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The geometry of Euclidean surfaces with conical singularities

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Abstract

The geometry of closed surfaces equipped with a Euclidean metric with finitely many conical points of arbitrary angle is studied. The main result is that the image of a non-closed geodesic has 0 distance from the set of conical points. Dynamical properties for the space of geodesics are also proved.
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1 Preliminaries

Let S be a closed surface of genus ≥ 1 equipped with a euclidean metric with finitely many *conical singularities* (or *conical points*), say $s_1, ..., s_n$. Every point which is not conical will be called a *regular point* of S. Denote by $\theta(s_i)$ the angle at each s_i.

In this work we show that if g is a non-closed geodesic in S, then its image has 0 distance from the set of conical points $\{s_1, ..., s_n\}$ (see Theorems 11 and 22 below). This is done by analyzing the existence of flat strips in the universal cover $\widetilde{S}$ of S. In fact we show that a geodesic line $\widetilde{g}$ in $\widetilde{S}$, whose projection is not a closed geodesic in S, cannot be contained in a (Euclidean) flat strip. In particular, any two elements ξ, η in $\partial\widetilde{S}$ determine a unique geodesic line in $\widetilde{S}$ provided that ξ, η are not the limit points of a hyperbolic element $\phi \in \pi_1(S)$. We then use Theorem 11 to show a classical result in this setup, namely, that closed geodesics form a dense subset in the space GS of geodesics in S. In the final section we generalize to surfaces with $\theta(s_i) \in (0, +\infty) \setminus \{2\pi\}$ by employing saddle connections.

We write $C(v,\theta)$ for the standard cone with vertex v and angle θ, namely, $C(v,\theta)$ is the set $\{(r,t) : 0 \leq r, t \in \mathbb{R}/\theta\mathbb{Z}\}$ equipped with the metric $ds^2 = dr^2 + r^2dt^2$.

Definition 1 *A Euclidean surface with conical singularities $s_1, ..., s_n$ is a closed surface S equipped with a metric $d(\cdot,\cdot)$ such that at*

- *Every point $p \in S \setminus \{s_1, ..., s_n\}$ has a neighborhood isometric to a disk in the Euclidean plane*
- *Each $s_i \in \{s_1, ..., s_n\}$ has a neighborhood isometric to a neighborhood of vertex v of the standard cone $C(v, \theta(s_i))$.*

From now on and until the final section we assume that the angle $\theta(s_i)$ of the conical point s_i satisfies $\theta(s_i) \in (2\pi, +\infty)$.

Clearly, the metric on S is a length metric and the surface S will be written *e.s.c.s.* for brevity. Note that if the genus g is ≥ 2, such Euclidean structures exist, see [9].

Definition 2 *A geodesic segment is an isometric map $h : [a,b] \to S$. If $x = h(a)$ and $y = h(b)$ then a geodesic segment joining x and y will be denoted by $[x,y]$. Let $I = [0,+\infty)$ or $I = (-\infty,+\infty)$. A geodesic line (resp. geodesic ray) in S is a local isometric map $h : I \to S$ where $I = (-\infty,+\infty)$ (resp. $I = [0,+\infty)$). A closed geodesic is a local isometric map $h : I \to S$ which is a periodic map. A metric space is called geodesic if every two points can be joined by a geodesic segment.*

As every locally compact, complete length space is geodesic (see Th. 1.10 in [6]) we immediately have

Proposition 3 *If S is a e.s.c.s. then S is a geodesic space.*

2 The universal covering and the limit set

Assume that S is a closed *e.s.c.s.* of genus $g \geq 2$ and denote by d the euclidean metric on S. Let $\widetilde{S}$ be the universal covering of S and let $p_S : \widetilde{S} \to S$ be the covering projection. Obviously, the universal covering $\widetilde{S}$ is homeomorphic to $\mathbb{R}^2$ and by requiring p_S to be a local isometric map we may lift d to a metric $\widetilde{d}$ on $\widetilde{S}$ so that $(\widetilde{S}, \widetilde{d})$ becomes an *e.s.c.s.* There is a discrete group of isometries Γ of $\widetilde{S}$ which is isomorphic to $\pi_1(S)$, acting freely on $\widetilde{S}$ so that $S = \widetilde{S}/\Gamma$.

The group Γ with the word metric is *hyperbolic in the sense of Gromov* since Γ is isomorphic to $\pi_1(S)$. On the other hand, Γ acts co-compactly on $\widetilde{S}$; this implies that $\widetilde{S}$ is itself a hyperbolic space in the sense of Gromov (see for example Th.

4.1 in [3]) which is complete and locally compact. Hence, $\widetilde{S}$ is a proper space i.e. each closed ball in $\widetilde{S}$ is compact (see [6] Th. 1.10). Therefore, the visual boundary $\partial_{vis}\widetilde{S}$ of $\widetilde{S}$ is defined by means of geodesic rays and is homeomorphic to $\mathbb{S}^1$ (see [3], p.19). Furthermore, as all conical singularities in S are assumed to have angle $> 2\pi$, we deduce that S has curvature ≤ 0, that is, S satisfies CAT(0) inequality locally (see for example [7, Theorem 3.15]). Since $\widetilde{S}$ is simply connected it follows that $\widetilde{S}$ satisfies CAT(0) globally, i.e., $\widetilde{S}$ is a Hadamard space (for the definition and properties of CAT(0) spaces see [1]). Note that since $\widetilde{S}$ is a CAT(0) space, geodesic lines and geodesic rays in $\widetilde{S}$ are global isometric maps.

In the next proposition we state the following important property of $\partial_{vis}\widetilde{S}$. For a proof see Proposition 2.1 in [3].

Proposition 4 *For every pair of points $x \in \widetilde{S}$, $\xi \in \partial_{vis}\widetilde{S}$ (resp. $\eta, \xi \in \partial_{vis}\widetilde{S}$) there is a geodesic ray $r : [0, \infty) \to \widetilde{S} \cup \partial_{vis}\widetilde{S}$ (resp. a geodesic line $l : (-\infty, \infty) \to \widetilde{S} \cup \partial_{vis}\widetilde{S}$) such that, $r(0) = x$, $r(\infty) = \xi$ (resp. $l(-\infty) = \eta$, $l(\infty) = \xi$).*

Since $\widetilde{S}$ is a hyperbolic space in the sense of Gromov, the isometries of $\widetilde{S}$ are classified as elliptic, parabolic and hyperbolic [**?**]. On the other hand, Γ is a hyperbolic group, thus Γ does not contain parabolic elements with respect to its action on its Cayley graph (see Th. 3.4 in [3]). From this, it follows that all elements of Γ are hyperbolic isometries of $\widetilde{S}$. Therefore, for each $\varphi \in \Gamma$ and each $x \in \widetilde{S}$ the sequence $\varphi^n(x)$ (resp. $\varphi^{-n}(x)$) has a limit point $\varphi(+\infty)$ (resp. $\varphi(-\infty)$) when $n \to +\infty$ and $\varphi(+\infty) \neq \varphi(-\infty)$. The point $\varphi(+\infty)$ is called *attractive* and the point $\varphi(-\infty)$ *repulsive* point of φ.

The *limit set* $\Lambda(\Gamma)$ of Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial_{vis}\widetilde{S}$, where x is an arbitrary point in $\widetilde{S}$. Since the action of Γ on $\widetilde{S}$ is co-compact, it is a well known fact that $\Lambda(\Gamma) = \partial_{vis}\widetilde{S}$, and hence $\Lambda(\Gamma) = \mathbb{S}^1$. Note that the action of Γ on $\widetilde{S}$ can be extended to $\partial_{vis}\widetilde{S}$ and that the action of Γ on $\partial_{vis}\widetilde{S} \times \partial_{vis}\widetilde{S}$ is given by the product action.

Denote by F_h the set of points in $\partial_{vis}\widetilde{S}$ which are fixed by hyperbolic elements of Γ. Since $\Lambda(\Gamma) = \partial_{vis}\widetilde{S}$, the following three results can be derived from [4].

Proposition 5 *The set F_h is $\Gamma-$invariant and dense in $\partial_{vis}\widetilde{S}$.*

Proposition 6 *There exists an orbit of Γ dense in $\partial_{vis}\widetilde{S} \times \partial_{vis}\widetilde{S}$.*

Proposition 7 *The set $\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma \text{ is hyperbolic}\}$ is dense in $\partial_{vis}\widetilde{S} \times \partial_{vis}\widetilde{S}$.*

Lemma 8 *Let φ be an element of Γ and let $\eta = \varphi(-\infty)$ and $\xi = \varphi(\infty)$ be the repulsive and attractive points of φ in $\partial\widetilde{S}$. Then any geodesic line c joining η and ξ projects to a closed geodesic in S.*

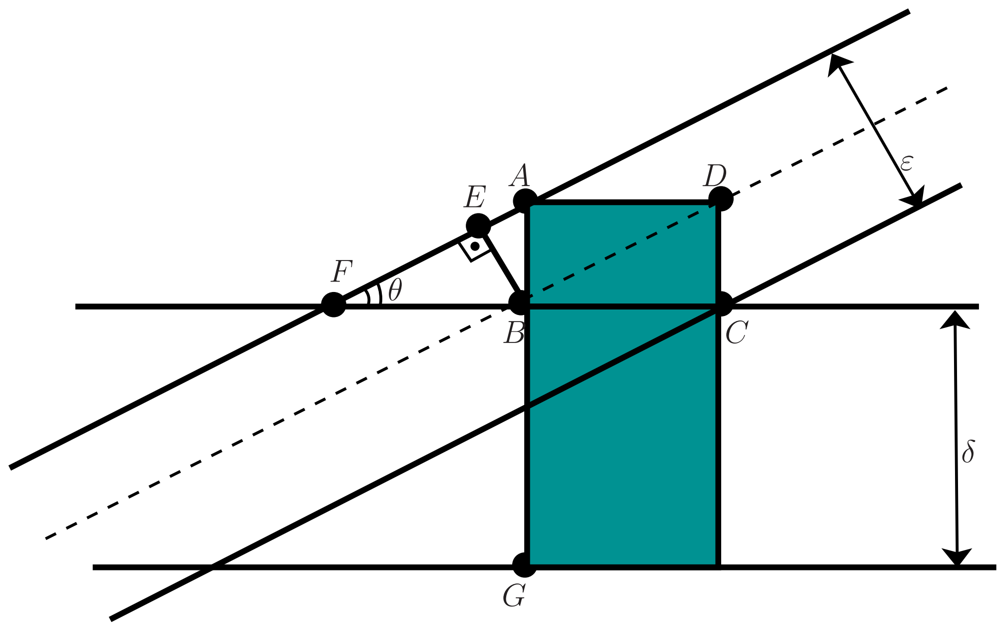

Figure 1: The quadrangle in the intersection of two flat strips

Proof. S is a $CAT(0)$ space and φ is an axial isometry (following Definition 3.1 of [1]). Therefore, from Proposition 3.3, p. 31 of [1], there is an axis c_0 of φ in $\widetilde{S}$ which projects to a closed geodesic in S.

Let c be a geodesic line of $\widetilde{S}$ joining the points η, ξ. Then c and c_0 are parallel in $\widetilde{S}$ i.e. they bound a flat strip in $\widetilde{S}$, (see Pr. 5.8, p. 25 in [1]). Therefore, by Proposition 3.3 of [1], c is also an axis of φ and thus it projects to a closed geodesic in S. ■

3 Flat strips and closed geodesics

Let GS be the space of all local isometric maps $\gamma : \mathbb{R} \to S$. The image of such a γ will be referred to as a geodesic in S. The geodesic flow is defined by the map

$$\Phi : \mathbb{R} \times GS \to GS$$

where the action of $\mathbb{R}$ is given by right translation, i.e. for each $t \in \mathbb{R}$ and $\gamma \in GS$, $\Phi(t, \gamma) = t \cdot \gamma$, where $t \cdot \gamma : \mathbb{R} \to S$ is the geodesic defined by $t \cdot \gamma(s) = \gamma(t + s)$, $s \in \mathbb{R}$.

Consider also the space $G\widetilde{S}$ of all isometric maps $g : \mathbb{R} \to \widetilde{S}$. Both spaces GS and $G\widetilde{S}$ are equipped with the compact-open topology. Moreover the space

$G\widetilde{S}$ with the compact-open topology is metrizable and its metric is given by the formula (see 8.3.B in [**?**])

$$|g_1 - g_2|_{G\widetilde{S}} = \int_{-\infty}^{+\infty} |g_1(t) - g_2(t)|_{\widetilde{S}} \, e^{-|t|} dt.$$

Definition 9 *A flat strip in $\widetilde{S}$ is a subset of $\widetilde{S}$ isometric, with respect to the induced metric, to a strip $[0, \varepsilon] \times \mathbb{R}$ in $\mathbb{R}^2$, for an appropriate $\varepsilon > 0$ such that for each $r \in [0, \varepsilon]$, $\{r\} \times \mathbb{R}$ is the image of a geodesic line. If in addition, for $r = 0, \varepsilon$ the distance $d(\operatorname{Im}(\{r\} \times \mathbb{R}), \{s_1, ..., s_n\})$ of the images of the geodesic lines $\{r\} \times \mathbb{R}$ from the set of conical points is 0, the flat strip will be called maximal flat strip. Observe that a maximal flat strip may contain conical points only in its boundary lines corresponding to $\{0\} \times \mathbb{R}$ and $\{\varepsilon\} \times \mathbb{R}$.*
By substituting $\mathbb{R}$ with $[0, \infty)$ and geodesic lines by geodesic rays, the notion of flat half strip is defined.

Notation 10 *The number $\varepsilon > 0$ will be called the width of the flat strip (resp. flat half strip for geodesic rays). A flat strip of width ε will be denoted by $FS(\varepsilon)$ (resp. $FHS(\varepsilon)$).*
For $\widetilde{g}$ in $G\widetilde{S}$ we will be writing $\widetilde{g} \in FS(\varepsilon)$ to indicate that $\widetilde{g}$ is identified with a line $\{r\} \times \mathbb{R}$ in $[0, \varepsilon] \times \mathbb{R} \equiv FS(\varepsilon)$.
For g in GS we will be writing $g \in FS(\varepsilon)$ to indicate that there exists a flat strip $FS(\varepsilon) \subset \widetilde{S}$ such that for some lift $\widetilde{g}$ of g, $\widetilde{g} \in FS(\varepsilon)$. Similarly, we write $\widetilde{g} \in FHS(\varepsilon)$ and $g \in FHS(\varepsilon)$ for geodesic rays.
If $\widetilde{g} \in G\widetilde{S}$ we write $\varepsilon(\widetilde{g})$ for the $\sup\{\varepsilon | \widetilde{g} \in FS(\varepsilon)\}$. Set $\varepsilon(\widetilde{g}) = 0$ if there is no such ε positive. Given $g \in GS$, pick a lift $\widetilde{g} \in G\widetilde{S}$ of g and set $\varepsilon(g) := \varepsilon(\widetilde{g})$. Clearly, $\varepsilon(g)$ does not depend on the choice of $\widetilde{g}$.
By writing $FS(g)$ (resp. $FS(\widetilde{g})$) we mean the maximal flat strip of width $\varepsilon = \varepsilon(g)$ (resp. $\varepsilon = \varepsilon(\widetilde{g})$) containing g (resp. $\widetilde{g}$). Similarly, $FHS(g)$ and $FHS(\widetilde{g})$ for geodesic rays.

We may parametrize each geodesic line $\{r\} \times \mathbb{R}$ by a mapping $\gamma^r : (-\infty, \infty) \to \widetilde{S}$ such that $\{\gamma^r(s) | r \in [0, \varepsilon]\}$ is identified with $[0, \varepsilon] \times \{s\}$. We will call such a parametrization a *normal parametrization* of E.

We show below (see Proposition 11) that the image of a non-closed geodesic in S does not have positive distance from the set of conical points. This implies that uniqueness of geodesic lines in $\widetilde{S}$ does hold for all geodesic which do not project to closed geodesics in S (cf Lemma 8). In other words, (maximal) flat strips in $\widetilde{S}$ correspond precisely to closed geodesics in S. In view of the latter, we will use occasionally the following quotient space of geodesics.

For each flat strip E isometric to $[0, \varepsilon] \times \mathbb{R}$, we identify each family of geodesics $g^r, r \in [0, \varepsilon]$ which forms a normal parametrization of E to a unique geodesic line g_E and denote the resulting quotient space by $G_0\widetilde{S}$. Thus, for every $r \in [0, \varepsilon]$ the points $g^r(s)$ are identified to $g_E(s)$, where $s \in \mathbb{R}$. Moreover, if a family $g^r, r \in [0, \varepsilon]$ forms a normal parametrization of E, so does the family $t \cdot g^r, r \in [0, \varepsilon]$. Thus, the geodesic flow on GS restricts to an action of $\mathbb{R}$ on $G_0\widetilde{S}$. Observe that for any two points $\xi, \eta \in \partial\widetilde{S}$ there exists a unique (up to parametrization) geodesic g in $G_0\widetilde{S}$ with $g(-\infty) = \xi$ and $g(+\infty) = \eta$. Similarly, we identify all local geodesics in GS which are closed and parallel and, thus, obtain the corresponding space G_0S.

The following proposition establishes the above mentioned uniqueness of (non-closed) geodesic lines in $\widetilde{S}$.

Theorem 11 *Let g be a non-closed geodesic or geodesic ray in S. Then*

$$d(\operatorname{Im} g, \{s_1, ..., s_n\}) = 0.$$

For its proof we need the following elementary lemma.

Lemma 12 *Let $FS(\varepsilon)$, $FS(\delta)$ be two infinite flat strips in $\widetilde{S}$ of width ε, δ respectively with $0 < \varepsilon \leq \delta$. Assume that $FS(\varepsilon)$, $FS(\delta)$ intersect at an angle θ, with $0 < \theta < \pi/2$. Then, $FS(\varepsilon) \cup FS(\delta)$ contains a right angle quadrangle of width $\delta + \frac{\varepsilon}{2\cos\theta}$ and length $\frac{\varepsilon}{2\sin\theta}$.*

Proof. The proof is elementary. Figure 1 exhibits the details, where the segment

$$AD = \frac{1}{2}FC = \frac{1}{2}\frac{\varepsilon}{\sin\theta}$$

is understood as the length and

$$GA = GB + BA = \delta + \frac{EB}{\cos\theta} = \delta + \frac{\varepsilon}{2\cos\theta} > \delta + \frac{\varepsilon}{2}$$

the width. ■

Remark 13 *Clearly, by letting $\theta \to 0$ the length of the (finite) flat quadrangle given by the above Lemma tends to ∞, while its width is bounded below by $\delta + \frac{\varepsilon}{2}$.*

Lemma 14 *Let $\widetilde{S}$ be the universal cover of e.s.c.s S. Let $[x_0, y_0]$ be a geodesic segment containing a conical point s in its interior. Assume that both angles subtended by $[x_0, s]$ and $[s, y_0]$ at s (notation $\measuredangle_s(x_0, y_0)$) are strictly bigger than π, say, $\theta_1 + \pi$ and $\theta_2 + \pi$ with $\theta_1 + \theta_2 + 2\pi = \theta(s)$. Then there exists an $\varepsilon > 0$ such that*

$$\forall x, y \quad \text{with} \quad d(x, x_0) < \varepsilon \quad \text{and} \quad d(y, y_0) < \varepsilon \Rightarrow [x, y] \ni s.$$

Proof. Set $\theta_0 = \frac{1}{3}\min\{\theta_1, \theta_2, \pi\}$. We may choose a neighborhood around x_0 of radius $\varepsilon > 0$ such that the angle $\measuredangle_s(x_0, x) < \theta_0$ for all x with $d(x, x_0) < \varepsilon$. Similarly for y_0. Then for any x, y with $d(x, x_0), d(y, y_0) < \varepsilon$ we have that both angles subtended by $[x, s]$ and $[s, y]$ at s are $> \pi$. Thus $[x, s] \cup [s, y]$ is a geodesic segment. ■

Lemma 15 *Assume that a sequence of geodesic rays $\{g_n\}$ converges to a geodesic ray g. If for some $\varepsilon > 0$, $g_n \in FHS(\varepsilon)$ for all n, then $g \in FHS(\varepsilon)$.*
The same result holds for a sequence of geodesic segments converging to a geodesic ray.

Proof. We write the proof for geodesic rays as the proof for segments is identical.
Assume that $\varepsilon(g) < \varepsilon$. Note that $\varepsilon(g)$ may be 0. We first treat the case where g contains a conical singularity. In this case, we have that, for some lift $\widetilde{g}$ of g and pre-image $\widetilde{s}$ of a conical singularity $s \in \{s_1, ..., s_n\}$, $\widetilde{g}(t_s) = \widetilde{s}$ for some $t_s \in (0, \infty)$. There are two angles formed by the segments of $\widetilde{g}$ at $\widetilde{g}(t_s)$. One of them must be equal to π, otherwise, by Lemma 14 and large enough N we have that the image of $\widetilde{g_N}$ contains $\widetilde{s}$, a contradiction. Without loss of generality, assume that the angle on the left, according to the parametrization of the ray $\widetilde{g}$, is equal to π.

If $\operatorname{Im}\widetilde{g}$ contains another pre-image $\widetilde{s}' \neq \widetilde{s}$ of a conical singularity $s' \in \{s_1, ..., s_n\}$, that is, $\widetilde{g}(t_{s'}) = \widetilde{s}'$ for some $t_{s'} \in (0, \infty)$, then, for the same reason as above, one of the two angles formed by the segments of $\widetilde{g}$ at $\widetilde{g}(t_{s'})$ must be equal to π.
Claim: The angle subtended at $\widetilde{g}(t_{s'})$ which is equal to π is also on the left, according to the parametrization, of the ray $\widetilde{g}$.
Proof of Claim: Assume the Claim does not hold and without loss of generality assume $t_{s'} > t_s$. Pick $N \in \mathbb{N}$ such that

$$d(\widetilde{g}(t), \widetilde{g}_N(t)) < \varepsilon/4 \quad \text{for all} \quad t \in [0, 2t_{s'}].$$

Then, $\widetilde{g}_N(t_s)$ is on the left of $\operatorname{Im}\widetilde{g}$ and $\widetilde{g}_N(t_{s'}) = \widetilde{s}'$ is on the right of $\operatorname{Im}\widetilde{g}$. Thus, the singularity $\widetilde{s}$ lies on the right of $\operatorname{Im}\widetilde{g}_N$ at distance $< \varepsilon/2$ and $\widetilde{s}'$ lies on the left of $\operatorname{Im}\widetilde{g}_N$ at distance $< \varepsilon/2$. This is a contradiction since $\widetilde{g}_N$ is contained in $FHS(\varepsilon)$. This completes the proof of the Claim.

Using the Claim we have that the assumption $\varepsilon(g) < \varepsilon$ implies that there must exist a conical singularity $\widetilde{s_1} \in \widetilde{S}$ on the left of the ray $\widetilde{g}$ such that

$$0 < d(\widetilde{s_1}, \operatorname{Im}\widetilde{g}) = \varepsilon_1' < \varepsilon.$$

Otherwise, we would be able to find a flat half strip containing $\widetilde{g}$ of width $\varepsilon' > \varepsilon(g)$. Let $[\widetilde{s_1}, \widetilde{g}(t_1)]$, $t_1 \in [0, \infty)$ be the geodesic segment of length ε_1' realizing the distance of $\widetilde{s_1}$ from $\operatorname{Im}\widetilde{g}$. For simplicity we assume that $t_1 > t_s$. The other case is treated similarly.

Each geodesic g_n splits the flat half strip $FHS(\varepsilon)$ to which it is contained into two flat half strips whose intersection is $\operatorname{Im}\widetilde{g}_n$. Denote by $\varepsilon_{1,n}$ and $\varepsilon_{2,n}$ the widths of these flat half strips with $\varepsilon_{1,n}+\varepsilon_{2,n}=\varepsilon$. Clearly, we may assume that $\widetilde{s_1}$ is on the left of the ray $\widetilde{g_n}$ and, moreover, by Lemma 14 it follows that for large enough n the conical singularity $\widetilde{s}=\widetilde{g}(t_s)$ is on the right of the ray $\widetilde{g_n}$, otherwise $\widetilde{g_n}$ cannot converge to $\widetilde{g}$. Choose $N\in\mathbb{N}$ such that

$$\forall t\in[0,t_1], d(\widetilde{g}_N(t),\widetilde{g}(t))<\frac{\varepsilon-\varepsilon_1'}{3}. \tag{1}$$

Since $\widetilde{s}$ does not belong to the flat half strip containing $\operatorname{Im}\widetilde{g_N}$, we have $\varepsilon_{2,N}<d(\widetilde{g_N}(t_s),\widetilde{s})$, thus

$$\varepsilon_{2,N}<d(\widetilde{g_N}(t_s),\widetilde{g}(t_s))<\frac{\varepsilon-\varepsilon_1'}{3}. \tag{2}$$

Similarly,

$$\varepsilon_{1,N}<d(\widetilde{g_N}(t_1),\widetilde{s_1})\leq d(\widetilde{g_N}(t_1),\widetilde{g}(t_1))+d(\widetilde{g}(t_1),\widetilde{s_1})$$

which implies that

$$\varepsilon_{1,N}<\frac{\varepsilon-\varepsilon_1'}{3}+\varepsilon_1'. \tag{3}$$

Combining inequalities 2,3 we obtain

$$\varepsilon=\varepsilon_{1,N}+\varepsilon_{2,N}<\frac{2}{3}(\varepsilon-\varepsilon_1')+\varepsilon_1'$$

a contradiction, which completes the proof in the case g contains a conical singularity.

Assume now that g does not contain a conical singularity. Then the assumption $\varepsilon(g)<\varepsilon$ implies that for some (hence any) lift $\widetilde{g}$ of g, there must exist conical singularities $\widetilde{s_1},\widetilde{s_2}\in\widetilde{S}$ such that

- $d(\widetilde{s_1},\operatorname{Im}\widetilde{g})=\varepsilon_1'>0$, $d(\widetilde{s_2},\operatorname{Im}\widetilde{g})=\varepsilon_2'>0$,
- $0<\varepsilon_1'+\varepsilon_2'<\varepsilon$, and
- $\widetilde{s_1},\widetilde{s_2}$ do not belong to the same side of $\operatorname{Im}\widetilde{g}$.

Other wise, we would be able to extend the flat half strip containing $\widetilde{g}$ to a flat half strip of width $\varepsilon'>\varepsilon(g)$. Let $[\widetilde{s_1},\widetilde{g}(t_1)]$ (resp. $[\widetilde{s_2},\widetilde{g}(t_2)]$), where $t_1\in[0,\infty)$ (resp. $t_2\in[0,\infty)$), be the geodesic segment of length ε_1' (resp. ε_2') realizing the distance of $\widetilde{s_1}$ (resp. $\widetilde{s_2}$) from $\operatorname{Im}\widetilde{g}$. We may assume that $t_2>t_1$.

Choose $N\in\mathbb{N}$ such that

$$\forall t\in[0,t_2], d(\widetilde{g}_N(t),\widetilde{g}(t))<\frac{\varepsilon-(\varepsilon_1'+\varepsilon_2')}{3}. \tag{4}$$

If both inequalities

$$d\left(\widetilde{s_1}, \widetilde{g_N}\left(t_1\right)\right) < \varepsilon_{1,N} \quad \text{and} \quad d\left(\widetilde{s_1}, \widetilde{g_N}\left(t_1\right)\right) < \varepsilon_{2,N} \tag{5}$$

hold, then $\widetilde{s_1}$ belongs to the flat half strip $FHS\left(\varepsilon\right)$ containing $\widetilde{g}_N$, a contradiction. Thus, at least one of the inequalities in 5 does not hold. Without loss of generality we may assume that the inequality

$$d\left(\widetilde{s_1}, \widetilde{g_N}\left(t_1\right)\right) > \varepsilon_{1,N}$$

holds and, therefore, the inequality

$$\varepsilon_1' + d\left(\widetilde{g}\left(t_1\right), \widetilde{g_N}\left(t_1\right)\right) = d\left(\widetilde{s_1}, \widetilde{g}\left(t_1\right)\right) + d\left(\widetilde{g}\left(t_1\right), \widetilde{g_N}\left(t_1\right)\right) > \varepsilon_{1,N} \tag{6}$$

holds. Similarly, and using the assumption that $\widetilde{s_1}, \widetilde{s_2}$ do not belong to the same side of $\operatorname{Im}\widetilde{g}$ we have

$$\varepsilon_2' + d\left(\widetilde{g}\left(t_2\right), \widetilde{g_N}\left(t_2\right)\right) = d\left(\widetilde{s_2}, \widetilde{g}\left(t_2\right)\right) + d\left(\widetilde{g}\left(t_2\right), \widetilde{g_N}\left(t_2\right)\right) > \varepsilon_{2,N} \tag{7}$$

Combining the above inequalities we reach

$$\varepsilon_1' + \varepsilon_2' + d\left(\widetilde{g}\left(t_1\right), \widetilde{g_N}\left(t_1\right)\right) + d\left(\widetilde{g}\left(t_2\right), \widetilde{g_N}\left(t_2\right)\right) > \varepsilon_{1,N} + \varepsilon_{2,N} = \varepsilon$$

which implies that

$$2\max\left\{d\left(\widetilde{g}\left(t_1\right), \widetilde{g_N}\left(t_1\right)\right), d\left(\widetilde{g}\left(t_2\right), \widetilde{g_N}\left(t_2\right)\right)\right\} > \varepsilon - \left(\varepsilon_1' + \varepsilon_2'\right)$$

Thus for some $t \in \{t_1, t_2\}$ we have that

$$d\left(\widetilde{g}_N\left(t\right), \widetilde{g}\left(t\right)\right) > \frac{\varepsilon - \left(\varepsilon_1' + \varepsilon_2'\right)}{2}$$

a contradiction by (4). ■

For the proof of Theorem 11 we also need the following proposition which is of interest in its own right.

Proposition 16 *Let g be a geodesic which does not contain a conical point, $\mathcal{U} = \{t \cdot g | t \in \mathbb{R}\}$ and $\overline{\mathcal{U}}$ its closure in GS under the compact open topology. Then the following are equivalent:*
(a) g is not periodic.
(b) $\overline{\mathcal{U}} \setminus \mathcal{U} \neq \emptyset$, that is, $\mathcal{U}$ is not closed.

Proof. Clearly, if g is periodic then $\mathcal{U}$ is closed. For the converse, assume, on the contrary, that for a non-periodic g we have $\overline{\mathcal{U}} = \mathcal{U}$. Pick a sequence $\{t_n\}$ converging to $+\infty$. Then, since GS is compact we have, up to a subsequence, that the sequence $\{t_n \cdot g\}$ converges to a point in $\mathcal{U}$, say,

$$t_n \cdot g \to t_0 \cdot g$$

for some $t_0 \in \mathbb{R}$. Clearly, by the definition of the compact open topology, for any $C \in \mathbb{R}$, we have

$$(t_n + C) \cdot g \to (t_0 + C) \cdot g$$

In other words, we have the following property:

($*$) every point $t \cdot g$ in $\mathcal{U}$ is the limit of a sequence of geodesics $\{t_n \cdot g\} \subset \mathcal{U}$ for some sequence $t_n \to \infty$.

Since $\operatorname{Im} g$ does not contain a conical point, we may pick an open disk D such that

$$D \cap \operatorname{Im} g \neq \emptyset \text{ and } \overline{D} \cap \{s_1, ..., s_n\} = \emptyset.$$

Then $\operatorname{Im} g \cap \overline{D}$ consists of geodesic segments $\sigma_j, j \in J$ of the form $g|_{[t_j^-, t_j^+]}$ for some $t_j^- < t_j^+ \in \mathbb{R}$ with endpoints $\sigma_j(-) = g\left(t_j^-\right)$ and $\sigma_j(+) = g\left(t_j^+\right)$ contained in ∂D. The degenerate cases $t_j^- = t_j^+$ where the segment σ_j is just one point are excluded from the collection. As the lengths of the non-null homotopic loops in S are bounded away from 0, the set

$$\left\{ \left|t_j^+ - t_{j'}^+\right| \,\middle|\, j \neq j' \in J \right\}$$

is bounded below which implies that the set $\left\{t_j^+ | j \in J\right\}$ is a discrete subset of $\mathbb{R}$, hence J is countable. Thus, we may enumerate the segments σ_j by considering $J = \mathbb{Z}$ and $t_j^- < t_j^+ < t_{j+1}^- < t_{j+1}^+$ for all j.
Observation: If $\sigma_j \equiv \sigma_{j'}$ for some $j < j'$ then, as $\operatorname{Im} g$ does not contain conical points, g must be periodic with period $t_{j'}^+ - t_j^+$.

By the above observation (which is not valid when $\operatorname{Im} g$ contains conical points), J is a countably infinite set.

By property ($*$) each geodesic segment σ_j is the limit of a sequence of segments in $\operatorname{Im} g \cap \overline{D} = \{\sigma_j, j \in J\}$. It follows that the set

$$\partial^2 J := \{(\sigma_j(-), \sigma_j(+)) \,|\, j \in J\}$$

is a perfect subset of $\partial D \times \partial D$, thus, uncountable, a contradiction. ■

Using standard compactness arguements and the natural projection $GS \to S$ given by $\gamma \to \gamma(0)$ it can be easily seen that $\mathcal{U} \subset GS$ is not closed if and only if $\operatorname{Im} g \subset S$ is not closed. Consequently, we have the following

Corollary 17 *Let g be a geodesic which does not contain a conical point. Then g is closed if and only if $\operatorname{Im} g$ is a closed set.*

Proof of Theorem 11 Assume, on the contrary, that

$$d(\operatorname{Im} g, \{s_1, ..., s_n\}) > 0.$$

Then the maximal flat half strip $FHS(\widetilde{g}) \subset \widetilde{S}$ containing a lift $\widetilde{g}$ of g has width $\varepsilon(g) = \varepsilon > 0$. Let $\mathcal{U} = \{t \cdot g | t \in \mathbb{R}\}$ and consider its closure $\overline{\mathcal{U}}$ in the space of geodesics in S under the compact open topology.

Set

$$\overline{\varepsilon} = \sup \left\{ \varepsilon(h) | h \in \overline{\mathcal{U}} \right\}.$$

By Lemma 15, $\overline{\varepsilon} \geq \varepsilon$. By Proposition 16 we may choose $h_0 \in \overline{\mathcal{U}} \setminus \mathcal{U}$ such that $\varepsilon(h_0) > \overline{\varepsilon} - \frac{\varepsilon}{3}$ and let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{U}$ such that $h_n = t_n \cdot g \to h_0$.

Pick a lift $\widetilde{h}_0$ of h_0 and a sequence $\left\{ \widetilde{h}_n \right\}$ approximating $\widetilde{h}_0$ in the compact open topology. We first claim that for all $n \in \mathbb{N}$ large enough the geodesic segment $\widetilde{h}_n([\overline{\varepsilon}, T])$, for any time $T > \overline{\varepsilon}$, cannot be parallel to the geodesic line $\operatorname{Im} \widetilde{h}_0$. For, if $\widetilde{h}_k([\overline{\varepsilon}, T])$ is parallel to the geodesic line $\operatorname{Im} \widetilde{h}_0$, then $\operatorname{Im} \widetilde{h}_0$ would have to be parallel to the whole geodesic line $\operatorname{Im} \widetilde{h}_k$ at distance, say $M \geq 0$. The same will also be true for any translation $t \cdot \widetilde{h}_k$. If $M = 0$ then $\widetilde{h}_k$ is a translation of $\widetilde{h}_0$ which implies that $h_0 \in \mathcal{U}$, is a contradiction since h_0 was chosen in $\overline{\mathcal{U}} \setminus \mathcal{U}$. In the case $M > 0$, as h_n, $n > k$ is a translation of h_k, namely $h_n = (t_n - t_k) \cdot h_k$, we would have that $h_n(0) = h_k(t_n - t_k)$ is at distance at least $M > 0$ from $h_0(0)$. This is a contradiction to the fact that $h_n = t_n \cdot g \to h_0$.

Pick a sequence $z_k \to \infty$ and for each $k \in \mathbb{N}$, let $N(k) \in \mathbb{N}$ be large enough so that for the time interval $[\overline{\varepsilon}, z_k]$ the geodesic segment $\widetilde{h}_{N(k)}([\overline{\varepsilon}, z_k])$ lies inside the flat half strip $FHS\left(\widetilde{h}_0\right)$. As the geodesic segment $\widetilde{h}_{N(k)}([\overline{\varepsilon}, z_k])$ is not parallel to the geodesic line $\operatorname{Im} \widetilde{h}_0$, we may apply Lemma 12 to the flat strips $FHS\left(\widetilde{h}_0\right)$ and $FHS\left(\widetilde{h}_{N(k)}\right)$, $k \in \mathbb{N}$ to obtain sequences

$$\{r_k\}_{k \in \mathbb{N}}, \{q_k\}_{k \in \mathbb{N}} \subset [0, \infty)$$

such that

- $q_k \to \infty$, and
- the geodesic segment $r_k \cdot \widetilde{h}_0([0, q_k])$ is contained in a right angle quadrangle of width $\varepsilon\left(\widetilde{h}_0\right) + \frac{\varepsilon}{2}$ and length q_k.

The sequence $\left\{ p_S\left(r_k \cdot \widetilde{h}_0\right) \right\}$ converges, up to a subsequence, to a geodesic $f \in \overline{\mathcal{U}}$. By Lemma 15 and Remark 13, it follows that $f \in FHS\left(\varepsilon\left(\widetilde{h}_0\right) + \frac{\varepsilon}{2}\right)$ which is a contradiction because

$$\varepsilon\left(\widetilde{h}_0\right) + \frac{\varepsilon}{2} = \varepsilon(h_0) + \frac{\varepsilon}{2} > \varepsilon(h_0) + \frac{\varepsilon}{3} > \overline{\varepsilon}$$

and $\overline{\varepsilon}$ is chosen to be $\sup \left\{ \varepsilon(h) | h \in \overline{\mathcal{U}} \right\}$. ■

Remark 18 *As the image of any geodesic in* $\widetilde{S}$ *splits* $\widetilde{S}$ *into two convex sets, say* $\widetilde{S}_L$ *and* $\widetilde{S}_R$*, we may speak of the distance to the left of* $\operatorname{Im}\widetilde{g}$ *from the set* $\left\{\widetilde{s}\in\widetilde{S}_L \mid \widetilde{s} \text{ conical point}\right\}$, *namely,*

$$d_L = \inf\left\{d\left(x,\widetilde{s}\right) \mid x\in\operatorname{Im}\widetilde{g}, \widetilde{s}\in\widetilde{S}_L\right\}$$

and similarly for d_R*. For a geodesic* $\widetilde{g}$ *in* $\widetilde{S}$ *which*
(a) does not contain any conical point
(b) projects to a non-closed geodesic in S
Theorem 11 asserts that $d_L = d_R = 0$*. This is clear in the above proof by working with a maximal flat strip to the left (right) of* $\operatorname{Im} g$.

Using the uniqueness of the non-closed geodesic lines in $\widetilde{S}$ (see Theorem 11), one can obtain a short proof of the following theorem shown in [1] in the context of Hadamard spaces.

Theorem 19 *There exists a geodesic* γ *in* GS *whose orbit* $\mathbb{R}\gamma$ *under the geodesic flow is dense in* GS.

4 Density of closed geodesics

In this section we show the following

Theorem 20 *The closed geodesics are dense in* GS.

Set $\partial^2\widetilde{S} = \{(\xi,\eta)\in\partial\widetilde{S}\times\partial\widetilde{S} : \xi\neq\eta\}$ and let $H : G_0\widetilde{S}\to\partial^2\widetilde{S}\times\mathbb{R}$ be the homeomorphism given as follows: choose a base point $x_0\in\widetilde{S}$. Let $\gamma_E\in G_0\widetilde{S}$. If the class γ_E consists of a single geodesic line γ joining $\gamma\left(+\infty\right), \gamma\left(+\infty\right)\in\partial\widetilde{S}$ set

$$H(\gamma_E) = (\gamma(-\infty), \gamma(\infty), s) \qquad (1)$$

where s is the real number such that $d(x_0, \gamma(\mathbb{R})) = d(x_0, \gamma(-s))$. If γ_E is a class of parallel geodesics $\{\gamma^r | r\in[0,\varepsilon]\}$ forming a flat strip E, then there exist a unique geodesic segment starting from x_0 and realizing the distance of x_0 from E. We may extend this geodesic segment to a geodesic segment which is perpendicular to all lines $\operatorname{Im}\gamma^r$, $r\in[0,\varepsilon]$. In other words, there exists a unique $s\in\mathbb{R}$ such that

$$d\left(x_0, \gamma^r(\mathbb{R})\right) = d(x_0, \gamma^r(-s)), \forall r\in[0,\varepsilon].$$

We use this $s\in\mathbb{R}$ to define $H(\gamma_E)$ as in equation (1). The map H is a homeomorphism (see [2, Th. 4.8]).

Proof of Theorem 20 Let $\beta\in GS$ be a non-closed geodesic and $\widetilde{\beta}\in G\widetilde{S}$ a lifting. Set $\eta = \widetilde{\beta}(-\infty)$, $\xi = \widetilde{\beta}(+\infty)$. Then by Proposition 7 there exists a sequence

$\{\phi_n\}$ of hyperbolic isometries such that $\phi_n(-\infty) \to \eta$ and $\phi_n(+\infty) \to \xi$. Set $\eta_n \equiv \phi_n(-\infty)$ and $\xi_n \equiv \phi_n(+\infty)$ and, clearly, $\phi_n(\eta_n) = \eta_n$, $\phi_n(\xi_n) = \xi_n$. Since $\partial\widetilde{S}$ is topologically a circle, we may choose the sequence $\{(\eta_n, \xi_n)\} \subset \partial_{vis}\widetilde{S} \times \partial_{vis}\widetilde{S}$ so that, in addition, for all n the following condition holds:

$$\eta_n, \xi_n \text{ belong to the same component of } \partial\widetilde{S} \setminus \{\eta, \xi\}.$$

As β is non-closed, the chosen lift $\widetilde{\beta} \in G\widetilde{S}$ determines a class $\widetilde{\beta}_E \in G_0\widetilde{S}$ which is a singleton. We have $H(\widetilde{\beta}_E) = (\eta, \xi, s)$ for some $s \in \mathbb{R}$. Denote by $\widetilde{\beta_{nE}}$ the geodesic $H^{-1}(\eta_n, \xi_n, s_n) \in G_0\widetilde{S}$ where s_n is chosen so that $s_n \to s$. As H is a homeomorphism, $\widetilde{\beta_{nE}} \to \widetilde{\beta}_E$ in the compact-open topology of $G_0\widetilde{S}$. If the class $\widetilde{\beta_{nE}}$ is not a singleton, choose $\widetilde{\beta_n}$ to be $\widetilde{\beta_n}^r$ where $r = 0$ or 1 according to which of the two geodesic lines $\operatorname{Im}\widetilde{\beta_n}^0$, $\operatorname{Im}\widetilde{\beta_n}^1$ has smaller distance (as a set) from $\operatorname{Im}\widetilde{\beta_n}$. We then have that the sequence $\left\{\widetilde{\beta_{nE}}\right\}$ determines a sequence $\left\{\widetilde{\beta_n}\right\}$ such that $\widetilde{\beta_n} \to \widetilde{\beta}$ in the compact-open topology of $G\widetilde{S}$. Clearly, each $\widetilde{\beta_n}$ is translated by ϕ_n and, thus, projects to a closed geodesic $\beta_n = \pi_S(\widetilde{\beta_n})$ in S. By continuity of π_S, we have the desired convergence $\beta_n \to \beta$. ■

5 Application to surfaces with conical singularities $< 2\pi$.

From now on we consider surfaces with conical singularities of arbitrary angle $\theta(s) \in (0, \infty) \setminus \{2\pi\}$. We will show that any geodesic in S can be approximated, in the compact open topology, by closed geodesics and/or appropriate saddle connections.

In what follows we will need a certain existence theorem of branched coverings which follows from Theorem 2.1 in [8]. The latter is the result of efforts of many mathematicians. Originally, the classical problem dating back to Hurewitz asks necessary and sufficient conditions for the existence of branched covering (see in [8] and its bibliography for supplementary information). We next recall basic definitions.

A branched covering is a map $\psi : S' \to S$, where S' and S are closed connected surfaces and ψ is locally modeled on maps of the form $\mathbb{C} \ni z \to z^k \in \mathbb{C}$ for some $k \geq 1$. The integer k is called the local degree at the point of S' corresponding to 0 in the source $\mathbb{C}$. If $k > 1$ then the point of S corresponding to 0 in the target $\mathbb{C}$ is called a branching point. The branching points are isolated, hence there are finitely many, say n, of them. Removing the branching points in S and all their pre-images in S', the restriction of ψ gives a genuine covering, whose degree we will denote by d. If the i-th branching point on S has m_i pre-images, the local degrees $(d_{ij})_{j=1,\dots,m_i}$ at these points give a partition of d i.e., $d_{ij} \geq 1$ and $\sum_{j=1}^{m_i} d_{ij} = d$.

In our context, we consider only the conical points $s_1, \ldots, s_l$, $l \leq n$ in S which are conical singularities of angle smaller than 2π and choose

- $m_i = 1$ for all $i = 1, \ldots, n$
- $d_{i1} = d$ odd natural number so that $d_{i1}\theta(s_i) > 2\pi$.

The above branching data is realizable by a branched covering (see Theorem 2.1 in [8]) yielding the following theorem

Theorem 21 *For every e.s.c.s. S of genus ≥ 0, there a (finite) branched covering $\psi : S' \to S$ such that:*
(1) S' is an e.s.c.s., with all conical singularities being of angle larger than 2π,
(2) the branch set on S are the conical singularities of angle smaller than 2π.
In particular, each conical singularity of angle smaller than 2π has one pre-image in S'.

Let S be a *e.s.c.s.* with conical singularities $\{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}$ with $\theta(s_i) \in (0, 2\pi)$ for $i = 1, \ldots, l$ and $\theta(s_i) \in (2\pi, \infty)$ for $i = l+1, \ldots, n$, $l \leq n$. Observe that the image of a geodesic in such surfaces, cannot contain a singular point s with $\theta(s) \in (0, 2\pi)$.

We consider the following notion in S : a local geodesic segment in S with endpoints in $\{s_1, \ldots, s_l\}$ will be called a *generalized saddle connection*. This terminology follows the notion of a saddle connection originally introduced in the study of translation surfaces (see [10]). Observe that a generalized saddle connection can only contain in its interior conical points s_i, for $i = l+1, \ldots, n$. We also allow the endpoints to coincide. We will also use the notion of a *closed piece-wise geodesic* by which we mean a finite union of generalized saddle connections which form a closed curve. Clearly, a closed piece-wise geodesic is a closed curve γ which is a local geodesic except at the points conical points $\operatorname{Im}\gamma \cap \{s_1, \ldots, s_l\}$.

Theorem 22 *Let S be a Euclidean surface with conical singularities*

$$\{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}$$

with $\theta(s_i) \in (0, 2\pi)$ for $i = 1, \ldots, l$ and $\theta(s_i) \in (2\pi, \infty)$ for $i = l+1, \ldots, n$, $l \leq n$. Then,
(a) for any non-closed geodesic or geodesic ray g in S,

$$d(\operatorname{Im} g, \{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}) = 0.$$

(b) every element in GS can be approximated, in the compact open topology, either by a sequence of closed geodesics or, by a sequence of closed piece-wise geodesics.

Proof. Let S' be the *e.s.c.s.* and $\psi : S' \to S$ the branched covering posited by the above theorem. The branched set of ψ is $\{s_1, \ldots, s_l\}$ and denote by $\{s'_1, \ldots, s'_l\}$ their pre-images in S'.

Let g be a non-closed geodesic (or geodesic ray). Observe that $\operatorname{Im} g$ does not contain any of the singularities $s_1, \ldots, s_l$. Clearly, g is covered by finitely many geodesics in GS'. Let g' be one of them.

For (a), if

$$d\left(\operatorname{Im} g, \{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}\right) > 0$$

then, since ψ is distance decreasing, we have

$$d\left(\operatorname{Im} g', \left\{s'_1, \ldots, s'_l, s'_{l+1}, \ldots, s'_n\right\}\right) > 0.$$

This is a contradiction because Theorem 11 asserts that

$$d\left(\operatorname{Im} g', \left\{s'_1, \ldots, s'_l, s'_{l+1}, \ldots, s'_n\right\}\right) = 0$$

For (b), Theorem 20 asserts that there exists a sequence of closed geodesics $\{g'_n\}$ converging, in the compact open topology, to g'. If

$$\operatorname{Im} g'_n \cap \{s'_1, \ldots, s'_l\} = \varnothing \tag{8}$$

holds for infinitely many n, then by passing, if necessary, to a subsequence we have that $\{g'_n\}$ converges to g' and, therefore, the sequence of closed geodesics $\{\psi(g'_n)\}$ converges to $\psi(g') = g$.

If (8) holds only for finitely many n, then we have that $\{\psi(g'_n)\}$ is a sequence of closed piece-wise geodesics which converges, in the compact-open topology, to g. ■

We now turn our attention to Euclidean surfaces with conical singularities $\{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}$which satisfy $\theta(s_i) \in (0, \pi)$ for $i = 1, \ldots, l$ and $\theta(s_i) \in (2\pi, \infty)$ for $i = l+1, \ldots, n$, $l \leq n$. Note that a geodesic, which, as usual, is defined to be a local isometric map, may have homotopically trivial self intersections, that is,

$$\exists t_1, t_2 \in \mathbb{R} \text{ with } g(t_1) = g(t_2) \text{ such that the loop } g|_{[t_1,t_2]} \text{ is contractible.}$$

Clearly, the lift $\widetilde{g}$ to the universal cover $\widetilde{S}$ of S of a geodesic g with homotopically trivial self intersections is not a global isometric map. In view of this and the following Lemma, we restrict our attention to geodesics which do not have homotopically trivial self intersections, equivalently, from now on the word geodesic will mean that its lift to the universal cover $\widetilde{S}$ is a (global) geodesic.

Lemma 23 *There exists a positive real number C such that for any geodesic g*

$$d(g(t), s_i) \geq C \quad \text{for all } i = 1, \ldots, l \text{ and for all } t \in \mathbb{R}.$$

Proof. It suffices to show that for each $i = 1, \ldots, l$ there exists $C_i > 0$ depending on $\theta(s_i)$, such that for any geodesic g

$$d(g(t), s_i) \geq C_i \text{ for all } t \in \mathbb{R}.$$

Choose C_i' such that $d(s_i, s_j) > C_i'$ for all $j \neq i$. Set $C_i = C_i' \cos \frac{\theta(s_i)}{2}$. We will show that if $d(\operatorname{Im} g, s_i) < C_i$ then g has a homotopically trivial self intersection.

Let $g(t_i)$ be the point on $\operatorname{Im} g$ of minimum distance, say C_0, from s_i. Clearly, $d(g(t_i), s_i) \equiv C_0 < C_i$. Then, the geodesic segment $[s_i, g(t_i)]$ is perpendicular to $\operatorname{Im} g$. Let r be the geodesic ray emanating from s_i such that both angles subtended by r and $[s_i, g(t_i)]$ at s_i are equal to $\frac{\theta(s_i)}{2}$. Set

$$T = \frac{C_0}{\cos \frac{\theta(s_i)}{2}} \text{ and } t_i' = C_0 \tan \frac{\theta(s_i)}{2}.$$

Then the geodesic segments

$$[s_i, g(t_i)], [g(t_i), g(t_i + t_i')] \text{ and } r|_{[0,T]}$$

and the geodesic segments

$$[s_i, g(t_i)], [g(t_i), g(t_i - t_i')] \text{ and } r|_{[0,T]}$$

form two (equal) right triangles with common hypotenuse $r|_{[0,T]}$ and common side $[s_i, g(t_i)]$. Thus,

$$g(t_i + t_i') = r(T) = g(t_i - t_i')$$

and g has a self intersection which is clearly homotopically trivial. ■

In the next Theorem GS consists, as mentioned above, of geodesics with no homotopically trivial self intersections.

Theorem 24 *Let S be a Euclidean surface with conical singularities*

$$\{s_1, \ldots, s_l, s_{l+1}, \ldots, s_n\}$$

with $\theta(s_i) \in (0, \pi)$ for $i = 1, \ldots, l$ and $\theta(s_i) \in (2\pi, \infty)$ for $i = l+1, \ldots, n$, $l \leq n$. Then, every element in GS can be approximated, in the compact open topology, either by a sequence of closed geodesics or, by a sequence of generalized saddles.

Proof. As in the previous theorem, consider the branched covering $\psi : S' \to S$ with branched set $\{s_1, \ldots, s_l\}$ with pre-images $\{s_1', \ldots, s_l'\}$ in S'. Let $\beta \in GS$ be a non-closed geodesic. By the previous lemma,

$$\exists C > 0 \text{ such that } d(\beta(t), s_i) \geq C, \text{ for all } i = 1, \ldots, l, \text{ and for all } t \in \mathbb{R}.$$

An analogous statement follows for any lift β' of β and the singularities $\{s_1', \ldots, s_l'\}$

$$\exists C > 0 \text{ such that } d(\beta'(t), s_i') \geq C, \text{ for all } i = 1, \ldots, l, \text{ and } t \in \mathbb{R}. \quad (9)$$

Theorem 20 asserts that there exists a sequence of closed geodesics $\{\beta'_n\}$ converging, in the compact open topology, to β'. If

$$\operatorname{Im}\beta'_n \cap \{s'_1, \ldots, s'_l\} = \varnothing \tag{10}$$

holds for infinitely many n, then we obtain as in the proof of the previous theorem that the sequence of closed geodesics $\{\psi(\beta'_n)\}$ converges to $\psi(\beta') = \beta$.

If (10) holds only for finitely many n, we may assume that for all n, $\operatorname{Im}\beta'_n$ contains at least one singularity from $\{s'_1, \ldots, s'_l\}$. Choose a sequence of positive $\varepsilon_k \to 0$ with $\varepsilon_k < C$. Then, for each $k \in \mathbb{N}$, there exists a closed geodesic $\beta'_{n(k)}$ which $(\varepsilon_k, [-k,k])$-approximates β', that is

for $t \in [-k,k]$, $d\left(\beta'_{n(k)}(t), \beta'(t)\right) < \varepsilon_k$

By (9), $\beta'_{n(k)}|_{[-k,k]}$ does not contain a singularity $s_i, i \leq l$. Restrict $\beta'_{n(k)}$ to a compact set $[t_{-k}, t_k]$ containing $[-k,k]$ such that

$$\beta'_{n(k)}(t_{-k}), \beta'_{n(k)}(t_k) \in \{s'_1, \ldots, s'_l\}$$

and $\beta'_{n(k)}|_{(t_{-k},t_k)}$ does not contain a singularity $s_i, i \leq l$. Clearly, $\gamma'_n \equiv \beta'_{n(k)}|_{[t_{-k},t_k]}$ is a sequence of generalized saddles (each with endpoints $\beta'_{n(k)}(t_{-k})$, $\beta'_{n(k)}(t_k)$) approximating β'. It follows that $\gamma_n = \psi(\gamma'_n)$ is a sequence of generalized saddles approximating β. ■

References

[1] W. Ballmann, *Lectures on spaces of non positive curvature*, Birkhäuser, 1995.

[2] Ch. Champetier, *Petite simplification dans les groupes hyperboliques*, Ann. Fac. Sci. Toulouse, VI. Ser., Math. 3, No.2, 161-221 (1994).

[3] M. Coornaert, T. Delzant, A. Papadopoulos, *Géometrie et théorie des groupes,* Lecture Notes in Mathematics, 1441, Sringer-Verlag, 1980.

[4] M. Coornaert, *Sur les gropes proprement discontinus d'isometries des espaces hyperboliques au sens de Gromov*, Thèse de U.L.P., Publication de I.R.M.A., 1990.

[5] M. Gromov, *Hyperbolic groups* in "Essays in group theory", M.S.R.I. Publ. 8 (Springer 1987), 75-263.

[6] M. Gromov, J. Lafontaine, P. Pansu, *Structures metriques pour les variétés Riemanniennes*, Fernand Nathan, Paris, 1981.

[7] F. Paulin, *Constructions of hyperbolic groups via hyperbolization of polyhedra,* In: Group Theory from a Geometrical Viewpoint, ICTP, Trieste, Italy, March 26–April 6, 1990, (eds. E. Ghys and A. Haefliger), 1991.

[8] E. Pervova, C. Petronio, *On the existence of branched coverings between surfaces with prescribed branch data I,* Algebraic & Geometric Topology 6, 1957-1985, 2006.

[9] M. Troyanov, *Les surfaces euclidiennes a singularites coniques*, Enseign. Math. (2) 32 (1986), no. 1-2, 79-94.

[10] A. Zorich, *Flat surfaces*, Frontiers in Number Theory, Physics, and Geometry Vol.I, P. Cartier, B. Julia, P. Moussa, P. Vanhove (Editors), Springer Verlag, 2006, 439–586.