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LETTERS TO THE EDITOR

Dear Editor,

Single vehicle routing problem with a predefined customer sequence and stochastic continuous demands

1. Introduction

The vehicle routing problem (VRP) consists of determining the optimal routing of a fleet of vehicles, based at one or several depots, which deliver goods to a set of n customers comprising the nodes of a predefined network. The vehicles may also pick up expired products from the customers. For a survey of vehicle routing models we refer to Simchi-Levi *et al.* (2005). We describe below a special vehicle routing model in which a single vehicle with limited capacity delivers goods to n customers according to a predefined customer sequence.

Consider a set of nodes $V = \{0, \dots, n\}$, with node 0 denoting the depot and the nodes $1, \dots, n$ corresponding to customers, and a set of arcs

$$A = \{(i, i+1), (i+1, 0) : i \in V - \{n\}\}$$

that join the customers along the route $1 \to 2 \to \cdots \to n$, as well as joining all customers with the depot. The travel cost (distance) of each arc $(i, j) \in A$ is defined by $c_{i,j} > 0$. The costs $c_{i,j}$, $(i, j) \in A$, satisfy the triangular inequality, i.e. $c_{i,j} < c_{i,k} + c_{k,j}$. We assume that a single vehicle must serve all customers according to the predefined sequence $1, \dots, n$. The vehicle is at the depot initially and after serving all customers it returns to the depot. Let d_i, $i = 1, \dots, n$, be the demand of customer i for a particular product. We suppose that the demands d_i, $i = 1, \dots, n$, are independent continuous random variables with known probability density functions $f_i(x)$, such that $f_i(x) = 0$ for $x \geq Q$, where Q is the capacity of the vehicle. Upon completion of service to each customer, the vehicle has either to travel to the next customer or to return to the depot for stock replenishment and resume the route. The actual demand of each customer is revealed only upon the vehicle's visit to the customer.

Our goal is to find the policy that minimizes the total expected cost. A realistic example of this model could be the situation in which a vehicle delivers petrol to a sequence of petrol stations. In this case the demand of each customer is stochastic, since, when the order is issued, it is unknown how much petrol will be sold during the time between the order and the delivery. Note that Yang *et al.* (2000) considered the case in which the demands of the customers are discrete random variables and developed a suitable dynamic programming algorithm for the determination of the optimal policy. They also proved that the optimal policy has the following form. For each customer $i \in \{1, \dots, n-1\}$ there exists a critical quantity h_i such that the optimal decision, after serving customer i, is to continue to customer $i+1$ if the remaining quantity in the vehicle is greater than or equal to h_i, or return to the depot for stock replenishment if it is less than h_i. In Section 2 we prove an analogous result for the case in which the demands are continuous random variables. We also give an algorithm for the determination of the optimal policy and a numerical example in which the demands are uniformly distributed.

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Note that Tsirimpas *et al.* (2008) investigated three practical variations of the VRP if a single vehicle serves n customers according to a predefined sequence and the respective quantities d_i, delivered to customer $i \in \{1, \dots, n\}$, and p_i, picked up from customer $i \in \{1, \dots, n\}$, are not random variables but constant numbers. Suitable dynamic programming algorithms that find the optimal routing of the vehicle were developed for each case.

2. The optimal policy

Let $V_i(q)$, $q \in [0, Q]$, be the minimum total expected cost from customer i to the end of the route, if customer i has been served and the remaining quantity in the vehicle is q. This quantity satisfies the following dynamic programming equation (see, for example, Ross (1983, Chapter I)):

$$V_i(q) = \min\{H_i(q), \tilde{H}_i\}, \qquad i = 1, \dots, n-1,$$

where

$$\begin{aligned} H_i(q) = c_{i,i+1} &+ \int_q^Q [2c_{i+1,0} + V_{i+1}(q + Q - x)] f_{i+1}(x)\,\mathrm{d}x \\ &+ \int_0^q V_{i+1}(q - x) f_{i+1}(x)\,\mathrm{d}x \end{aligned} \tag{1}$$

and

$$\tilde{H}_i = c_{i,0} + c_{0,i+1} + \int_0^Q V_{i+1}(Q - x) f_{i+1}(x)\,\mathrm{d}x. \tag{2}$$

The boundary condition is

$$V_n(q) = c_{n,0}, \qquad q \in [0, Q].$$

If $\tilde{H}_i < H_i(q)$, then the optimal decision is to return to the depot for replenishment and then to go to customer $i + 1$. If $H_i(q) \leq \tilde{H}_i$, then the optimal decision is to go directly to the customer $i + 1$. In this case, if the demand x of the customer is greater than q, then the vehicle supplies the customer $i + 1$ with the quantity q, returns to the depot for stock replenishment, and then returns to customer $i + 1$ in order to deliver the remaining quantity $x - q$. It can be proved by induction on $i \in \{1, \dots, n-1\}$ that $H_i(q)$, $q \in [0, Q]$, is decreasing with respect to q. The proof is similar to the proof of an analogous result in Section 3 of Yang *et al.* (2000).

It can be easily seen that, for $i = 1, \dots, n-1$, $H_i(Q) < \tilde{H}_i$ and $H_i(0) > \tilde{H}_i$, since the costs $c_{i,j}$ satisfy the triangular inequality. From the monotonicity of $H_i(q)$, $q \in [0, Q]$, it follows that, for $i = 1, \dots, n-1$, there exists a critical number $h_i \in (0, Q)$ such that $H_i(h_i) = \tilde{H}_i$. The optimal policy chooses the action of going directly to the next customer $i + 1$ if $q \geq h_i$, whereas it chooses the action of returning to the depot for stock replenishment if $q < h_i$. The critical number h_i that corresponds to customer $i \in \{1, \dots, n-1\}$ can be found as follows. First we discretize the set of possible demands by dividing the interval $[0, Q]$ into small intervals of length ξ. Then we compute $\tilde{H}_i$, using (2), and, for $j = Q/\xi - 1, Q/\xi - 2, \dots$, we compute $H_i(j\xi)$, using (1), until $H_i(j\xi) > \tilde{H}_i$. The critical number h_i is equal to $(\tilde{j} + 1)\xi$, where $\tilde{j}$ is the maximum value of $\{0, \dots, Q/\xi - 1\}$ that satisfies the above inequality. The integrals in (1) and (2) are computed numerically. The minimum expected cost is evaluated at the points

$j\xi$, $j = 0, \ldots, Q/\xi$. The algorithm for the determination of h_i, $i = 1, \ldots, n-1$, is described in detail below. An analogous algorithm for the determination of the critical numbers in the case of discrete demands was presented by Yang *et al.* (2000).

Algorithm for the determination of h_i, $i = 1, \ldots, n-1$
Set $V_n(j\xi) = c_{0,n}$, $j = 0, \ldots, Q/\xi$.
For $i = n-1, \ldots, 1$:

Find $V_i(0)$ and $V_i(Q)$ from the following equations:

$$V_i(0) = \tilde{H}_i = c_{i,0} + c_{0,i+1} + \sum_{j=0}^{Q/\xi-1} V_{i+1}(Q - j\xi) f_{i+1}(j\xi)\xi,$$

$$V_i(Q) = c_{i,i+1} + \sum_{j=0}^{Q/\xi} V_{i+1}(Q - j\xi) f_{i+1}(j\xi)\xi,$$

and for $j = Q/\xi - 1, Q/\xi - 2, \ldots, 0$ evaluate the following quantity:

$$H_i(j\xi) = c_{i,i+1} + \sum_{r=j}^{Q/\xi-1} [2c_{i+1,0} + V_{i+1}(j\xi + Q - r\xi)] f_{i+1}(r\xi)\xi + \sum_{r=0}^{j-1} V_{i+1}(j\xi - r\xi) f_{i+1}(r\xi)\xi,$$

until $H_i(j\xi) > V_i(0)$. The critical number h_i is equal to $(\tilde{j} + 1)\xi$, where $\tilde{j}$ is the maximum value of $\{0, \ldots, Q/\xi - 1\}$ that satisfies the above inequality. For $j = 1, \ldots, \tilde{j}\xi$, $V_i(j\xi) = V_i(0)$ and for $j = (\tilde{j}+1)\xi, \ldots, Q/\xi - 1$, $V_i(j\xi) = H_i(j\xi)$.

As an illustration we present the following example. Suppose that the capacity of the vehicle is $Q = 10$ and the number of customers is $n = 10$. The demands d_i, $i = 1, \ldots, 10$, of the customers are independent continuous random variables uniformly distributed in the interval $[0, 10]$. We choose $\xi = 0.005$ so that the interval $[0, Q]$ is divided into $Q/\xi = 2000$ small subintervals of length ξ. The travel costs (distances) between the depot (node 0) and the nodes $1, \ldots, 10$ are assumed to be

$$c_{0,1} = 25, \quad c_{0,2} = 20, \quad c_{0,3} = 15, \quad c_{0,4} = 22, \quad c_{0,5} = 18,$$
$$c_{0,6} = 12, \quad c_{0,7} = 17, \quad c_{0,8} = 20, \quad c_{0,9} = 18, \quad c_{0,10} = 13.$$

The distances between the nodes i and $i+1$, $i = 1, \ldots, 9$, are taken to be

$$c_{1,2} = 18, \quad c_{2,3} = 12, \quad c_{3,4} = 16,$$
$$c_{4,5} = 20, \quad c_{5,6} = 14, \quad c_{6,7} = 13,$$
$$c_{7,8} = 10, \quad c_{8,9} = 15, \quad c_{9,10} = 19.$$

The critical numbers obtained by the algorithm are

$$h_1 = 3.25, \quad h_2 = 2.335, \quad h_3 = 5.23,$$
$$h_4 = 4.445, \quad h_5 = 3.335, \quad h_6 = 5.295,$$
$$h_7 = 3.25, \quad h_8 = 3.615, \quad h_9 = 5.385.$$

The minimum total expected cost is equal to

$$c_{0,1} + \sum_{j=0}^{Q/\xi-1} V_1(Q - j\xi) f_1(j\xi)\xi,$$

and is found to be approximately equal to 303.14. This is considerably smaller than the total cost,

$$2\sum_{i=1}^{9} c_{0,i} + c_{0,10} = 347,$$

that we would have if the vehicle, after serving each customer i, $1 \le i \le 9$, returns to the depot for stock replenishment and then goes to the next customer.

3. Conclusions and further problems

We have considered a variation of the vehicle routing problem that was studied in Section 2 of Yang *et al.* (2000). A vehicle is assumed to serve n customers according to a predefined customer sequence, with the quantity d_i that must be delivered to each customer $i \in \{1, \dots, n\}$ being a continuous random variable. We proved that the policy that minimises the total expected cost is characterized by some critical numbers h_i, $i \in \{1, \dots, n\}$, such that the optimal decision, after serving customer i, is to continue to customer $i+1$ if the remaining quantity in the vehicle is greater than or equal to h_i, or return to the depot for stock replenishment if it is less than h_i. An algorithm for the determination of these critical numbers was also given.

A complementary problem is the problem of finding the optimal routing if the vehicle must pick up from each customer $i \in \{1, \dots, n\}$ a random quantity p_i of a particular product. In this case, after serving each customer, the vehicle either has to travel to the next customer or return to the depot to unload all products that have been picked up, and then go to the next customer. The actual quantity that must be picked up from each customer is revealed only upon the visit to the customer. It can be proved in the same way as in Section 2 that the optimal policy is characterized by some critical numbers $\tilde{h}_i$, $i \in \{1, \dots, n\}$, such that the optimal decision, after serving customer i, is to continue to customer $i + 1$ if the quantity in the vehicle is less than or equal to $\tilde{h}_i$, or return to the depot to unload it if it is greater than $\tilde{h}_i$.

The more general problem, in which the vehicle must deliver a random quantity d_i to each customer $i \in \{1, \dots, n\}$ and also pick up a random quantity p_i from each customer $i \in \{1, \dots, n\}$, seems interesting. Note that, if d_i and p_i, $i = 1, \dots, n$, are constant, it is possible to develop a suitable dynamic programming algorithm for the determination of the optimal policy (see Tsirimpas *et al.* (2008, Section 3)). When the quantities d_i and p_i, $i = 1, \dots, n$, are random, it seems difficult to develop a suitable dynamic programming algorithm. This problem could be the subject of future research.

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References

Ross, S. M. (1983). *Introduction to Stochastic Dynamic Programming.* Academic Press, New York.

Simchi-Levi, D., Chen, X. and Bramel, J. (2005). *The Logic of Logistics. Theory, Algorithms, and Applications for Logistics and Supply Chain Management,* 2nd edn. Springer, New York.

TSIRIMPAS, P., TATARAKIS, A., MINIS, I. AND KYRIAKIDIS, E. G. (2008). Single vehicle routing with a predefined customer sequence and multiple depot returns. *Europ. J. Operat. Res.* **187,** 483–495.

YANG, W.-H., MATHUR, K. AND BALLOU, R. H. (2000). Stochastic vehicle routing problem with restocking. *Transportation Sci.* **34,** 99–112.

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