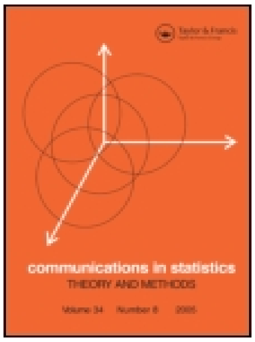

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A semi-Markov decision model for the optimal control of a simple immigration-birth-death process through the introduction of a predator

E. G. Kyriakidis[a] and T. D. Dimitrakos[b]

[a]Department of Statistics, Athens University of Economics and Business, Athens, Greece; [b]Department of Mathematics, University of the Aegean, Karlovassi, Samos, Greece

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ABSTRACT
This article is concerned with the problem of controlling a simple immigration-birth-death process, which represents a pest population, by the introduction of a predator in the habitat of the pests. The optimization criterion is the minimization of the expected long-run average cost per unit time. It is possible to construct an appropriate semi-Markov decision model with a finite set of states if and only if the difference between the per capita birth rate and the per capita death rate of the pests is smaller than half of the rate at which the predator is introduced in the habitat.

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Minimum average cost; Semi-Markov decision process; Simple immigration-birth-death process.

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1. Introduction

The Markov decision and semi-Markov decision models are flexible and powerful tools for solving probabilistic sequential decision problems with an infinite planning horizon. The relevant theory combines concepts from the theory of Markov and semi-Markov models and from the dynamic programming method. The aim is the determination of the optimal policy, i.e., the optimal rule for choosing decisions as the process evolves over time. The most widely used optimization criteria are the minimization of the expected total discounted cost and the minimization of the expected long-run average cost per unit time. The expected total discounted cost of a policy π is defined as the expected total cost during an infinite-time horizon if the costs are discounted at a rate $\alpha \in (0, 1)$ per unit time given that the policy π is employed. The expected long-run average cost per unit time of a policy π is defined as the limit as $n \to \infty$ of the expected cost incurred until the n-th decision epoch divided by n, given that the policy π is employed. We refer to the books of Ross (1983), Puterman (1994), Bather (2000), Tijms (2003), and Hu and Yue (2008), where the theory of Markov and semi-Markov decision processes is presented together with many applications from various areas. Most of the applications deal with queueing systems, manufacturing systems, inventory models, maintenance problems, replacement problems, and problems for controlling biological populations.

CONTACT E. G. Kyriakidis ✉ ekyriak@aueb.gr Department of Statistics, Athens University of Economics and Business, 76 Patission Street, 10434 Athens, Greece.
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Various deterministic and stochastic models for the optimal control of damaging biological populations have been developed during the last years. A recent article, which also gives references to earlier work, is that of Pathak and Maiti (2012). Individuals whose presence is damaging are referred as pests. Insects which destroy a crop or spread a disease can be considered as pests. Pests may be also bacteria or diseased cells which multiply in a living organism. The amount of damage done by the pests at any time generally depends on the initial number of pests in the habitat, their birth and death rates, and their migration rates to and from the habitat. The damage done by the pests can be usually represented by a cost which increases as their population size increases. For that reason one is faced with the problem of controlling the growth of the pest population. Spraying the crop or the insects with some insecticide, and treating an organism with drugs or X-rays may be appropriate controlling actions for the population growth in the above cases. Another possibility could be the introduction in the habitat of a predator that captures the pests. A controlling action gives rise to costs due to labor, materials, risk etc. and it is therefore necessary to balance the cost caused by the pests against the cost of the controlling action. We present below a pest control problem that we will formulate as a semi-Markov decision model under the criterion of the minimization of the average cost.

Consider a population of pests which grows stochastically in a habitat according to a simple immigration-birth-death process with immigration rate $\nu > 0$ and per capita birth and death rates λ and μ, respectively ($\nu > 0, \lambda \geq 0, \mu \geq 0$). We assume that the deaths of the pests are caused by a predator that lives permanently in their habitat and captures the pests with rate proportional to their population size. We refer to this predator as the internal predator. We define the unit of cost as the cost per unit time of the damage caused by each pest and we assume that the damage caused by the pests during one unit of time is proportional to their population size. It follows that the cost of the damage caused by the pests is i per unit time, if i is their current population size. Note that a great number of deterministic and stochastic population models for predator-prey interactions can be found in the literature (see, e.g., Hassell, 1978; Renshaw, 1991).

The pest population may be controlled by some action that introduces an external predator in the habitat of the pests after some random time that is exponentially distributed with mean ρ^{-1}. The presence of the external predator in the habitat prevents the immigrations and the births of the pests. The external predator dominates the internal predator in the sense that the internal predator does not capture the pests as long as the external predator is in the habitat. The external predator captures the pests one at a time with constant rate $\sigma > 0$ until their population size is annihilated and, then leaves the habitat after some random time that is exponentially distributed with mean θ^{-1}. As soon as the external predator emigrates from the habitat the internal predator becomes again active causing the deaths of the pests with per capita rate μ.

The cost of taking controlling action that introduces the external predator in the habitat of the pests is equal to $k > 0$ per unit of time. The decision epochs include the epochs at which an immigration or a birth or a death of a pest occurs and the epochs at which the predator emigrates. Let i and i' denote the states of the process at which the population size of the pests is $i (i \geq 0)$ and the predator is absent from their habitat or present, respectively. A stationary policy f is defined by a sequence $\{f_i\}$, $i \geq 0$, where f_i is the action chosen when the process is at state i. It is assumed that $f_i = 1$ when the controlling action, which introduces the external predator in the habitat, is being taken and $f_i = 0$ when the controlling action is not being taken. If the stationary policy $f = \{f_i\}$, $i \geq 0$, is used, our assumptions imply that we have a continuous-time Markov chain model for the population growth of the pests with state space

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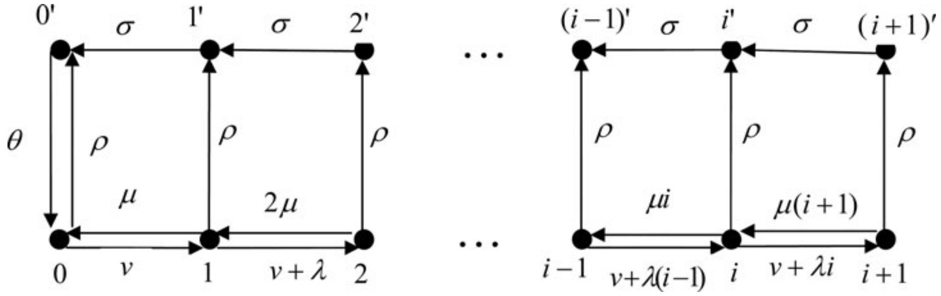

Figure 1. The transition rates of the process under P_0.

$\{0, 0', 1, 1', \ldots\}$ and the following transitions in a small time interval $(t, t + \delta t)$:

$$
\begin{array}{lll}
i \to i+1 & \text{with probability} & (\nu + \lambda i)\delta t + o(\delta t),\ i \geq 0, \\
i \to i-1 & \text{with probability} & \mu i \delta t + o(\delta t),\ i \geq 1, \\
i' \to (i-1)' & \text{with probability} & \sigma \delta t + o(\delta t),\ i \geq 1, \\
i \to i' & \text{with probability} & f_i \rho \delta t + o(\delta t),\ i \geq 0, \\
0' \to 0 & \text{with probability} & \theta \delta t + o(\delta t).
\end{array}
$$

Our objective is to find a stationary policy which minimizes the expected long-run average cost per unit time among all stationary policies. Let P_x, $x = 0, 1, \ldots$ be the stationary policy that takes the controlling action that introduces the external predator in the habitat if and only if the population size of the pests is greater or equal to x. The policy P_x is called control-limit policy. It seems intuitively reasonable to suppose that the optimal policy is of control-limit type. The transition rates of the process under the policy P_0 are depicted in Figure 1.

In Kyriakidis (1995b) we studied the same problem when $\lambda = \mu = 0$. In that work, first, an expression for the average cost g_x under the policy P_x, $x = 0, 1, \ldots$ was derived using a regenerative argument, then the optimal policy P_{x^*} among all control-limit policies P_x, $x = 0, 1, \ldots$ was found analytically by minimizing g_x with respect to $x = 0, 1, \ldots$ and finally the optimality of P_{x^*} among all stationary policies was established by applying Bather's (1976) general results. Specifically, it was shown that (i) the policy P_{x^*} satisfied the average-cost optimality equations and (ii) some extra conditions on the cost and transition rates were valid. In Kyriakidis (2003), we generalized the cost structure of the problem studied in Kyriakidis (1995b) by assuming that the cost rate caused by the pests is an increasing function of their population size. In that work the existence of an optimal control-limit policy was established by following the Federgruen-So technique (see Federgruen and So, 1989, 1990, 1991; So, 1992) which is based on the variation of a fictitious parameter over the entire real line. Specifically, we introduced a fictitious cost r incurred each unit of time the process is occupying state $0'$. It was shown that an optimal control-limit policy exists when r takes small values. This assertion was then extended inductively from interval to interval of the values of the parameter r.

In the present problem the process under the control-limit policy P_x, $x = 0, 1, \ldots$ is a regenerative process, where the successive entries into state x can be considered as regenerative epochs between successive cycles. From a well known regenerative argument (see Proposition 5.8 in Ross, 1992) the average cost under the policy P_x is equal to the expected cost during a cycle divided by the expected time of the cycle. Note that, if the initial state is $x \in \{0, 1, \ldots\}$, the process under P_x may move from the set of states $\{x, x + 1, \ldots\}$, in which the controlling action is exerted, into the state $x - 1$, in which the controlling action is not exerted, before the external predator is introduced in the habitat of the pests. Hence, if the initial state is x, the time until the external predator is introduced in the habitat is not exponentially distributed with mean ρ^{-1}, as in the case in which $\mu = 0$. Therefore, the regenerative argument used in Kyriakidis (1995b) for the determination of the average cost under the policy P_x cannot be applied in the present problem. If $\mu = 0$ and $0 < \lambda < \rho/2$ it is possible to apply the regenerative argument and to derive an expression for the average cost under the policy P_x, $x = 0, 1, \ldots$ with respect to x. However, this expression is too complicated and it is impossible to minimize it with respect to x. Consequently, even in this case it does not seem possible to prove the optimality of a control-limit policy by applying Bather's (1976) theory, as it was achieved in Kyriakidis (1995b). Furthermore, if we introduce a fictitious cost incurred each unit of time the process is in state $0'$, it seems difficult to apply the Federgruen-So method as in the problem that was studied in Kyriakidis (2003). This difficulty is due to the fact that the one-step expected time, if the external predator is not in the habitat of the pests, tends to 0 as the pest population size tends to infinity.

From the above remarks it becomes clear that it is difficult to prove the structure of the optimal policy for the present problem. However a computational treatment of the problem is possible. In Sec. 2 a suitable semi-Markov decision model with a finite set of decision epochs is constructed provided that the birth rate, the death rate and the rate at which the external predator is introduced satisfy a certain condition. In Sec. 3, numerical results obtained by the value iteration algorithm are presented providing strong evidence that the optimal policy is of control-limit type. Note that an analogous numerical treatment was presented in Kyriakidis (1995a) for the problem of controlling a simple immigration-birth-death process through total catastrophes. In Sec. 4, a summary of the results of the article is given.

Our model can also have the following alternative interpretation. Suppose that computer viruses are installed in a computer system according to a Poisson process with rate $\nu > 0$. The cost rate due to the damages caused by the viruses is equal to i when the number of viruses is equal to i. Each virus reproduces itself with rate $\lambda > 0$. An antivirus software 1 is installed permanently in the system and removes the viruses with rate proportional to the number of viruses. The removal rate per virus is equal to $\mu > 0$. The controller may install in the system the antivirus software 2 and the installation lasts a random period of time that is exponentially distributed with mean ρ^{-1}. The cost rate of this action is equal to $k > 0$. The antivirus software 2: (i) stops the arrivals and the reproduction of the viruses, (ii) prevents the operation of software 1, and (iii) removes the viruses one at a time with rate $\sigma > 0$ until their annihilation. Then it is uninstalled after an exponentially distributed period of time with mean θ^{-1}. The objective is to find the average-cost optimal policy and the decision epochs include the epochs at which the viruses are installed in the system, the epochs at which the viruses are reproduced, the epochs at which the viruses are removed by antivirus software 1 and the epochs at which antivirus software 2 is uninstalled. Note that deterministic and stochastic immigration-birth-death processes for the spread of computer viruses in a computer system have been considered in the literature (see, e.g., Jones and White, 1990; Wierman and Marchette, 2004; Amador and Artalejo, 2013).

2. Formulation as a semi-Markov decision model

In the present problem the set of states of the process at the decision epochs is the set $\{0, 1, \ldots\}$. This set is infinite and therefore, a direct application of the standard semi-Markov decision algorithms (i.e. the policy iteration algorithm, the value iteration algorithm, the linear programming algorithm (see Tijms, 2003, Ch. 7)) is not possible. This difficulty could be circumvented by eliminating the effect of immigrations and deaths if the population size

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of the pests exceeds a sufficiently large level. This modification would lead to a very large finite state space enabling us to implement the algorithms. However, such a truncation of the state space would change essentially the original model. It is preferable to obtain a more efficient semi-Markov decision formulation for our problem by restricting the set of admissible stationary policies rather than truncating the state space. Similar approaches were followed in a queueing problem in Tijms (2003, p. 291) and in the pest control problem studied in Kyriakidis (1995a).

We restrict our attention only to the stationary policies that always take the controlling action that introduces the external predator in the habitat when the pest population size is greater than or equal to M, where M is a sufficiently large integer. This restriction is intuitively reasonable and it is consistent to Theorem 3 in Kyriakidis (1995a) that states that if $\lambda = \mu = 0$ the optimal policy is of control-limit type. It also enables us to obtain a semi-Markov decision model for our problem with finite state space $S = \{0, \ldots, M\}$ and action sets $A(i) = \{0, 1\}$, $0 \le i \le M-1$, and $A(M) = \{1\}$. The standard semi-Markov decision algorithms can be implemented directly under this formulation. The semi-Markov decision model is determined by the one-step transition probabilities, the expected one-step transition times and the expected one-step transition costs. These quantities are needed for the implementation of the algorithms and are defined as follows. Let $p_{ij}(a)$ be the probability that at next decision epoch the process will be in state $j \in S$ if action $a \in \{0, 1\}$ is chosen in the present state $i \in S$, and let $T(i, a)$ and $C(i, a)$ be the corresponding expected time and expected cost, respectively. We point out that in the above definition if $i = M$ the action $a = 1$ is the only one that it is admissible. The non-zero, one-step transition probabilities, one-step expected transition times and one-step expected costs that correspond to the states $0, \ldots, M-1$ are given below:

- *non-zero one-step transition probabilities*

$$\begin{aligned}
p_{i,i+1}(a) &= \frac{\nu + \lambda i}{\nu + (\lambda + \mu)i + a\rho}, \, 0 \le i \le M-1, \, a \in \{0, 1\}, \\
p_{00}(1) &= \frac{\rho}{\nu + \rho}, \\
p_{i,i-1}(a) &= \frac{\mu i}{\nu + (\lambda + \mu)i + a\rho}, \, 2 \le i \le M-1, \, a \in \{0, 1\}, \\
p_{i0}(1) &= \frac{\rho}{\nu + (\lambda + \mu)i + \rho}, \, 2 \le i \le M-1, \\
p_{10}(a) &= \frac{\mu + a\rho}{\nu + \lambda + \mu + a\rho}, \, a \in \{0, 1\},
\end{aligned}$$

- *one-step expected transition times*

$$\begin{aligned}
T(i, 0) &= \frac{1}{\nu + (\lambda + \mu)i}, \, 0 \le i \le M-1, \\
T(i, 1) &= \frac{\sigma\rho + (\sigma + \rho i)\theta}{\sigma\theta[\nu + (\lambda + \mu)i + \rho]}, \, 0 \le i \le M-1,
\end{aligned}$$

- *one-step expected costs*

$$\begin{aligned}
C(i, 0) &= \frac{i}{\nu + (\lambda + \mu)i}, \, 0 \le i \le M-1, \\
C(i, 1) &= \frac{2\sigma(i + k) + \rho i(i + 1)}{2\sigma[\nu + (\lambda + \mu)i + \rho]}, \, 0 \le i \le M-1.
\end{aligned}$$

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The only non-zero probabilities $p_{Mj}(1), 0 \leq j \leq M$, are the probabilities $p_{M,M-1}(1)$, $p_{MM}(1)$ and $p_{M0}(1)$. Note that

$$\begin{aligned} p_{M,M-1}(1) &= \frac{\mu M}{\nu + (\lambda + \mu)M + \rho}, \\ p_{MM}(1) &= \frac{\nu + \lambda M}{\nu + (\lambda + \mu)M + \rho} p_{M+1}, \\ p_{M0}(1) &= 1 - p_{M,M-1}(1) - p_{MM}(1), \end{aligned} \tag{1}$$

where p_{M+1} in (1) is the probability that the process returns to M before the introduction of the external predator in the habitat occurs, given that the initial state is $M + 1$ and the controlling action is being taken. The probability p_{M+1} is the limit of $p^{(n)}_{M+1}$, as $n \to \infty$, where $p^{(n)}_{M+1}$, $n = 0, 1, \ldots$, is the probability that the process returns to state M in at most n steps before the introduction of the external predator in the habitat, given that the initial state is i and the controlling action is being taken. The probabilities $p^{(n)}_{M+1}$, $n = 0, 1, \ldots$ can be found through the following recursive scheme:

$$\begin{aligned} p^{(0)}_i &= 0, i \geq M + 1, \\ p^{(n)}_M &= 0, n \geq 0, \\ p^{(r)}_i &= \frac{(\nu + \lambda i) p^{(r-1)}_{i+1} + \mu i p^{(r-1)}_{i-1}}{\nu + (\lambda + \mu)i + \rho}, \end{aligned}$$

where, $i = M + n - r + 1$ for each $r \in \{1, \ldots, n\}$. The probability p_{M+1} can be approximated by $p^{(n_1)}_{M+1}$, where n_1 is a positive number such that $(p^{(n_1)}_{M+1} - p^{(n_1-1)}_{M+1})/p^{(n_1)}_{M+1} < \varepsilon$ and ε is a prespecified tolerance number.

Conditioning on the outcome of the next transition from state M, we obtain

$$T(M, 1) = \frac{\sigma\rho + (\sigma + \rho M)\theta}{\sigma\theta[\nu + (\lambda + \mu)M + \rho]} + \frac{\nu + \lambda M}{\nu + (\lambda + \mu)M + \rho} T_{M+1}$$

and

$$C(M, 1) = \frac{2\sigma(M + k) + \rho M(M + 1)}{2\sigma[\nu + (\lambda + \mu)M + \rho]} + \frac{\nu + \lambda M}{\nu + (\lambda + \mu)M + \rho} C_{M+1},$$

where T_{M+1} and C_{M+1} are the expected time and cost, respectively, until the process enters either state 0 or state M given that the initial state is $M + 1$ and the controlling action is being taken. The result of the following lemma is needed for the proof of Proposition 2.1.

Lemma 2.1.

(i) If $\lambda - \mu \geq \rho$ *then* $T_{M+1} = \infty$.

(ii) If $\lambda - \mu \geq \rho/2$ *then* $C_{M+1} = \infty$.

Proof. Let $\tilde{T}_{M+1}$ and $\tilde{C}_{M+1}$ be the expected time and the expected cost, respectively, until entry either into state 0 or state M, if the initial state is $M + 1$ and the controlling action that introduces the external predator in the habitat is being taken, in a modified process for the growth of the pest population in which the effect of immigrations is eliminated and the state M is absorbing. From the definitions of T_{M+1}, $\tilde{T}_{M+1}$, C_{M+1}, and $\tilde{C}_{M+1}$ we have that

$$T_{M+1} \geq \tilde{T}_{M+1} \tag{2}$$

and

$$C_{M+1} \geq \tilde{C}_{M+1}. \tag{3}$$

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Let $p_i(t)$, $i \geq M$, be the probability that the pest population size is i at time t for the modified (uncontrolled) process, given that the initial size is $M+1$. The probability that the modified (uncontrolled) process is absorbed ultimately in state M is μ/λ. This result is a consequence of the inequality $\lambda > \mu$ and can be shown by considering the corresponding embedded random walk as in the computation of the probability of ultimate absorption into state 0 in a simple birth-death process (Taylor and Karlin, 1994, pp. 351–352)). From the definition of $p_M(t)$ we have that

$$p_M(t) \leq \frac{\mu}{\lambda}. \tag{4}$$

The forward equations (see, e.g., Taylor and Karlin, 1994, Ch. 6) for the probabilities $p_i(t)$, $i \geq M$, are:

$$\begin{aligned}
\frac{dp_i(t)}{dt} &= -(\lambda+\mu)ip_i(t) + \mu(i+1)p_{i+1}(t) + \lambda(i-1)p_{i-1}(t),\, i \geq M+2,\\
\frac{dp_{M+1}}{dt} &= -(\lambda+\mu)(M+1)p_{M+1}(t) + \mu(M+2)p_{M+2}(t),\\
\frac{dp_M(t)}{dt} &= \mu(M+1)p_{M+1}(t).
\end{aligned}$$

Let $D(t) = \sum_{i=M}^{\infty} ip_i(t)$.

Multiplying the forward equations by i, $i \geq M$, and summing over i we obtain a differential equation for $D(t)$. After solving it and making use of (4) we obtain the inequality below. For details we refer to Kyriakidis (1995a, pp. 351–352):

$$D(t) \geq (M+1-\mu M/\lambda)e^{(\lambda-\mu)t} + Mp_M(t). \tag{5}$$

Let $t^*(\leq t)$ be the time until the modified (uncontrolled) process returns to state M, given that the initial state is $M+1$ and the state at time t is M. Assume that the controlling action is taken even when the process enters the absorbing state M. The expected time until entry into state 0 is equal to $t + i\sigma^{-1} + \theta^{-1}$ if the predator is introduced in the habitat at time t and the number of pests at time t is $i \geq M+1$, while the expected time until entry in state M is equal to $E(t^*)$ if the predator is introduced in the habitat at time t and the number of pests at time t is M. Conditioning on the time until the introduction of the external predator, which is exponentially distributed with mean ρ^{-1}, we have that

$$\tilde{T}_{M+1} = \int_0^{\infty} \left[\sum_{i=M+1}^{\infty} (t + i\sigma^{-1} + \theta^{-1})p_i(t) + E(t^*)p_M(t) \right] \rho e^{-\rho t}\,dt.$$

From the above equation we deduce that

$$\tilde{T}_{M+1} > \sigma^{-1} \int_0^{\infty} \left[\sum_{i=M+1}^{\infty} ip_i(t) \right] \rho e^{-\rho t}\,dt.$$

Part (i) of the lemma is an immediate consequence of (2), (5) and the above inequality. We define

$$\begin{aligned}
F(t) &= \sum_{i=M}^{\infty} i^2 p_i(t),\\
G(t) &= F(t) - M^2 p_M(t).
\end{aligned} \tag{6}$$

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Note that

$$F(0) = (M+1)^2. \tag{7}$$

Multiplying the forward equations by $i^2 (i \geq M)$ and summing over i we have that

$$\sum_{i=M}^{\infty} i^2 \frac{dp_i(t)}{dt} = -(\lambda+\mu) \sum_{i=M+1}^{\infty} i^3 p_i(t) + \mu \sum_{i=M}^{\infty} (i+1) i^2 p_{i+1}(t) + \lambda \sum_{i=M+2}^{\infty} (i-1) i^2 p_{i-1}(t)$$

$$= -(\lambda+\mu) \sum_{i=M+1}^{\infty} i^3 p_i(t) + \mu \sum_{i=M+1}^{\infty} i(i-1)^2 p_i(t) + \lambda \sum_{i=M+1}^{\infty} i(i+1)^2 p_i(t).$$

Using the definition of $F(t)$ the above equation can be written as

$$\frac{dF(t)}{dt} = 2(\lambda-\mu)[F(t) - M^2 p_M(t)] + (\lambda+\mu) \sum_{i=M+1}^{\infty} i p_i(t).$$

Making use of (6) and (7) we obtain

$$\frac{dF(t)}{dt} \geq 2(\lambda-\mu)F(t) - \frac{2(\lambda-\mu)M^2\mu}{\lambda} + (\lambda+\mu)\left(M+1-\frac{\mu M}{\lambda}\right) e^{(\lambda-\mu)t}.$$

Multiplying the above inequality by $e^{-2(\lambda-\mu)t}$, integrating between 0 and t and using initial condition (7) we obtain:

$$F(t) \geq H(t) + \frac{\mu M^2}{\lambda},$$

where

$$H(t) = \frac{[(\lambda-\mu)^2 M^2 + (3\lambda+\mu)(\lambda-\mu)M + 2\lambda^2] e^{2(\lambda-\mu)t} - (\lambda+\mu)[(\lambda-\mu)M+\lambda] e^{(\lambda-\mu)t}}{\lambda(\lambda-\mu)}.$$

From (4), (6), and the above inequality we have

$$G(t) \geq H(t). \tag{8}$$

Let $m(s|i,t)$ be the mean pest population size at time $s(\leq t)$ for the modified (uncontrolled) process, given that the initial size is $M+1$ and the size at time t is $i(i \geq M)$. Assume that the controlling action that introduces the external predator in the habitat is exerted even when the process enters the absorbing state M. Conditioning on the time until the introduction of the predator, which is exponentially distributed with mean ρ^{-1}, and taking into account the cost structure of the problem we have

$$\tilde{C}_{M+1} = \int_0^{\infty} \left[\sum_{i=M+1}^{\infty} [\int_0^t m(s|i,t) ds + \frac{i(i+1)}{2\sigma}] p_i(t) + p_M(t) \int_0^{t^*} m(s|M,t) ds \right] \rho e^{-\rho t} dt$$
$$+ k\tilde{T}_{M+1}.$$

From the above equation we deduce that

$$\tilde{C}_{M+1} > (2\sigma)^{-1} \int_0^{\infty} \left[\sum_{i=M+1}^{\infty} i^2 p_i(t) \right] \rho e^{-\rho t} dt.$$

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Part (ii) of lemma is an immediate consequence of (3), (8), and the above inequality.

Proposition 2.1.
(i) T_{M+1} *is finite if and only if* $\lambda - \mu < \rho$.
(ii) C_{M+1} *is finite if and only if* $\lambda - \mu < \rho/2$.

Proof. If $T_{M+1} < \infty$ then the relation $\lambda - \mu < \rho$ follows from Part (i) of Lemma 2.1. If $C_{M+1} < \infty$ then the relation $\lambda - \mu < \rho/2$ follows from Part (ii) of Lemma 2.1.

Let T'_{M+1} and C'_{M+1} be the expected time and expected cost, respectively, until entry into state 0 through state $0'$, given that the initial state is $M+1$ and the controlling action is being taken even if the process returns to M before the introduction of the predator. T'_{M+1} and C'_{M+1} can be found by conditioning on the time until the introduction of the predator, which is exponentially distributed with mean ρ^{-1}. Thus,

$$\begin{aligned} T_{M+1} \le T'_{M+1} &= 1 + \int_0^\infty \left[\sum_{i=0}^{\infty} (i\sigma^{-1} + \theta^{-1}) p'_i(t)\right] \rho e^{-\rho t} dt \\ &= 1 + \theta^{-1} + \sigma^{-1} \int_0^\infty m(t) \rho e^{-\rho t} dt \qquad (9) \end{aligned}$$

and

$$\begin{aligned} C_{M+1} \le C'_{M+1} &= k + \int_0^\infty \left[\int_0^t m(s) ds\right] \rho e^{-\rho t} dt + \int_0^\infty \left[\sum_{i=0}^{\infty} i(i+1)(2\sigma)^{-1} p'_i(t)\right] \rho e^{-\rho t} dt \\ &= k + [1 + (2\sigma)^{-1}] \int_0^\infty m(s) \rho e^{-\rho t} dt + (2\sigma)^{-1} \int_0^\infty m_2(t) \rho e^{-\rho t} dt, \qquad (10) \end{aligned}$$

where $p'_i(t)$, $i \ge 0$, is the probability that the population size is i at time t for a simple (uncontrolled) immigration-birth-death process with initial state $M+1$ and, $m(t)$ and $m_2(t)$ are the corresponding mean and second moment. The quantities $m(t)$ and $m_2(t)$ can be represented as $c_1 e^{(\lambda-\mu)t} + c_0$ and $c_2 e^{2(\lambda-\mu)t} + c_1 e^{(\lambda-\mu)t} + c_0$, respectively (see Bailey, 1964, p.99). Therefore, (9) implies that T_{M+1} is finite if $\lambda - \mu < \rho$ and (10) implies that C_{M+1} is finite if $\lambda - \mu < \rho/2$.

From the above proposition it follows that a semi-Markov decision formulation of the problem with finite state space $S = \{0, \dots, M\}$ is possible if and only if $\lambda - \mu < \rho/2$. The quantities T_{M+1} and C_{M+1} can be approximated as follows. Let $T_i^{(n)}$ and $C_i^{(n)}$, $n = 0, 1, \dots$, $i \ge M+1$, denote the expectations of $\min\{t_i, t_i^{(n)}\}$ and $\min\{c_i, c_i^{(n)}\}$, respectively, where t_i and c_i are the time and cost, respectively, until the process reaches either state 0 or state M given that the controlling action is being taken and the initial state is i, and $t_i^{(n)}$ and $c_i^{(n)}$ are the time and cost, respectively, until the n-th transition under the policy P_M given that the initial state is i. The quantities T_{M+1} and C_{M+1} are the limits as $n \to \infty$ of the sequences $T_{M+1}^{(n)}$, $n = 0, 1, \dots$ and $C_{M+1}^{(n)}$, $n = 0, 1, \dots$ that are defined by the following recursive schemes:

$$T_i^{(0)} = C_i^{(0)} = 0, \; i \ge M+1,$$

$$T_M^{(n)} = C_M^{(n)} = 0, \; n \ge 0,$$

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$$T_i^{(r)} = \frac{\sigma\rho + (\sigma + \rho i)\theta}{\sigma\theta[\nu + (\lambda + \mu)i + \rho]} + \frac{(\nu + \lambda i)T_{i+1}^{(r-1)} + \mu i T_{i-1}^{(r-1)}}{\nu + (\lambda + \mu)i + \rho},$$

$$C_i^{(r)} = \frac{2\sigma(i + k) + \rho i(i + 1)}{2\sigma[\nu + (\lambda + \mu)i + \rho]} + \frac{(\nu + \lambda i)C_{i+1}^{(r-1)} + \mu i C_{i-1}^{(r-1)}}{\nu + (\lambda + \mu)i + \rho},$$

where $i = M + n - r + 1$ for each $r \in \{1, \ldots, n\}$. Hence, T_{M+1} and C_{M+1} can be approximated by $T_{M+1}^{(n_2)}$ and $C_{M+1}^{(n_3)}$ where n_2 and n_3 are positive numbers such that $(T_{M+1}^{(n_2)} - T_{M+1}^{(n_2-1)})/T_{M+1}^{(n_2)} < \varepsilon$ and $(C_{M+1}^{(n_3)} - C_{M+1}^{(n_3-1)})/C_{M+1}^{(n_3)} < \varepsilon$, where ε is a prespecified tolerance number.

Remark 2.1. If $\lambda \geq \mu$ the uncontrolled process (i.e., the process under the policy that never introduces the external predator in the habitat of the pests) does not have an equilibrium distribution and consequently, in view of Lemma 2.2.2 in Bather (1976), the expected long-run average cost per unit time is infinite for every initial state. If $\lambda < \mu$, the process that is never controlled has an equilibrium distribution. Given the cost structure of the problem we can see that the average cost in this case is equal to the mean of the equilibrium distribution, i.e., $\nu/(\mu - \lambda)$ (see Bailey, 1964, p.99). It seems difficult to find a condition that guarantees the optimality of the policy that never initiates the controlling action in this case. In all numerical examples that we have tested with $\mu > \lambda$ the optimal policy was of control-limit type with average cost smaller than $\nu/(\mu - \lambda)$.

Remark 2.2. Suppose that we consider a modified problem in which (i) the cost rate caused by i pests is c_i, $i \geq 1$, where $\{c_i\}$ is non decreasing and $c_i \leq Ai$, $i \geq 1$, for some positive number A, (ii) the external predator captures the pests at a rate σi, when the pest population size is i, and (iii) the deaths of the pests with per capita rate μ are not necessarily caused by an internal predator and continue to occur even when the external predator is in their habitat. Assumptions (ii) and (iii) imply that the per capita death rate of the pests is equal to $\mu + \sigma$ when the external predator is present in their habitat. If $\lambda - \mu < \rho$, it is possible to construct a finite-state semi-Markov decision model for this problem in a similar way as in the original problem. However, it seems difficult to prove that the above condition is a necessary condition for the construction of a finite-state semi Markov decision model.

3. Numerical results

In this section, we present numerical results obtained by the value-iteration algorithm (see Tijms, 2003, pp. 285–286). In all examples tested, the optimal stationary policy was found to be of control-limit type. We choose $M = 50$ as the truncation point and the numbers 3, 0.4, 0.3, 2, 6 as the initial values of the parameters ν, λ, μ, ρ, k, respectively. Note that the condition $\lambda - \mu < \rho/2$ holds and, therefore, a semi-Markov decision formulation with state space $S = \{0, \ldots, M\}$ is possible. We also choose $\varepsilon = 10^{-3}$ as the tolerance number for the computation of $p_{M+1}^{(n_1)}$, $T_{M+1}^{(n_2)}$, $C_{M+1}^{(n_3)}$, and as the accuracy number for the stopping criterion of the value-iteration algorithm. In Tables 1–3, we present the critical number x^*, that corresponds to the optimal control-limit policy P_{x^*}, the minimum average cost and the required number of iterations. In Table 1 the effect of varying σ is studied, in Table 2 the effect of varying θ is studied and in Table 3 the effect of varying both σ and θ is studied.

It can be seen from Tables 1 and 2 that the optimal critical level x^* increases as each of the parameters σ or θ increases. This can be explained intuitively since as σ or θ increases,

Table 1. The effect of varying σ for $\theta = 10$.

σ	x^*	minimum average cost	number of iterations
0.1	1	2.9473	139943
0.5	1	3.1416	28118
1	1	3.3273	14063
5	2	3.7374	2932
10	2	3.8560	1544
30	3	3.9270	625
50	3	3.9378	442
80	3	3.9442	339
100	3	3.9463	306
500	3	3.9534	203

Table 2. The effect of varying θ for $\sigma = 10$.

θ	x^*	minimum average cost	number of iterations
0.1	0	0.3985	5023
0.5	0	1.5981	2222
1	1	2.4159	1864
5	2	3.6387	1580
10	2	3.8560	1544
30	3	4.0097	1519
50	3	4.0359	1515
80	3	4.0509	1512
100	3	4.0559	1512
500	3	4.0720	1509

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the required expected time for the extermination of the pest population decreases and, consequently, it is reasonable to initiate the controlling action that introduces the predator in the habitat only if the population size exceeds higher values. It can also be seen from Tables 1 and 2 that the minimum average cost increases as each of the parameters σ or θ increases. This seems to be reasonable since (i) the presence of the predator in the habitat of the pests prevents their immigrations and births and (ii) as σ or θ increases the time period, during which the predator is in the habitat, decreases. From Table 3 we see that if $(\sigma, \theta) = (500, 500)$ then x^* and the minimum average cost coincide with those obtained in the problem studied in Kyriakidis (1995a). This is reasonable since, when $\sigma \to \infty$ and $\theta \to \infty$ the introduction of the external predator in the habitat of the pests is equivalent to a total catastrophe that annihilates their population size.

Table 3. The effect of varying both σ and θ.

σ	θ	x^*	minimum average cost	number of iterations
0.1	0.1	0	1.8800	150118
0.5	0.5	0	2.2832	29566
1	1	1	2.6688	14671
5	5	2	3.5640	2973
10	20	2	3.9747	1526
30	20	3	4.0401	610
50	50	3	4.1298	419
80	80	3	4.1603	315
100	100	3	4.1709	280
500	500	3	4.2058	174

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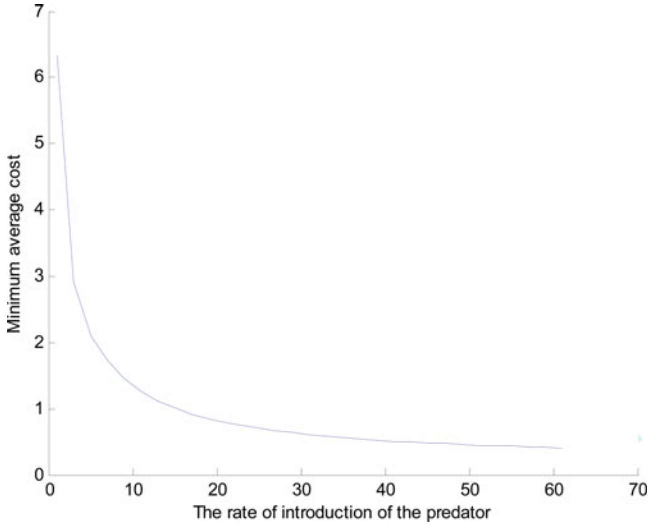

Figure 2. The minimum average cost as parameter ρ varies.

In Figure 2, for $\sigma = \theta = 10$, we present a graph that shows the variation of the minimum average cost as the rate ρ of the introduction of the external predator in the habitat of the pests varies in the set $\{1, \ldots, 60\}$. We see that, as ρ increases, the minimum average cost decreases. This can be explained, since, as ρ increases, the time period until the introduction of the external predator in the habitat of the pests decreases. When ρ takes values in the set $\{1, \ldots, 20\}$ the minimum average cost decreases rapidly. When ρ takes values in the set $\{21, \ldots, 60\}$ the minimum average cost decreases slowly. In Figure 3 below, for $\sigma = \theta = 10$ and $\rho = 5$, we present a graph that shows the variation of the minimum average cost as the cost rate k of taking the controlling action that introduces the predator in the habitat of the pests varies in the set $\{1, \ldots, 20\}$. We see that, as k increases, the minimum average cost increases rather linearly. In Figure 4 below, for $\sigma = \theta = 5$, $\rho = 3$, and $k = 20$, we present a graph that shows the variation of the critical number x^* that corresponds to the optimal control-limit policy P_{x^*}, as the per capita birth rate λ of the pests takes values in the set $\{0.1, 0.2, \ldots, 4.1\}$. We

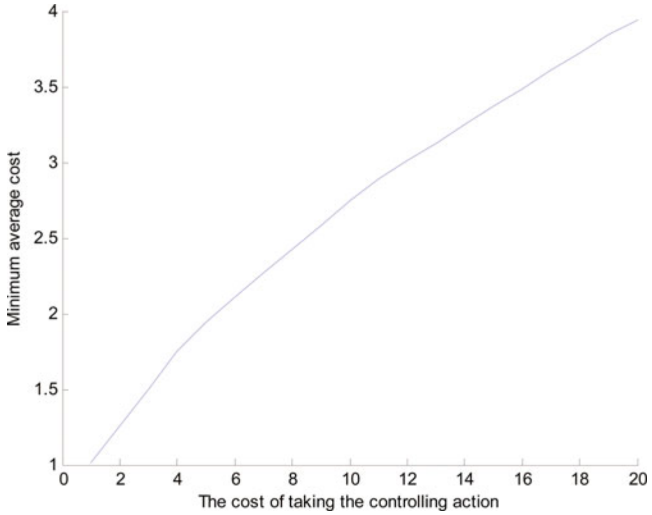

Figure 3. The minimum average cost as parameter k varies.

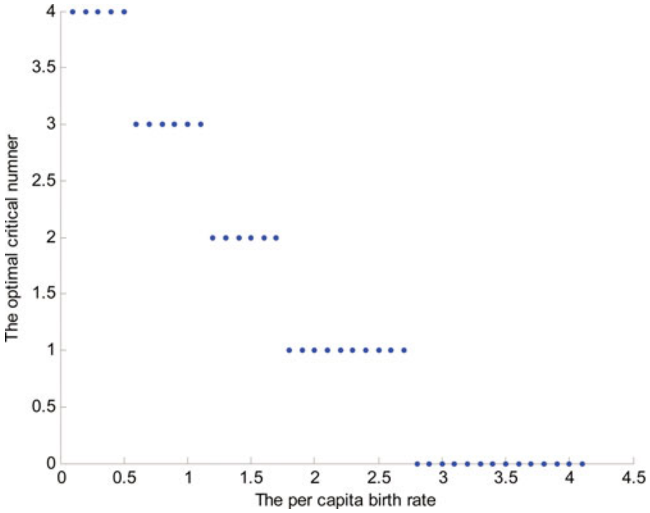

Figure 4. The variation of the optimal critical number x^* as parameter λ varies.

observe that, as λ increases, the critical number x^* that corresponds to the optimal control-limit policy P_{x^*}, decreases. This is plausible since, as λ increases, it is preferable to take the controlling action that introduces the external predator in the habitat of the pests when the pest population size takes smaller values.

4. Summary

In this article we considered the problem of the optimal control of a population of individuals, that are referred as pests, by the introduction of a predator in their habitat. The pest population grows according to a simple immigration-birth-death process and the predator captures the pests one at a time until their population size is annihilated and then emigrates from the habitat of the pests. The cost structure includes the cost of the damage caused by the pests, which increases linearly with respect to their population size, and the cost of the controlling action that introduces the predator in the habitat.

A suitable finite-state semi-Markov decision model was constructed for the determination of the policy that minimizes the expected long-run average cost per unit time among all stationary policies. This construction is possible only if a specific condition is valid on the per capita birth and death rates and the rate at which the predator is introduced. A great number of numerical results lead us to the conjecture that the optimal policy is of control-limit type, i.e., it initiates the controlling action that introduces the predator if and only if the pest population size exceeds some critical level. A proof of this conjecture seems to be difficult.

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