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# An improved algorithm for the computation of the optimal repair/replacement policy under general repairs

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**Abstract**

We consider a system which deteriorates with age and may experience a failure at any time instant. On failure, the system may be replaced or repaired. The repair can partially reset the failure intensity of the unit. Under a suitable cost structure it has been proved in the literature that the average-cost optimal policy is of control-limit type, i.e. it conducts a replacement if and only if, on the *n*th failure, the real age of the system is greater than or equal to a critical value. We develop an efficient special-purpose policy iteration algorithm that generates a sequence of improving control-limit policies. The value determination step of the algorithm is based on the embedding technique. There is strong numerical evidence that the algorithm converges to the optimal policy.


*Keywords:* Dynamic programming; Repair; Replacement; Average cost

## 1. Introduction

A great number of stochastic models have been introduced to describe the behavior of a repairable system that is subject to failure. In most of these models (see e.g. [1,7,11,9,10]) it was assumed that there are only two types of repair, the perfect repair and the minimal repair. The former results in a functioning system, which is as good as new, while the latter restores the system to its functioning condition just prior to failure.

Kijima et al. [3] studied the general repair model, in which the repair brings the state of the system to a certain better state. The minimal repair and the perfect repair are two special cases. They assumed that the repair and replacement costs are constant and considered a periodical replacement problem where the system is replaced at only scheduled times $kT$, $k = 0, 1, \ldots$ and is repaired whenever it fails. The long-run average cost per unit time was derived and an approximation procedure, which can be used to find the optimal replacement period, was proposed.

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Makis and Jardine [6] generalized the above repair/replacement process by formulating a suitable semi-Markov decision model. The system failure instants are the decision epochs. A two dimensional infinite state space was utilized, where the state of the process consists of the number of failures and the real age of the system. The first state variable is discrete while the second one is continuous. The replacement cost was assumed to be fixed and the repair cost was assumed to depend on the number of failures and the age of the system. Under some conditions on the costs and the failure rate of the system, Makis and Jardine showed that the policy that minimizes the long-run expected average cost per unit time is of control-limit type, i.e. it replaces the system at the $n$th failure if and only if its age is greater than or equal to some critical value that depends on $n$.

Love et al. [5] approximated the semi-Markov decision model of Makis and Jardine with a finite-state discrete semi-Markov decision model, by truncating the state space and by discretizing the second state variable. They determined the one-step transition probabilities and transition times for the reformulated model and developed an algorithm that generates a sequence of strictly improving control-limit policies. In the last paragraph of Section 3 of their paper, they mention that the algorithm determines the optimal control-limit policy. As we will explain in Section 3 of the present article, there is no proof of this assertion, though there is strong numerical evidence that it is true.

The main purpose of the present paper is to improve the algorithm of Love et al. by applying Tijms's [13, p. 234] embedding technique. In our problem it is possible to find explicit expressions for all quantities that are needed for the application of the embedding technique. This technique reduces considerably the calculations in the value determination step of the algorithm, which have been increased considerably because of the discretization of the second state variable.

The rest of the paper is organized as follows. The description of the problem and the relevant finite-state semi-Markov decision model are given in the next section. In Section 3 we develop the special-purpose policy iteration algorithm by applying Tijms's embedding technique. Some numerical results are presented in Section 4.

## 2. Model formulation

Consider a system that deteriorates over time and is subject to failures. The state of the system is represented by the pair of variables $(n, t_n)$, where $n$ denotes the $n$th failure and $t_n$ the real age of the system at that instant. The first variable ($n$) is discrete and the second one ($t_n$) is continuous. The state space is the following continuous set:

$$I = \{(n, t_n) | n = 1, 2, \ldots, t_n \geqslant 0\}.$$

A maximum number $N$ of failures before replacement and a number $B$, as the upper bound on the real age of the system, are assumed. Following Love et al. [5] we discretize the second state variable as follows. The real age axis is divided into a set of equally spaced age slices and we regard that the $n$th failure occurs at the age slice $i_n$. A scaling parameter $\xi$ is also defined as the number of time slices in a real time unit to relate the time slice $i_n$ to the real time $t_n$. Thus, if the $n$th failure occurs at the age slice $i_n$, we assume that the real age of the system at that failure instant is between $i_n/\xi$ and $(i_n + 1)/\xi$.

Henceforth we suppress the subscript $n$ on the age slices $i_n$ and simply refer to the age slice as $i$. The state space $I$ is approximated by the following finite and discrete set:

$$S = \{(n, i) | 1 \leqslant n \leqslant N, 0 \leqslant i \leqslant M\},$$

where $M = B\xi$. Note that we can increase the accuracy of this approximation by increasing the values of $N$, $B$, $\xi$.

The decision epochs are the system failure instants. At each decision epoch two actions $a \in \{0, 1\}$ may be taken. The system either can be repaired ($a = 1$) or can be replaced by a new identical one ($a = 0$). It is assumed that both maintenance activities are executed in negligible time.

In states $(N, i)$,$0 \leqslant i \leqslant M$, and $(n, M)$,$1 \leqslant n \leqslant N$, the action of replacement is mandatory and brings the system to the states $(1, j)$, where $0 \leqslant j \leqslant M$. If at a decision epoch, the system is in states $(n, i)$, $1 \leqslant n \leqslant N - 1$, $0 \leqslant i \leqslant M - 1$, and the action of repair is selected, the system makes a transition to the states $(n + 1, j)$, where

$i \leqslant j \leqslant M$. If at a decision epoch, the system is in states $(n,i)$, $1 \leqslant n \leqslant N-1$, $0 \leqslant i \leqslant M-1$, and the action of replacement is selected, the system makes a transition to the states $(1,j)$, where $0 \leqslant j \leqslant M$.

We suppose that a fixed cost $C_0$ is incurred when at a decision epoch the system is replaced by a new one. A cost $C_1(n,i)$ is also incurred if at a decision epoch the system is in state $(n,i)$ and the repair action is selected. We assume that $C_1(n,i)$ is non-decreasing in $n$ and $i$.

Let $X_n$,$2 \leqslant n \leqslant N-1$, be the time between the $(n-1)$th and the $n$th failure of the system. Let also $f(x)$ and $F(x)$ be the probability density function and the distribution function, respectively, of the time to the first failure. Following Kijima et al. [3], Makis and Jardine [6] and Love et al. [5], if the repair action $a = 1$ is chosen in state $(n,i)$,$1 \leqslant n \leqslant N-1$, $0 \leqslant i \leqslant M-1$, the virtual age of the system after the repair is assumed to be $(\theta i)/\xi$, where $0 \leqslant \theta \leqslant 1$, and the probability density function of $X_{n+1}$ is

$$f_{\frac{\theta i}{\xi}}(x) = \frac{f\left(x + \frac{\theta i}{\xi}\right)}{\overline{F}\left(\frac{\theta i}{\xi}\right)}, \quad x > 0,$$

where $\overline{F}(x) = 1 - F(x), x > 0$. Note that the parameter $\theta$ represents the degree of repair. If $\theta = 1$, the virtual age of the system after the $n$th repair is equal to its real age which means that a minimal repair was performed. If $\theta < 1$, the virtual age of the system is strictly smaller than $i$, meaning that the system is rejuvenated by the repair.

Our assumptions imply that we have a semi-Markov decision model with state space $S$. Our goal is to find the policy that minimizes the expected long-run average cost per unit time. Note that the state $(1,0)$ can be reached from every state under every stationary policy. Hence, there exists an optimal stationary policy (see [12, p. 149]). Makis and Jardine [6] proved that, if the function $f(x)$ and the failure rate of the system satisfy some suitable conditions, the optimal policy is of control-limit type, i.e. it replaces the system at the $n$th failure, $n = 1,\ldots,N-1$, if and only if the corresponding age slice $i$ is greater than or equal to some critical value $s_n^*$. They also proved that $s_1^* \geqslant s_2^* \geqslant \cdots \geqslant s_{N-1}^*$.

In Section 3 we will present an efficient semi-Markov decision algorithm to compute the optimal stationary policy.

## 3. The algorithm

We can compute the optimal policy by implementing the standard policy iteration, value iteration and linear programming algorithms. For a description and various applications of these algorithms we refer to Chapter 3 of Tijms's [13] book. It is also possible to develop a computationally tractable algorithm which operates on the class of control-limit policies and generates a sequence of improving control-limit policies. The algorithm generates policies such that for each $n$, $1 \leqslant n \leqslant N-1$, and given a degree of repair $\theta$, $0 \leqslant \theta \leqslant 1$, initiate the replacement of the system if and only if the corresponding age slice is equal to or greater than some critical number $s_n$. By comparing the results of our algorithm with those of the standard policy iteration and value iteration algorithms there is strong numerical evidence that the final policy obtained by the algorithm is the optimal one, since it coincides with the final policy obtained by the standard algorithms. The design of the algorithm is based on the embedding technique of Tijms (see [13, p. 234]). Similar algorithms have been proposed in various queueing, inventory, maintenance and pest control models. We refer to the book of Tijms [13, pp. 234–248] and the papers of Nobel and Tijms [8] and Kyriakidis [4].

The description of the algorithm follows. We consider a particular control-limit policy $R$ which for each number of failures $n$, $1 \leqslant n \leqslant N-1$, and given a degree of repair $\theta$ is characterized by the critical numbers $s_n$, $1 \leqslant n \leqslant N-1$. We define the set of states $E$ as

$$E = \{(n,i) | 1 \leqslant n \leqslant N-1, 0 \leqslant i \leqslant s_n - 1\}.$$

Note that the set $E$ can be reached from every initial state $(n,i) \in S$ if the policy $R$ is employed. The embedding technique can be applied if we take the set $E$ as the embedded set of states. Let $g(R)$ be the long-run expected average cost per unit time and, $\tau_s^E$ and $c_s^E$ be the expected time and the expected cost, respectively, until the first entry in the set $E$, if the initial state of the system is $s \in S$ and the policy $R$ is employed. Let also $p_{sr}^E$ be the probability that the first entry state in the set $E$ equals $r$ given that the policy $R$ is employed and the initial state of the system is $s \in S$. It is assumed that for the initial state $s \in E$ the first entry state in the set $E$ is the state at the next return to the set $E$.

According to the relation (3.6.1) in [13, p. 235] the relative values $h(s)$, $s \in S$, associated with the policy $R$ satisfy the following system of linear equations:

$$h(s) = c_s^E - g(R)\tau_s^E + \sum_{r\in E} p_{sr}^E h(r), \quad s \in S, \tag{1}$$

$$h(1,0) = 0. \tag{2}$$

The quantities $\tau_s^E$, $c_s^E$ and $p_{sr}^E$, $s \in S$, $r \in E$, can be computed by using simple conditional arguments and by taking into account the transitions of the system under the policy $R$. The expressions for these quantities are given below. The derivation of the expression (3) is given in detail in the appendix. The other expressions can be obtained similarly.

$$\begin{aligned}
\tau_{(n,i)}^E &= \int_0^\infty x f_{\frac{0i}{\xi}}(x)\,\mathrm{d}x + \int_{\frac{s_{n+1}-i}{\xi}}^\infty f_{\frac{0i}{\xi}}(x)\,\mathrm{d}x \frac{\int_0^\infty x f(x)\,\mathrm{d}x}{\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x}, \quad (n,i) \in E,\\
\tau_{(n,i)}^E &= \frac{\int_0^\infty x f(x)\,\mathrm{d}x}{\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x}, \quad (n,i) \in S - E,\\
c_{(n,i)}^E &= C_1(n,i) + C_0 \frac{\int_{\frac{s_{n+1}-i}{\xi}}^\infty f_{\frac{0i}{\xi}}(x)\,\mathrm{d}x}{\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x}, \quad (n,i) \in E,\\
c_{(n,i)}^E &= \frac{C_0}{\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x}, \quad (n,i) \in S - E,\\
p_{(n,i)(n+1,j)}^E &= \int_{\frac{j-i}{\xi}}^{\frac{j-i+1}{\xi}} f_{\frac{0i}{\xi}}(x)\,\mathrm{d}x, \quad 0 \leqslant i < s_n,\ i \leqslant j < s_{n+1},\ 1 \leqslant n \leqslant N-2,\\
p_{(n,i)(1,j)}^E &= \int_{\frac{s_{n+1}-i}{\xi}}^\infty f_{\frac{0i}{\xi}}(x)\,\mathrm{d}x \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x \left(\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x\right)^{-1}, \quad 0 \leqslant i < s_n,\ 0 \leqslant j < s_1,\ 1 \leqslant n \leqslant N-2,\\
p_{(n,i)(1,j)}^E &= \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x \left(\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x\right)^{-1}, \quad s_n \leqslant i \leqslant M,\ 0 \leqslant j < s_1,\ 1 \leqslant n \leqslant N-2,\\
p_{(N-1,i)(1,j)}^E &= \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x \left(\int_0^{\frac{s_1}{\xi}} f(x)\,\mathrm{d}x\right)^{-1}, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant j < s_1.
\end{aligned} \tag{3}$$

The relative values $h(s)$, $s \in E$, and the average cost $g(R)$ can be computed by solving the system of linear equations:

$$h(s) = c_s^E - g(R)\tau_s^E + \sum_{r\in E} p_{sr}^E h(r), \quad s \in E, \tag{4}$$

together with Eq. (2). The so-called policy improvement quantity $Q_R(s;a)$ associated with the policy $R$ is defined by

$$Q_R(s;a) = c_s(a) - g(R)\tau_s(a) + \sum_{r\in S} p_{sr}^{(a)} h(r), s \in \{1,\ldots,N-1\} \times \{0,\ldots,M-1\}, a \in \{0,1\}, \tag{5}$$

where $p_{sr}(a)$ is the probability that the next state of the process is the state $r \in S$, given that the present state of the system is $s$ and the action $a \in \{0,1\}$ is chosen, and $\tau_s(a)$, $c_s(a)$ are the corresponding one-step expected transition time and cost, respectively. The transition probabilities $p_{sr}(a)$ are given by the following relations:

$$p_{(n,i)(n+1,j)}(1) = \int_{\frac{j-i}{\xi}}^{\frac{j-i+1}{\xi}} f_{\frac{\theta i}{\xi}}(x)\,\mathrm{d}x, \quad 1 \leqslant n \leqslant N-1, \ 0 \leqslant i \leqslant j \leqslant M-1, \tag{6}$$

$$p_{(n,i)(n+1,M)}(1) = \int_{\frac{M-i}{\xi}}^{\infty} f_{\frac{\theta i}{\xi}}(x)\,\mathrm{d}x, \quad 1 \leqslant n \leqslant N-1, \ 0 \leqslant i \leqslant M-1, \tag{7}$$

$$p_{(n,i)(1,j)}(0) = \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x, \quad 1 \leqslant n \leqslant N, \ 0 \leqslant i \leqslant M, \ 0 \leqslant j \leqslant M-1, \tag{8}$$

$$p_{(n,i)(1,M)}(0) = \int_{\frac{M}{\xi}}^{\infty} f(x)\,\mathrm{d}x, \quad 1 \leqslant n \leqslant N, \ 0 \leqslant i \leqslant M. \tag{9}$$

The expression (6) can be easily explained by noting that, if the system is repaired when it is in state $(n,i)$, then the next state is $(n+1,j)$ if the time between the $n$th failure of the system and the $(n+1)$th failure of the system belongs to the interval $[(j-i)/\xi, (j-i+1)/\xi]$. The expression (8) is derived similarly. In Eqs. (7) and (9) we have consolidated all cases where $j \geqslant M$ into the states $(n+1,M)$ and $(1,M)$, respectively, such that the failures occurring beyond $M$ are put on $M$. The one-step expected transition times are given by expressions (8) and (9) of Love et al.[5]. The quantities $c_s(a)$, $s \in S$, $a \in \{0,1\}$ are given by $c_{(n,i)}(1) = C_1(n,i)$ and $c_{(n,i)}(0) = C_0$.

Suppose that for some number of failures $n(1 \leqslant n \leqslant N-1)$, there exists a number $\tilde{s}_n$ such that $0 \leqslant \tilde{s}_n < s_n$ and $Q_R((n,i);0) < h(n,i)$, $\tilde{s}_n \leqslant i < s_n$. Then, according to the semi-Markov version of Theorem 3.2.1 in Tijms [13, p. 192], the control-limit policy that is characterized by the critical numbers: $s_1,\ldots,s_{n-1},\tilde{s}_n,s_{n+1},\ldots,s_{N-1}$, achieves smaller average cost than $g(R)$. Similarly, if for some number of failures $n(1 \leqslant n \leqslant N-1)$ there exists a number $\tilde{s}_n$ such that $s_n < \tilde{s}_n \leqslant M$ and $Q_R((n,i);1) < h(n,i), s_n \leqslant i < \tilde{s}_n$, then, according again to the semi-Markov version of Theorem 3.2.1 in [13, p. 192], the control-limit policy that is characterized by the critical numbers: $s_1,\ldots,s_{n-1},\tilde{s}_n,s_{n+1},\ldots,s_{N-1}$ achieves smaller average cost than $g(R)$.

The above remarks lead us to design the following special-purpose policy iteration algorithm which generates a sequence of strictly improving control-limit policies. The corresponding Matlab program can be found in the Web site: http://www.actuar.aegean.gr/Faculty/Algorithm2.html.

### 3.1. Special-purpose algorithm

*Step 1:* (Initialization). Choose a limit on the number of repairs $N$, an upper bound of the real age of the system $B$, a scaling parameter $\xi$ and a degree of repair $\theta \in [0,1]$. Choose an initial control-limit policy $R$ characterized by the critical numbers $s_n$, $1 \leqslant n \leqslant N-1$.

*Step 2:* (Value-determination step). For the current control-limit policy $R$ compute the average cost $g(R)$ and the associated relative values $h(s)$, $s \in E$, by solving the system of linear Eqs. (4) and (2).

*Step 3:* (Policy improvement step). For each $n = 1,\ldots,N$:

(a) Find, if it exists, the smallest number $\tilde{s}_n$ such that $0 \leqslant \tilde{s}_n < s_n$ and $Q_R((n,i);0) < h(n,i), \tilde{s}_n \leqslant i < s_n$. Otherwise,

(b) find, if it exists, the largest number $\tilde{s}_n$ such that $s_n < \tilde{s}_n \leqslant M$ and $Q_R((n,i);1) < h(n,i), s_n \leqslant i < \tilde{s}_n$.

The quantities $Q_R((n,i);1)$ and $Q_R((n,i);0)$ are given by (5), where, if it is necessary, the relative values $h(s)$, $s \in S-E$, can be computed from Eq. (1).

Replace $s_n$ by $\tilde{s}_n$ for those $n$, $1 \leqslant n \leqslant N-1$, for which it is possible to find a number $\tilde{s}_n$ and go to Step 2.

*Step 4:* (Convergence test). If it is not possible to find any $\tilde{s}_n, 1 \leqslant n \leqslant N-1$, the algorithm is stopped. The final policy is $R$ with average cost $g(R)$.

The algorithm generates a sequence of strictly improving control-limit policies and stops after a finite number of iterations since the set of control-limit policies is finite. It differs from the algorithm proposed by Love et al. (see Section 3 in [5]) in the value determination step. Specifically, in the value determination step of our algorithm, the average cost of the current control-limit policy and the associated relatives values, that correspond to the set $E$, are computed, while in the value determination step of the algorithm of Love et al., the average cost of the current control-limit policy and the associated relative values, that correspond to the elements of the entire state space $S$, are computed. In the policy improvement step of our algorithm, the relative values that correspond to the set $S - E$ can be computed from (1), if it is necessary. Note also that in our algorithm the upper bound $B$of the real age of the system is not needed when computing the average cost and the relative values of a specific control-limit policy. There is strong numerical evidence that the algorithm converges to the optimal policy. The computational time required by our algorithm is smaller than the computational time required by the standard policy iteration algorithm and by the algorithm proposed by Love et al. This is due to the fact that the number of the unknowns in the value determination step of our algorithm is equal to the number of the elements of the embedded set of states $E$ while the number of the unknowns in the value determination step of the standard policy iteration algorithm and of the algorithm of Love et al. is equal to the number of elements of the entire state space $S$. Note also that from a great number of examples that we have tested, there is strong evidence that the number of iterations of the special-purpose policy iteration algorithm is not especially influenced by the initial control-limit policy.

Love et al. mention that the final policy of the algorithm is the optimal one. Although there is strong evidence, from a great number of examples that we have tested, that the assertion is true, a rigorous proof seems to be difficult. Note that in other Markov decision models (see e.g. [2,4]) it was possible to prove that the final policy obtained by similar special-purpose policy iteration algorithms is the optimal one. However in these models the state space was one-dimensional and it was possible to prove that the average cost under a control-limit policy is a unimodal function with respect to the critical point. A consequence of this result is the optimality of the final policy.

## 4. Numerical examples

As illustrations of the algorithm we present two examples. In Examples 1 and 2 below we assume that the lifetime distribution of a new system is Gamma and Weibull, respectively. For the determination of the one-step transition probabilities and of the one-step expected transition times in the following examples we refer to Appendix A of Love et al. [5].

**Example 1.** We assume that the lifetime distribution of a new system follows the Gamma ($\alpha$,$\lambda$) distribution, where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The probability density function is given by: $f(x) = \frac{\lambda^{\alpha}x^{\alpha-1}}{\Gamma(\alpha)}\exp(-\lambda x), x > 0$, where, $\Gamma(\cdot)$ denotes the gamma function. Suppose that $N = 9$, $B = 10$, $\theta = 0.3$, $\alpha = 3$, $\lambda = 3$. We further assume that the scaling parameter $\xi$ may take the values 10, 20, 50, 100 and $C_0 = 4$. Let also $C_1(n,i) = \sqrt{n+1}$, $1 \leqslant n \leqslant 8$, $0 \leqslant i \leqslant B\xi - 1$. As initial control-limit policy we choose the policy that is characterized by the critical numbers $s_n = 10\xi$, $1 \leqslant n \leqslant 8$. In Table 1 below we present the successive control-limit policies generated by the algorithm. We point out that for each value of $n$, $1 \leqslant n \leqslant 8$, a control-limit policy that is characterized by the numbers $s_n$, prescribes the replacement of the system if and only if the real age of the system is equal to or greater than $s_n/\xi$.

In Table 2 we present for each value of $\xi \in \{10, 20, 50, 100\}$ the CPU time (in seconds) of the Matlab program for our special-purpose algorithm that we run on a PC Acer Aspire 1605 DLC. We also give the required CPU times for the standard policy iteration algorithm and the algorithm of Love et al. Furthermore, the average cost of the final policy obtained by the three algorithms is given.

**Example 2.** We assume that the lifetime distribution of a new system is Weibull ($\alpha$,$\lambda$) where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The probability density function is given by: $f(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp[-(\lambda x)^{\alpha}]$, $x > 0$. Suppose that $N = 9$, $B = 5$, $\alpha = 5$, $\lambda = 2$, $\theta = 0.8$. We further assume that the scaling parameter $\xi$ may take the values 10, 20, 50, 100 and $C_0 = 6$. Let also $C_1(n,i) = i/\xi$,

Table 1
The critical numbers $s_n$, $1 \leqslant n \leqslant 8$ of the successive policies for Example1

| |
|---|
| $s_1 = s_2 = \cdots = s_8 = 10\xi$ |
| $s_1 = 5\xi$, $s_2 = 0.6\xi$, $s_3 = 0.3\xi$, $s_4 = \cdots = s_8 = 0$ |
| $s_1 = 5\xi$, $s_2 = 2.6\xi$, $s_3 = 1.7\xi$, $s_4 = 1.1\xi$, $s_5 = 0.8\xi$, $s_6 = 0.5\xi$, $s_7 = 0.3\xi$, $s_8 = 0.1\xi$ |
| $s_1 = 5\xi$, $s_2 = 2.3\xi$, $s_3 = 1.4\xi$, $s_4 = \xi$, $s_5 = 0.6\xi$, $s_6 = 0.4\xi$, $s_7 = 0.1\xi$, $s_8 = 0$ |

Table 2
The minimum average costs and CPU times for Example 1 for different values of $\xi$

| $\xi$ | Minimum average cost | Our algorithm (seconds) | Policy iteration algorithm (seconds) | Algorithm of Love et al. (seconds) |
|---|---|---|---|---|
| 10 | 2.8996 | 16 | 50 | 44 |
| 20 | 2.9033 | 34 | 135.1 | 123 |
| 50 | 2.9054 | 79.5 | 275.6 | 254.5 |
| 100 | 2.9096 | 166.2 | 565.8 | 550.3 |

Table 3
The critical numbers $s_n$, $1 \leqslant n \leqslant 8$ of the successive policies for Example 2

| |
|---|
| $s_1 = s_2 = \cdots = s_8 = 5\xi$ |
| $s_1 = 3.7\xi,\ s_2 = 3.3\xi,\ s_3 = 2.9\xi,\ s_4 = 2.5\xi,\ s_5 = 2.2\xi,\ s_6 = 1.9\xi,\ s_7 = 1.6\xi,\ s_8 = 1.4\xi$ |
| $s_1 = \cdots = s_5 = 2.5\xi,\ s_6 = 2.3\xi,\ s_7 = s_8 = 2.2\xi$ |
| $s_1 = \cdots = s_8 = 2.4\xi$ |

Table 4
The minimum average costs and CPU times for Example 2 for different values of $\xi$

| $\xi$ | Minimum average cost | Our algorithm (seconds) | Policy iteration algorithm (seconds) | Algorithm of Love et al. (seconds) |
|---|---|---|---|---|
| 10 | 2.0803 | 17 | 48 | 42 |
| 20 | 2.0828 | 36.7 | 135.3 | 123 |
| 50 | 2.0863 | 73.2 | 281.1 | 254.6 |
| 100 | 2.0898 | 152.6 | 574.5 | 557.2 |

$1 \leqslant n \leqslant 8, 0 \leqslant i \leqslant B\xi - 1$. As initial control-limit policy we choose the policy which is characterized by the critical numbers $s_n = 5\xi$, $1 \leqslant n \leqslant 8$. As in the previous example, in Table 3 we present the successive control-limit policies generated by the algorithm and in Table 4 we give the minimum average costs and the required CPU times for each value of $\xi \in \{10, 20, 50, 100\}$.

In both examples the successive policies generated by our algorithm are the same for all values of $\xi \in \{10, 20, 50, 100\}$. However, as $\xi$ varies, there is a slight difference between the values of the average cost of the final policy of the algorithms. This can be explained since the discrete state space $S$ approximates better the initial continuous state space $I$, as the scaling parameter $\xi$ increases. In Example 2 the critical numbers $s_n$ of the final policy are the same for all $n$, $1 \leqslant n \leqslant 8$, since the repair cost $C_1\ (n, i)$ does not depend on $n$.

It can be seen from Tables 2 and 4 that, for both examples, the computation time required by our algorithm is considerably smaller than the computational times required by the algorithm of Love et al. and by the standard policy iteration algorithm. This difference increases as the scaling parameter $\xi$ increases.

## Appendix. Derivation of expression (3)

Assume that the control-limit policy $R$ that is characterized by the critical points $s_1, \ldots, s_{N-1}$ is used and the initial state is $(n, i)$, $0 \leqslant i \leqslant s_n - 1$, $1 \leqslant n \leqslant N - 2$. For the first entry state into the set $E$ to be the state $(1, j)$, $0 \leqslant j \leqslant s_1 - 1$, it is necessary that the real age of the system at its $(n+1)$th failure is greater than $s_{n+1}/\xi$. The probability of this event is equal to

$$P_1 = \int_{\frac{s_{n+1}-i}{\xi}}^{\infty} f_{\frac{\theta i}{\xi}}(x)\,\mathrm{d}x. \tag{10}$$

Hence,

$$p^E_{(n,i)(1,j)} = P_1 P_2, \tag{11}$$

where $P_2$ is the probability that the first entry state in the set $E$ under the policy $R$ is the state $(1,j)$, given that the initial state is some state $(n+1,i')$, $i' > s_{n+1}$.

Suppose that $(n+1,i')$, $i' > s_{n+1}$, is the initial state. The system is replaced and its real age at its first failure may be greater than or smaller than $s_1/\xi$. If it is greater than $s_1/\xi$, it will be replaced again. The probability that the first entry state in $E$ is the state $(1,j)$ after one replacement is equal to $\int_{j/\xi}^{(j+1)/\xi} f(x)\,\mathrm{d}x$, the probability that the first entry state in $E$ is the state $(1,j)$ after two replacements is $P_3 \int_{j/\xi}^{(j+1)/\xi} f(x)\,\mathrm{d}x$, where $P_3 = \int_{s_1/\xi}^{\infty} f(x)\,\mathrm{d}x$, the probability that the first entry state in $E$ is the state $(1,j)$ after three replacements is $P_3^2 \int_{j/\xi}^{(j+1)/\xi} f(x)\,\mathrm{d}x$ and so on. Hence, summing all these probabilities we have that

$$P_2 = \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x + P_3 \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x + P_3^2 \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x + \cdots = \frac{1}{1-P_3} \int_{\frac{j}{\xi}}^{\frac{j+1}{\xi}} f(x)\,\mathrm{d}x. \tag{12}$$

The relations (10)–(12) yield the expression (3).

## References


[1] R.E. Barlow, F. Proschan, Mathematical Theory of Reliability, Wiley, New York, 1965.

[2] A. Federgruen, K.C. So, Optimal time to repair a broken server, Advances in Applied Probability 21 (1989) 376–397.

[3] M. Kijima, H. Morimura, Y. Suzuki, Periodical Replacement problem without assuming minimal repair, European Journal of Operational Research 37 (1988) 194–203.

[4] E.G. Kyriakidis, Optimal control of a truncated general immigration process through total catastrophes, Journal of Applied Probability 36 (1999) 461–472.

[5] C.E. Love, Z.G. Zhang, M.A. Zitron, R. Guo, A discrete semi-Markov decision model to determine the optimal repair/replacement policy under general repairs, European Journal of Operational Research 125 (2000) 398–409.

[6] V. Makis, A. Jardine, A note on optimal replacement policy under general repair, European Journal of Operational Research 69 (1993) 75–82.

[7] T. Nakagawa, M. Kowada, Analysis of a system with minimal repair and its application to replacement policy, European Journal of Operational Research 12 (1983) 176–182.

[8] R.D. Nobel, H.C. Tijms, Optimal control for an $M^X/G/1$ queue with two service modes, European Journal of Operational Research 113 (1999) 610–619.

[9] R.I. Phelps, Replacement policies under minimal repair, Journal of Operations Research Society 32 (1981) 549–554.

[10] R.I. Phelps, Optimal policy for minimal repair, Journal of Operations Research Society 34 (1983) 425–427.

[11] W.P. Pierskala, L.A. Voelker, A survey of maintenance model, Naval Research Logistics Quarterly 23 (1976) 353–388.

[12] S.M. Ross, Applied Probability Models with Optimization Applications, Dover, New York, 1992.

[13] H.C. Tijms, Stochastic Models: An Algorithmic Approach, Wiley, Chichester, 1994.