\zeta}$ with $c_1=A(\tau_1)\,e^{-\beta\tau_1}$ and this holds for time $t=\tau=\frac{1}{\beta}\ln(\frac{c}{\zeta\,c_1})$. Now we can obtain an upper estimate T_u^* for $t^*(u)$ which is $T_u^*=\tau+t_1^*>t^*=t^*(u)$, where t_1^* is the blow-up time of the problem:

$$u_t(x,t)+u_x(x,t)=\lambda\,\frac{f(u(x,t))}{\left(\int_0^1 f(u(x,t))\,dx\right)^2},\qquad 0<x<1,\quad t>\tau,$$

$$u(0,t)=0\,,\quad t>\tau,\qquad u(x,\tau)=w_2(x)+c\,\phi(x)\ge 0,\qquad 0<x<1,$$

and $t_1^*\ll\tau$.

6. Discussion

In the present work, we estimate the blow-up time t^* of the solution to problem (1.1). It is useful, from the point of view of applications, to know the time where the temperature u becomes infinity. In our model, this is the time that the food is burnt. Similar estimates are also known for local (the reaction diffusion problem) as well as for non-local (the Ohmic heating problem) problems, [16, 17]. Here the results are obtained for the case where there exists a steady-state solution $w^*=w(x;\lambda^*)$ at $\lambda=\lambda^*$, either for $0<\lambda-\lambda^*\ll 1$, with nonnegative initial data, or for $0<\lambda\le\lambda^*$ and initial data greater than the greatest steady-state solution. The methods applied, are comparison and asymptotic techniques, as well as numerical computations. Although the asymptotic estimate of t^* is found for $f(s) = e^s$, it is possible to be estimated for general function f satisfying (1.2).

Our main estimates, for given λ, λ^* and $0 < \lambda - \lambda^* \ll 1$, are: upper bound $\epsilon + c_1 \ln[c_2(\lambda - \lambda^*)^{-1}]$; lower bound $c_3(\lambda - \lambda^*)^{-1/2}$; asymptotic estimate $t^* \sim c_4(\lambda - \lambda^*)^{-1/2}$ as $\lambda \to \lambda^*+$. For $0 < \lambda \le \lambda^*$ and given initial data $u_0(x)$ greater than the greatest steady-state solution $w_2(x)$, we have the following upper estimates: either $c_5 \ln(c_6 A_0^{-1} + 1)$ or $\epsilon + c_7 \ln(c_8 \zeta^{-1})$, where A_0, ζ are measures of the difference $u_0 - w_2$, (if $u_0 \to w_2+$ then $A_0,\ \zeta \to 0+$), c_i are constants and $0 < \epsilon \ll 1$. Some numerical results are also represented in Section 4.

It still remains an open question how to estimate t^* when there is no regular solution w^* at $\lambda = \lambda^*$.

Acknowledgment. The authors would like to thank Professor A. A. Lacey for several fruitful discussions. Also we would like to thank Dr. A. Delis for fruitful discussion regarding the numerical scheme.

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