On surfaces with conical singularities of arbitrary angle

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Abstract

The geometry of closed surfaces equipped with a Euclidean metric with finitely many conical points of arbitrary angle is studied. The main result is that the set of closed geodesics is dense in the space of geodesics.

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1 Introduction

Let S be a closed surface of genus ≥ 1 equipped with a Euclidean metric with finitely many *conical singularities* (or *conical points*), denoted by $s_1, ..., s_n$. Every point which is not conical will be called a *regular point* of S. Denote by $\theta(s_i)$ the angle at each s_i, with $\theta(s_i) \in (0, +\infty) \setminus \{2\pi\}$ and denote by $C(S)$ the set $\{s_1, ..., s_n\}$. Existence of conical points with angle less than 2π results into major differences in the geometry of the surface S compared to the case where all angles are $> 2\pi$. An important property which fails in this class of spaces is that geodesic segments with specified endpoints and homotopy class are no longer unique and similarly for geodesic rays and lines in the universal cover $\widetilde{S}$ (see Example 5 below). Moreover, extension of geodesics also fails. More precisely, there exist geodesic segments σ in $\widetilde{S}$ not containing any singularity which cannot be extended to any geodesic segment σ' properly containing σ. These facts make the study of the geometry of $\widetilde{S}$ interesting. In fact, the tools for studying the geometry of Euclidean surfaces with conical points of arbitrary angle are, in principle, limited to the property that $\widetilde{S}$ is a hyperbolic space in the sense of Gromov.

In this note we first show that the set of points on the boundary $\partial\widetilde{S}$ to which there corresponds more than one geodesic ray is dense in $\partial\widetilde{S}$ (see Theorems 9 and 10 below). Moreover, it is shown that for any boundary point $\xi \in \partial\widetilde{S}$ there exists a base point $\widetilde{x}_0 \in \widetilde{S}$ such that at least two geodesic rays emanating from $\widetilde{x}_0$ correspond to ξ. We then show that the images of all geodesic rays corresponding to a boundary point $\xi \in \partial\widetilde{S}$ are contained in a convex subset of $\widetilde{S}$ whose boundary is geodesic consisting of two geodesic rays. Thus, in the class of geodesic rays corresponding to each point $\xi \in \partial\widetilde{S}$ there are associated two distinct outermost (left and right) geodesic rays. Similarly for geodesic lines.

We show that the set of closed geodesics is dense in the space of all geodesics GS in the following sense: for each pair of distinct points $\xi, \eta \in \partial\widetilde{S}$ and each outermost geodesic line γ joining them, there exists a sequence of geodesics $\{c_n\}$ in $\widetilde{S}$, with the projection of every c_n to S being a closed geodesic, such that $\{c_n\}$ converges in the usual uniform sense on compact sets to γ.

2 Preliminaries

Let $\widetilde{S}$ be the universal covering of S and let $p : \widetilde{S} \to S$ be the universal covering projection. Obviously, the universal covering $\widetilde{S}$ is homeomorphic to $\mathbb{R}^2$ and by requiring p to be a local isometric map we may lift d to a metric $\widetilde{d}$ on $\widetilde{S}$ so that $(\widetilde{S}, \widetilde{d})$ becomes an $e.s.c.s$. Clearly, $\pi_1(S)$ is a discrete group of isometries of $\widetilde{S}$ acting freely on $\widetilde{S}$ so that $S = \widetilde{S}/\pi_1(S)$.

Due to the existence of conical points with angle $< 2\pi$, a geodesic γ in S, usually defined to be a local isometric map, may have homotopically trivial self intersections, that is,

$$\exists t_1, t_2 \in \mathbb{R} \text{ with } \gamma(t_1) = \gamma(t_2) \text{ such that the loop } \gamma|_{[t_1,t_2]} \text{ is contractible.}$$

Clearly, any lift $\widetilde{\gamma}$ to the universal cover $\widetilde{S}$ of S of a local geodesic γ with homotopically trivial self intersections is not a global isometric map. In view of this and Lemma 1 below, we restrict our attention to geodesics and geodesic segments which do not have homotopically trivial self intersections.

Let GS be the space of all local isometric maps $\gamma : \mathbb{R} \to S$ so that its lift to the universal cover $\widetilde{S}$ is a (global) geodesic. The image of such a γ will be referred to as a geodesic in S. Similarly we define the notion of a geodesic segment, that is, a local isometric map whose domain is a closed interval which lifts to a geodesic segment in $\widetilde{S}$. The geodesic flow is defined by the map

$$\Phi : \mathbb{R} \times GS \to GS$$

where the action of $\mathbb{R}$ is given by right translation, i.e. for each $t \in \mathbb{R}$ and $\gamma \in GS$, $\Phi(t,\gamma) = t \cdot \gamma$, where $t \cdot \gamma : \mathbb{R} \to S$ is the geodesic defined by $t \cdot \gamma(s) = \gamma(t+s)$, $s \in \mathbb{R}$.

The group $\pi_1(S)$ with the word metric is *hyperbolic in the sense of Gromov.* On the other hand, $\pi_1(S)$ acts co-compactly on $\widetilde{S}$; this implies that $\widetilde{S}$ is itself a hyperbolic space in the sense of Gromov (see for example [2, Ch.4 Th. 4.1]) which is complete and locally compact. Hence, $\widetilde{S}$ is a proper space i.e. each closed ball in $\widetilde{S}$ is compact (see [5] Th. 1.10). Therefore, the visual boundary $\partial\widetilde{S}$ of $\widetilde{S}$ is defined by means of geodesic rays and is homeomorphic to $\mathbb{S}^1$ (see [2], p.19).

The following properties contain information concerning the images of geodesics with respect to the conical points.

Lemma 1 *There exists a positive real number C such that for any geodesic γ in $\widetilde{S}$*

$$d(\gamma(t), \widetilde{s}_i) \geq C \text{ for all } t \in \mathbb{R} \text{ and for all } s_i \in C(S) \text{ with } \theta(s_i) \in (0, 2\pi).$$

where $\widetilde{s}_i$ denotes a pre-image of s_i.

The proof of this Lemma is given in [1]. By considering, if necessary, a constant C' smaller than C, the above Lemma holds for geodesics in S.

Lemma 2 *If two geodesic segments σ_1, σ_2 in $\widetilde{S}$ intersect at two points x, y such that x, y are isolated in $\sigma_1 \cap \sigma_2$ then both x, y are conical points with angle $> 2\pi$. If $\sigma_1 \cap \sigma_2$ is a closed segment then its endpoints are conical points with angle $> 2\pi$.*
Similarly for homotopic with endpoints fixed geodesic segments in S.

Proof. Let $\sigma_1 = [w_1, z_1]$, $\sigma_2 = [w_2, z_2]$ be two geodesic segments in $\widetilde{S}$ intersecting at two points x, y which are isolated in $\sigma_1 \cap \sigma_2$. Clearly, $\sigma_1|_{[x,y]} \cup \sigma_2|_{[y,z_2]}$ realizes the distance from x to z_2. Therefore, the angle formed by $\sigma_1|_{[x,y]}$, $\sigma_2|_{[y,z_2]}$ at y is at least π. Similarly, the angle formed by $\sigma_2|_{[x,y]}$, $\sigma_1|_{[y,z_1]}$ at y is at least π, hence $\theta(y) > 2\pi$. ■

Since $\widetilde{S}$ is a hyperbolic space in the sense of Gromov, the isometries of $\widetilde{S}$ are classified as elliptic, parabolic and hyperbolic [4]. On the other hand, $\pi_1(S)$ is a hyperbolic group, thus $\pi_1(S)$ does not contain parabolic elements with respect to its action on its Cayley graph (see Th. 3.4 in [2]). From this, it follows that all elements of $\pi_1(S)$ are hyperbolic isometries of $\widetilde{S}$. Therefore, for each $\varphi \in \pi_1(S)$ and each $x \in \widetilde{S}$ the sequence $\varphi^n(x)$ (resp. $\varphi^{-n}(x)$) has a limit point $\varphi(+\infty)$ (resp. $\varphi(-\infty)$) when $n \to +\infty$ and $\varphi(+\infty) \neq \varphi(-\infty)$. The point $\varphi(+\infty)$ is called *attractive* and the point $\varphi(-\infty)$ *repulsive* point of φ.

The following important property for hyperbolic spaces (see Proposition 2.1 in [2]) holds for $\partial\widetilde{S}$.

Proposition 3 *For every pair of points $x \in \widetilde{S}$, $\xi \in \partial\widetilde{S}$ (resp. $\eta, \xi \in \partial\widetilde{S}$) there is a geodesic ray $r : [0, \infty) \to \widetilde{S} \cup \partial\widetilde{S}$ (resp. a geodesic line $\gamma : (-\infty, \infty) \to \widetilde{S} \cup \partial\widetilde{S}$) such that, $r(0) = x$, $r(\infty) = \xi$ (resp. $\gamma(-\infty) = \eta$, $\gamma(\infty) = \xi$).*

Remark that uniqueness does not hold in the above proposition. In fact, we have the following straightforward corollary to Lemma 2.

Corollary 4 *If a geodesic segment intersects a geodesic ray at two isolated points as in Lemma 2, then there exist two distinct geodesic rays defining the same point at infinity.*
Similarly for geodesic lines.

Thus, for each pair of points $x \in \widetilde{S}$, $\xi \in \partial\widetilde{S}$ there corresponds a class of geodesic rays r with $r(0) = x$, $r(\infty) = \xi$, the cardinality of which varies from a singleton to uncountable (see discussion following Example 5 below). It is well known that in hyperbolic metric spaces the stability property of quasi-geodesic rays (and lines) holds in the sense of bounded Hausdorff distance (see, for example, [3, Chapter I, § 6]). It follows that any two geodesic rays r_1, r_2 in the same class are asymptotic, that is, (see [2, Ch. 3, Thm. 3.1]) there exists a constant $A > 0$ which depends only on the hyperbolicity constant of $\widetilde{S}$ such that

$$\forall t \in [0, +\infty), d(r_1(t), r_2(t)) < A. \tag{1}$$

Similarly for geodesic lines. For a point $\xi \in \partial\widetilde{S}$ (and having fixed a base point in $\widetilde{S}$) we write $r \in \xi$ to indicate that the geodesic ray r belongs to the class of rays corresponding to ξ, that is, $r(\infty) = \xi$. We also say that ξ is the positive point of r.

Similarly, for a pair (η, ξ) of points in $\partial\widetilde{S}$ with $\eta \neq \xi$ we write $\gamma \in (\eta, \xi)$ to indicate that the geodesic line γ belongs to the class of lines with the property $\gamma(-\infty) = \eta$ and $\gamma(\infty) = \xi$. We say that ξ is the positive point of γ and η the negative.

By writing that the sequence $\{\xi_n\} \subset \partial\widetilde{S}$ converges to ξ in the visual metric, notation $\xi_n \to \xi$, we mean that there exist geodesic rays $r_n \in \xi_n$ and $r \in \xi$ such that the sequence $\{r_n\}$ converges in the usual uniform sense on compact sets to r.

The following example demonstrates a simple case where lifts of distinct closed geodesic (as well as non-closed geodesics) have the same negative and positive points in $\partial\widetilde{S}$.

Example 5 *Consider the genus 0 surface Σ obtained from the flat figures $ACEBZDA$ and $AC'E'B'Z'D'A$ by identifying AB_2C with $AB_2'C'$, AB_1D with $AB_1'D'$ and EBZ with $E'B'Z'$ (see Figure 1). The resulting cylinder Σ has two singular points A, B with angles $\theta(A) = \pi$ and $\theta(B) = 3\pi$. The segments BB_1 and $B_1'B'$ give rise to a simple closed geodesic σ in Σ. Similarly, the segments*

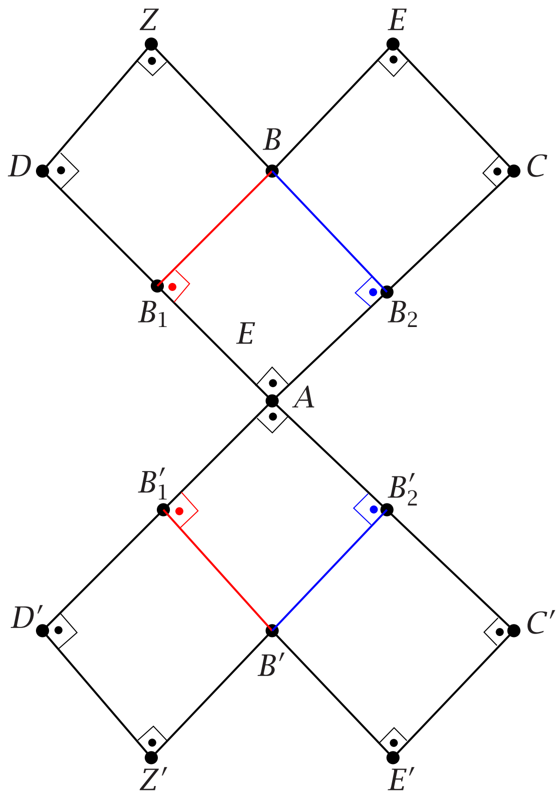

Figure 1: The surface Σ with two conical points of angle π and 3π.

BB_2 and $B_2'B'$ give rise to a simple closed geodesic τ in Σ. Both σ and τ contain B and their union bounds a convex subset of Σ with the same homotopy type as Σ.

Since Σ has geodesic boundaries, the described example can clearly occur in surfaces of any genus. Pick a lift $\tilde{\sigma}$ of σ in $\tilde{\Sigma}$. Then there is a countable number of points $\widetilde{B_i}$, $i \in \mathbb{Z}$, with the properties $\widetilde{B_i} \in \operatorname{Im}\tilde{\sigma}$ and $p\left(\widetilde{B_i}\right) = B$. Clearly, any lift $\tilde{\tau}$ of τ containing $\widetilde{B_{i_0}}$ for some i_0 must contain $\widetilde{B_i}$ for all i and, moreover, $\tilde{\tau}(+\infty) = \tilde{\sigma}(+\infty)$ and $\tilde{\tau}(-\infty) = \tilde{\sigma}(-\infty)$. Therefore, using σ and τ we may construct countably many pairwise distinct closed geodesics in Σ, as well as uncountably many non-closed geodesics, whose lifts in $\tilde{\Sigma}$ are contained in $\operatorname{Im}\tilde{\tau} \cup \operatorname{Im}\tilde{\sigma}$ and they all share the same positive (resp. negative) point $\tilde{\tau}(+\infty) = \tilde{\sigma}(+\infty)$ (resp. $\tilde{\tau}(-\infty) = \tilde{\sigma}(-\infty)$).

The *limit set* Λ of $\pi_1(S)$ is defined to be $\Lambda = \overline{\pi_1(S)x} \cap \partial\tilde{S}$, where x is an arbitrary point in $\tilde{S}$. Since the action of $\pi_1(S)$ on $\tilde{S}$ is co-compact, it is a well known fact that $\Lambda = \partial\tilde{S}$, and hence $\Lambda = \mathbb{S}^1$. Note that the action of $\pi_1(S)$ on $\tilde{S}$ can be extended to $\partial\tilde{S}$ and that the action of $\pi_1(S)$ on $\partial\tilde{S} \times \partial\tilde{S}$ is given by the product action.

Denote by F_h the set of points in $\partial\tilde{S}$ which are fixed by hyperbolic elements of $\pi_1(S)$. Since $\Lambda = \partial\tilde{S}$, the following three results can be derived from [3].

Proposition 6 *The set F_h is $\pi_1(S)-$invariant and dense in $\partial\tilde{S}$.*

Proposition 7 *There exists an orbit of $\pi_1(S)$ dense in $\partial\tilde{S} \times \partial\tilde{S}$.*

Proposition 8 *The set $\{(\phi(+\infty), \phi(-\infty)) : \phi \in \pi_1(S)\}$ is dense in $\partial\tilde{S} \times \partial\tilde{S}$.*

3 Density in $\partial\tilde{S}$.

In this section we first show that the set of points in $\partial\tilde{S}$ for which the class of the corresponding geodesic rays is not a singleton, forms a dense subset of $\partial\tilde{S}$. We fix throughout a base point $\widetilde{x_0} \in \tilde{S}$.

Proposition 9 *The set*

$$Y = \left\{ \xi \in \partial\tilde{S} \mid \begin{array}{c} \exists \text{ distinct geodesic rays } r_1, r_2 \text{ such that} \\ r_1(0) = \widetilde{x_0} = r_2(0) \text{ and } r_1(\infty) = \xi = r_2(\infty) \end{array} \right\}$$

is dense in $\partial\tilde{S}$.

Proof. As $\partial\widetilde{S}$ is homeomorphic to $\mathbb{S}^1$ we will be talking about intervals in $\partial\widetilde{S}$ and we will mean open (resp. closed) connected subsets of $\partial\widetilde{S}$ homeomorphic to open (resp. closed) intervals in $\mathbb{S}^1$. It suffices to show that for any interval $I \subset \partial\widetilde{S}$, $I \cap Y \neq \emptyset$.
Claim: Let $\xi \notin Y$, r_ξ the (unique) geodesic ray with $r_\xi(0) = \widetilde{x}_0$, $r_\xi(+\infty) = \xi$ and r an arbitrary geodesic ray with $r(0) = \widetilde{x}_0$ and $r(+\infty) \neq \xi$. Then $\operatorname{Im} r_\xi \cap \operatorname{Im} r$ is either, a geodesic segment of the form $[\widetilde{x}_0, \widetilde{x}_1]$ for some $\widetilde{x}_1 \in \operatorname{Im} r_\xi$ or, a singleton namely $\{\widetilde{x}_0\}$.
Similarly, if r is an arbitrary geodesic segment then $\operatorname{Im} r_\xi \cap \operatorname{Im} r$ is either, a geodesic sub-segment of r or, a singleton or, the empty set.

For the proof of the Claim observe that $\operatorname{Im} r_\xi \cap \operatorname{Im} r$ is necessarily connected. For, if x, y belong to distinct connected components of $\operatorname{Im} r_\xi \cap \operatorname{Im} r$ then $r|_{[x,y]}$ does not coincide with $r_\xi|_{[x,y]}$. Thus, the geodesic ray

$$r' = r_\xi|_{[\widetilde{x}_0, x]} \cup r|_{[x,y]} \cup r_\xi|_{[y,+\infty]}$$

is distinct from r_ξ and, clearly, $r'(+\infty) = \xi$, a contradiction. As both $\operatorname{Im} r_\xi, \operatorname{Im} r$ are homeomorphic to $[0, +\infty)$, the Claim follows. The proof in the case r is a geodesic segment is similar.

Returning to the proof of the Proposition, suppose, on the contrary, that for some closed interval $[\eta, \rho] \subset \partial\widetilde{S}$ we have $[\eta, \rho] \cap Y = \emptyset$. By the Claim, $\operatorname{Im} r_\eta \cap \operatorname{Im} r_\rho = \{\widetilde{x}_0\}$ or, $[\widetilde{x}_0, \widetilde{x}_1]$. We may assume that $\operatorname{Im} r_\eta \cap \operatorname{Im} r_\rho = \{\widetilde{x}_0\}$ otherwise, replace in the sequel the union $\operatorname{Im} r_\eta \cup \operatorname{Im} r_\rho$ by

$$\operatorname{Im} r_\eta \cup \operatorname{Im} r_\rho \setminus [\widetilde{x}_0, \widetilde{x}_1).$$

Then, inside the compact, convex set $\widetilde{S} \cup \partial\widetilde{S}$ the union

$$\operatorname{Im} r_\eta \cup \operatorname{Im} r_\rho \cup [\eta, \rho]$$

splits the set $\widetilde{S} \cup \partial\widetilde{S}$ into two convex (closed) subsets whose common boundary is the union $\operatorname{Im} r_\eta \cup \operatorname{Im} r_\rho \cup \{\eta, \rho\}$. Observe that convexity follows from the above Claim and the assumption $[\eta, \rho] \cap Y = \emptyset$. Denote by $\widetilde{S}([\eta, \rho])$ the subset of $\widetilde{S} \cup \partial\widetilde{S}$ which contains $[\eta, \rho]$, Pick and fix a conical point $\widetilde{s}$ in the interior of $\widetilde{S}([\eta, \rho])$ with $\theta(\widetilde{s}) < 2\pi$.

For each $\xi \in [\eta, \rho]$, consider the (unique, as $[\eta, \rho] \cap Y = \emptyset$) geodesic ray r_ξ with $r_\xi(0) = x_0$ and $r_\xi(+\infty) = \xi$. By the Claim, as above, $\operatorname{Im} r_\xi \cup \{\xi\}$ splits $\widetilde{S}([\eta, \rho])$ into two closed convex subsets $\widetilde{S}([\eta, \xi])$ and $\widetilde{S}([\xi, \rho])$, the former containing η and the latter containing ρ, whose intersection is $\operatorname{Im} r_\xi \cup \{\xi\}$. Define

$$I(\eta) = \left\{\xi \in [\eta, \rho] \mid \widetilde{s} \notin \widetilde{S}([\eta, \xi])\right\}.$$

Similarly, define $I(\rho)$. We will show that $I(\eta), I(\rho)$ are closed and disjoint subsets of $[\eta, \rho]$, thus contradicting the connectedness of $[\eta, \rho]$.

Let $\xi \in I(\eta) \cap I(\rho)$. Then, by definition, $\tilde{s} \in \tilde{S}([\eta, \xi]) \cap \tilde{S}([\xi, \rho])$. It follows that $\tilde{s} \in \operatorname{Im} r_\xi$ which contradicts Lemma 1. This shows that $I(\eta), I(\rho)$ are disjoint.

To see that $I(\eta)$ is closed, let $\{\xi_n\} \subset I(\eta)$ be a sequence converging to ξ with r_{ξ_n}, r_ξ the corresponding (unique) geodesic rays with positive points ξ_n, ξ. We want to show that $\xi \in I(\eta)$. Assume, on the contrary, that $\xi \in I(\rho)$, i.e $\tilde{s} \notin \tilde{S}([\xi, \rho])$. Let $[x_\eta, \tilde{s}]$ (resp. $[x_\rho, \tilde{s}]$) be geodesic segments realizing the distance, say d_η (resp. d_ρ) of $\tilde{s}$ form $\operatorname{Im} r_\eta$ (resp. $\operatorname{Im} r_\rho$) for some point $x_\eta \in \operatorname{Im} r_\eta$ (resp. $x_\rho \in \operatorname{Im} r_\rho$). Consider the following neighborhood $\mathcal{N}$ of geodesic rays around r_ξ determined by the positive number $C/2$ (cf. Lemma 1) and the compact set $[0, 2(d_\eta + d_\rho)]$:

$$\mathcal{N} = \{r \mid r(0) = \tilde{x}_0 \text{ and } \forall t \in [0, 2(d_\eta + d_\rho)], d(r(t), r_\xi(t)) < C/2\}.$$

Note that if $r \in \mathcal{N}$, then $\operatorname{Im} r|_{[2(d_\eta + d_\rho), +\infty)}$ does not intersect the union of segments $[x_\eta, \tilde{s}] \cup [x_\rho, \tilde{s}]$, otherwise, r would not be a (global) geodesic ray. Clearly, for all n large enough, $r_{\xi_n} \in \mathcal{N}$, thus, $r_{\xi_n}(t) \notin [x_\eta, \tilde{s}] \cup [x_\rho, \tilde{s}]$ for all $t > 2(d_\eta + d_\rho)$, which implies that $\tilde{s} \notin \tilde{S}([\xi_n, \rho])$. In other words, $\xi_n \in I(\rho)$, a contradiction. ■

We now show the analogous result for geodesic lines. We write $\partial^2 \tilde{S}$ for the product $\partial \tilde{S} \times \partial \tilde{S}$ with the diagonal excluded.

Proposition 10 *The set*

$$Z = \left\{ (\eta, \xi) \in \partial^2 \tilde{S} \;\middle|\; \begin{array}{c} \exists \text{ distinct geodesic lines } \gamma_1, \gamma_2 \text{ such that} \\ \gamma_1(-\infty) = \eta = \gamma_2(-\infty) \text{ and } \gamma_1(\infty) = \xi = \gamma_2(\infty) \end{array} \right\}$$

is dense in $\partial^2 \tilde{S}$.

Proof. It suffices to show that for arbitrary $\xi_0 \in \partial \tilde{S}$ and any closed interval $[\eta, \rho] \subset \partial \tilde{S}$ with $\xi_0 \notin [\eta, \rho]$, there exist two distinct geodesics γ_1, γ_2 with $\gamma_1(-\infty) = \xi_0 = \gamma_2(-\infty)$ and $\gamma_1(+\infty) = \xi = \gamma_2(+\infty)$. For, if $\forall \xi \in [\eta, \rho]$ there exist a unique geodesic line γ_ξ with $\gamma_\xi \in (\xi_0, \xi)$ we may repeat the argument in the proof of the previous proposition as follows: pick and fix a singular point $\tilde{s}$ with $\theta(\tilde{s}) < 2\pi$ in the interior of the (convex) set $\tilde{S}([\eta, \rho])$ bounded by the geodesic lines joining the pairs (ξ_0, η) and (ξ_0, ρ). For each $\xi \in [\eta, \rho]$, consider the (unique, as $[\eta, \rho] \cap Z = \emptyset$) geodesic γ_ξ with $\gamma_\xi(-\infty) = \xi_0$ and $\gamma_\xi(+\infty) = \xi$. By the Claim, as above, $\operatorname{Im} \gamma_\xi \cup \{\xi\}$ splits $\tilde{S}([\eta, \rho])$ into two closed convex subsets $\tilde{S}([\eta, \xi])$ and $\tilde{S}([\xi, \rho])$, the former containing η and the latter containing ρ, whose intersection is $\operatorname{Im} \gamma_\xi \cup \{\xi_0, \xi\}$. Define

$$I(\eta) = \left\{ \xi \in [\eta, \rho] \mid \tilde{s} \notin \tilde{S}([\eta, \xi]) \right\}.$$

Similarly, define $I(\rho)$. Then, as above, $I(\eta), I(\rho)$ are closed and disjoint subsets of $[\eta, \rho]$, a contradiction.

Proof of $I(\eta)$ closed: Let $\{\xi_n\} \subset I(\eta)$ be a sequence converging to ξ with $\gamma_{\xi_n}, \gamma_\xi$ the corresponding (unique) geodesic lines with $\gamma_{\xi_n}(-\infty) = \xi_0 = \gamma_\xi(-\infty)$ and $\gamma_{\xi_n}(\infty) = \xi_n$, $\gamma_\xi(\infty) = \xi$. Pick a parametrization for γ_ξ, for example, set $\gamma_\xi(0)$ to be a point of minimal distance from $\widetilde{s}$ and assume, on the contrary, that $\xi \in I(\rho)$.

As above, let $[x_\eta, \widetilde{s}]$ (resp. $[x_\rho, \widetilde{s}]$) be geodesic segments realizing the distance d_η (resp. d_ρ) of $\widetilde{s}$ from $\operatorname{Im}\gamma_\eta$ (resp. $\operatorname{Im}\gamma_\rho$) for some point $x_\eta \in \operatorname{Im}\gamma_\eta$ (resp. $x_\rho \in \operatorname{Im}\gamma_\rho$). Consider the neighborhood $\mathcal{N}$ of geodesic lines around γ_ξ determined by the positive number $C/2$ (cf. Lemma 1) and the compact set $K = [-2(d_\eta + d_\rho), 2(d_\eta + d_\rho)]$. Clearly, if $\gamma \in \mathcal{N}$ then $\gamma|_{\mathbb{R}\setminus K}$ does not intersect the union of segments $[x_\eta, \widetilde{s}] \cup [x_\rho, \widetilde{s}]$. The same holds for γ_n, n large enough, thus, $\widetilde{s} \notin \widetilde{S}([\gamma_n(+\infty), \rho])$, equivalently, $\xi_n \in I(\rho)$ a contradiction. ■

We conclude this Section with the following proposition which indicates that uniqueness of geodesic rays is a property which depends on the choice of base point.

Proposition 11 *Let $\xi \in \partial\widetilde{S}$ be arbitrary. Then for some point $x \in \widetilde{S}$ there exist at least two geodesic rays r_1, r_2 such that $r_1(0) = r_2(0) = x$ and $r_1(+\infty) = r_2(+\infty) = \xi$.*

Proof. Assume, on the contrary, that for every point $x \in \widetilde{S}$ there exists exactly one geodesic ray, denoted by r_x, with $r_x(0) = x$ and $r_x(+\infty) = \xi$. Observe that for two arbitrary distinct geodesic rays r_1, r_2 with $r_1(+\infty) = r_2(+\infty) = \xi$ (then, by assumption, $r_1(0), r_2(0)$ must be distinct) we have that

$$\operatorname{Im} r_1 \cap \operatorname{Im} r_2 \text{ is either a geodesic sub} - \text{ray of both or, } \emptyset. \tag{2}$$

otherwise for a base point in the intersection we would have two distinct geodesic rays corresponding to ξ.

Fix a point $\widetilde{s}_0$ where $s_0 = p(\widetilde{s}_0)$ is a conical point with angle $\theta(s_0) < 2\pi$. Let $D(\widetilde{s}_0, \varepsilon)$ be a closed disk of radius $\varepsilon > 0$ not containing any conical point except $\widetilde{s}_0$. The geodesic ray $r_{\widetilde{s}_0}$ intersects $\partial\overline{D}(\widetilde{s}_0, \varepsilon)$ at a single point denoted $\widetilde{x}_0$. We will reach a contradiction by defining a continuous surjective map from $\partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ to a space $\{+, -\}$ consisting of two points.

Let A be a positive number, see property (1) above, such that for any $x \in D(\widetilde{s}_0, \varepsilon)$ the (unique) geodesic ray r_x satisfies

$$\forall t \in [0, +\infty), d(r_{\widetilde{s}_0}(t), r_x(t)) < A.$$

Observe that we may adjust ε so that $\varepsilon < A$.

Let $D(r_{\widetilde{s}_0}(3A), A)$ be the closed disk of radius A centered at $r_{\widetilde{s}_0}(3A)$. Then, the set

$$D(r_{\widetilde{s}_0}(3A), A) \setminus \operatorname{Im} r_{\widetilde{s}_0}$$

consists of two connected components. Using the orientation of $r_{\tilde{s}_0}$ we may mark these components by saying that the component to the right is the positive component and the one to the left the negative, notation D^+ and D^- respectively. In order to define a map

$$R : \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\} \to \{+, -\}$$

we will distinguish 3 cases for each point $x \in \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$:
Case I: $\operatorname{Im} r_x \cap \operatorname{Im} r_{\tilde{s}_0} = \emptyset$
Case II: $\operatorname{Im} r_x \cap \operatorname{Im} r_{\tilde{s}_0} \neq \emptyset$ and the unique time $t_x \in [0, +\infty)$ so that $\operatorname{Im} r_{\tilde{s}_0}|_{[t_x, +\infty]} \subset \operatorname{Im} r_x$, which exists by (2), satisfies $t_x > 2A$.
Case III: as in Case II with $t_x \leq 2A$.

Observe that in Case III $\operatorname{Im} r_x$ intersects neither D^+ nor D^-. Clearly, in Cases I and II $\operatorname{Im} r_x$ intersects at least one component D^+, D^-. We claim that, in Cases I and II, $\operatorname{Im} r_x$ cannot intersect both components D^+ and D^-. To see this, let $\sigma : [a, b] \to \tilde{S}$ be a curve with the properties $\sigma(a) \in D^+, \sigma(b) \in D^-$ and $\operatorname{Im} \sigma \cap \operatorname{Im} r_{\tilde{s}_0} = \emptyset$. By standard triangle inequality arguments it follows that $lenght(\sigma) > 2A$. If $\operatorname{Im} r_x$ intersected both D^+, D^- then, being a geodesic, it must intersect $r_{\tilde{s}_0}$ transversely, a contradiction according to the assumptions in Case I and II. Thus, it follows (in Case I and II) that either $\operatorname{Im} r_x \cap D^+ \neq \emptyset$ or $\operatorname{Im} r_x \cap D^- \neq \emptyset$ but not both. For $x \in \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$ whose geodesic ray r_x falls into Case I or II we may now define

$R(x) := +$ if $\operatorname{Im} r_x \cap D^+ \neq \emptyset$ and

$R(x) := -$ if $\operatorname{Im} r_x \cap D^- \neq \emptyset$.

Let now $x \in \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$ so that r_x falls in to Case III. It is easy to see that $t_x > \varepsilon$. For, if $0 < t_x \leq \varepsilon$ then $r_{\tilde{s}_0}(t_x) \in D(\tilde{s}_0, \varepsilon)$ which is impossible because $r_{\tilde{s}_0}(t_x)$ is a conical point and ε is chosen so that $\tilde{s}_0$ is the unique conical point in $D(\tilde{s}_0, \varepsilon)$. If $t_x = 0$, then the conical point $\tilde{s}_0 = r_{\tilde{s}_0}(0)$ of angle $< 2\pi$ lies on the geodesic ray r_x, a contradiction by Lemma 1. Thus, $t_x > \varepsilon$ and there exists $\delta > 0$ sufficiently small so that the disk $D(r_{\tilde{s}_0}(t_x), \delta)$ does not contain $\tilde{x}_0 = r_{\tilde{s}_0}(\varepsilon)$. This disk $D(r_{\tilde{s}_0}(t_x), \delta)$ can be used to define $R(x)$ as above: $\operatorname{Im} r_x$ intersects exactly one of the two oriented components of $D(r_{\tilde{s}_0}(t_x), \delta) \setminus \operatorname{Im} r_{\tilde{s}_0}$ and define $R(x)$ accordingly.

We show that R is continuous. Let $\{x_n\}$ be a sequence in $\partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$ converging to a point x. The sequence of geodesic rays $\{r_{x_n}\}$ converges, up to a subsequence, to a geodesic ray q_x emanating from x. Since $r_{x_n}(+\infty) = \xi$ for all n, it follows that $q_x(+\infty) = \xi$. By assumption of uniqueness of geodesic rays we have $q_x = r_x$. Thus, $r_{x_n} \to r_x$ uniformly on compact sets. Without loss of generality we may assume that $R(x) = +$.

First assume that the geodesic ray r_x falls into Case I or II, that is, $\operatorname{Im} r_x \cap D^+ \neq \emptyset$. Since $r_{x_n} \to r_x$ uniformly on compact sets it follows that there exists N so that

$$\forall n \geq N, \operatorname{Im} r_{x_n} \cap D^+ \neq \emptyset$$

which means that $R(x_n) = +, \forall n \geq N$. Case II is treated similarly. This shows that R is continuous.

We show that R is onto. We may choose a sequence $\{x_n\} \subset \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$ converging to $\tilde{x}_0$ from the right in the following sense: for all sufficiently small $\delta > 0$ the set $D(\tilde{x}_0, \delta) \setminus \operatorname{Im} r_{\tilde{s}_0}$ consists of two connected components. We mark them as right (positive) and left (negative) according to the positive direction of $r_{\tilde{s}_0}$. We say that a sequence $\{x_n\} \subset \partial D(\tilde{s}_0, \varepsilon) \setminus \{\tilde{x}_0\}$ converges to $\tilde{x}_0$ from the right if x_n belongs to the right (positive) component of $D(\tilde{x}_0, \delta) \setminus \operatorname{Im} r_{\tilde{s}_0}$ for all but finitely any n. Clearly, for such a sequence $\{x_n\}$, the corresponding geodesic rays $r_{x_n} \to r_{\tilde{s}_0}|_{[\varepsilon,+\infty)}$ uniformly on compact sets. Choose $\varepsilon_1 < \varepsilon$ and set

$$\mathcal{N}(\varepsilon_1) = \left\{ y \in \tilde{S} \mid \exists t \in [\varepsilon, +\infty) : d(y, r_{\tilde{s}_0}(t)) < \varepsilon_1 \right\}.$$

As above, $\mathcal{N}(\varepsilon_1) \setminus \operatorname{Im} r_{\tilde{s}_0}$ consists of two components $\mathcal{N}^+$and $\mathcal{N}^-$. Pick a sequence $x_n \to \tilde{x}_0$ from the right. Then, for all n large enough, $x_n \in \mathcal{N}^+$ and by the uniform convergence of r_{x_n}, $\operatorname{Im} r_{x_n} \subset \mathcal{N}^+$. This implies that $R(x_n) = +$ for all large enough n. Similarly we show that R attains the value $-$. ■

4 Density of closed geodesics

We begin by showing that each class of geodesic rays (resp. lines) with the same boundary point at infinity contains a leftmost and a rightmost geodesic ray (resp. line) which bound a convex set containing the image of any other geodesic ray (resp. line) in the same class.

Proposition 12 *For every point $\xi \in \partial\tilde{S}$, there exist two geodesic rays r_L, r_R with $r_L(\infty) = \xi = r_R(\infty)$ and whose images bound a convex subset $\tilde{S}(\xi)$ of $\tilde{S}$ with the property*

$$\operatorname{Im} r \subset \tilde{S}(\xi) \text{ for all geodesic rays } r \text{ with } r(\infty) = \xi.$$

Proof. Observe that if the class of geodesic rays corresponding to ξ is a singleton, the above Proposition holds trivially with $r_L = r_R$ being the unique geodesic ray with positive point ξ.

Let A be the number posited in equation (1). Denote by $C(\tilde{x}_0, m)$ (resp. $D(\tilde{x}_0, m)$) the circle (resp. closed disc) of radius m centered at $\tilde{x}_0$. For each large enough $N \in \mathbb{N}$, the set

$$\xi(N) = \{r(N) \mid r \in \xi\}$$

is contained in an interval $I_{N,\xi}$ of diameter A inside the circle $C(\tilde{x}_0, N)$. For large enough N we may orient $I_{N,\xi}$ and speak of its left and right endpoint.

We claim that $\xi(N)$ is a closed set. To see this let $\{y_n\}$ be a sequence of points in $\xi(N)$ converging to $y \in I_{N,\xi}$. By definition of $\xi(N)$, for each y_n there exists a geodesic ray $r_n \in \xi$ (not necessarily unique) such that $r_n(N) = y_n$. By passing to a subsequence, if necessary, $\{r_n\}$ converges to a geodesic ray r and, clearly, $r \in \xi$. As $y_n \to y$, y must belong to $\operatorname{Im} r$ and, on the other hand, $y \in I_{N,\xi} \subset C(\tilde{x}_0, N)$. Thus, $y = r(N)$ which shows that $\xi(N)$ is closed.

By compactness, the leftmost and rightmost points of $\xi(N)$ inside $I_{N,\xi}$, denoted by y_L and y_R respectively, exist. As the number of conical points in $D(\tilde{x}_0, N)$ is finite, we may choose (cf Lemma 2) geodesic segments $\sigma_{L,N}$ and $\sigma_{R,N}$ with endpoints $\tilde{x}_0, y_L$ and $\tilde{x}_0, y_R$ respectively, satisfying the following property:

- the convex subset of $D(\tilde{x}_0, N)$ bounded by the union
$$\sigma_{L,N} \cup [y_L, y_R] \cup \sigma_{R,N} \tag{3}$$
where $[y_L, y_R]$ indicates the subinterval of $C(\tilde{x}_0, N)$ containing $\xi(N)$, contains all geodesic segments $r|_{[0,N]}$ for all $r \in \xi$.

The segment $\sigma_{L,N}$ (and similarly for $\sigma_{R,N}$) can be obtained by starting with a geodesic segment $\sigma'_{L,N}$ with endpoints $\tilde{x}_0, y_L$ and then if a geodesic ray intersects the segment $\sigma'_{L,N}$, it must do so at pairs of (conical) points (otherwise, the property of y_L being leftmost would be violated). As the intersection points are conical points, they are finitely many pairs of intersection points so we may replace (see Lemma 2) finitely many parts of the segment $\sigma'_{L,N}$ to obtain $\sigma_{L,N}$.

The sequences $\{\sigma_{L,N}\}_{N\in\mathbb{N}}$, $\{\sigma_{R,N}\}_{N\in\mathbb{N}}$ converge to geodesic rays $r_L, r_R \in \xi$ respectively. The required property in the statement of the Proposition for the convex set $\tilde{S}(\xi)$ bounded by $\operatorname{Im} r_L$, $\operatorname{Im} r_R$, now follows: for, if $r \in \xi$ with $\operatorname{Im} r \nsubseteq \tilde{S}(\xi)$ then, for some $M > 0$, $r(M) \notin \tilde{S}(\xi)$. Assume that the distance $d\left(r(M), \tilde{S}(\xi)\right) = C_0 > 0$ of $r(M)$ from $\tilde{S}(\xi)$ is realized by a point on $\operatorname{Im} r_R$. Then, for a compact set $K \supset [0, M]$ and the positive number $C_0/2$ there exist N_0 so that

$$d(r_R(t), \sigma_{R,N}(t)) < C_0/2 \text{ for all } t \in K \text{ and for all } N > N_0.$$

We may assume that N_0 satisfies $N_0 > [M] + 1$. It follows that $r(M)$ does not belong to the convex subset of $D(\tilde{x}_0, N_0)$ bounded by the union

$$\sigma_{L,N_0} \cup [y_L, y_R] \cup \sigma_{R,N_0}$$

contradicting (3). ■

In view of the above Proposition we introduce the following

Terminology: For each $\xi \in \partial\tilde{S}$, the geodesic rays posited in the above proposition will be called leftmost and rightmost geodesic rays in the class of ξ and will be denoted by $r_{L,\xi}$ and $r_{R,\xi}$ respectively.

Proposition 13 *For every pair of points $\eta, \xi \in \partial\widetilde{S}$ with $\eta \neq \xi$, there exist two geodesic lines $\gamma_L, \gamma_R \in (\eta, \xi)$, that is, $\gamma_L(-\infty) = \eta = \gamma_R(-\infty)$ and $\gamma_L(\infty) = \xi = \gamma_R(\infty)$, whose images bound a convex subset $\widetilde{S}(\eta, \xi)$ of $\widetilde{S}$ with the property*

$$\operatorname{Im}\gamma \subset \widetilde{S}(\eta, \xi) \text{ for all geodesic lines } \gamma \in (\eta, \xi).$$

Proof. The line of proof is similar to the previous proposition, however, we include it here because certain modifications are needed.

We may assume that there exist at least two geodesics in the class of (η, ξ), otherwise the statement is trivial. Moreover, each $\gamma \in (\eta, \xi)$ is considered oriented with positive the direction from η to ξ and then the left and right component of $\partial\widetilde{S} \setminus \{\eta, \xi\}$ is determined. Pick a base point $\tilde{x}_0$ on the image of an arbitrary $\gamma_0 \in (\eta, \xi)$ and set $\gamma_0(0) = \tilde{x}_0$. For large enough $N \in \mathbb{N}$, we may find intervals

$$I_N^+(\eta, \xi) := [\gamma_0(N) - A, \gamma_0(N) + A] \subset C(\tilde{x}_0, N)$$
$$I_N^-(\eta, \xi) := [\gamma_0(-N) - A, \gamma_0(-N) + A] \subset C(\tilde{x}_0, N)$$

where A is the constant posited in (1), with the property

$$\text{for all } \gamma \in (\eta, \xi), \operatorname{Im}\gamma \cap C(\tilde{x}_0, N) \subset I_N^+(\eta, \xi) \cup I_N^+(\eta, \xi).$$

For each γ in (η, ξ), the intersection $\operatorname{Im}\gamma \cap I_N^+(\eta, \xi)$ (resp. $I_N^-(\eta, \xi)$) is not necessarily a singleton. However, there exist unique numbers $t_{\gamma,N}^-, t_{\gamma,N}^+ \in \mathbb{R}$ such that

$$\gamma\left(t_{\gamma,N}^-\right) \in I_N^-(\eta, \xi), \gamma\left(t_{\gamma,N}^+\right) \in I_N^+(\eta, \xi)$$

and $\left|t_{\gamma,N}^- - t_{\gamma,N}^+\right|$ is minimal with respect to the above inclusions. Equivalently, $\gamma|_{\left(t_{\gamma,N}^-, t_{\gamma,N}^+\right)} \cap C(\tilde{x}_0, N) = \varnothing$. Set

$$\xi^+(N) = \left\{\gamma\left(t_{\gamma,N}^+\right) \mid \gamma \in (\eta, \xi)\right\}$$

and similarly for $\xi^-(N)$. We claim that both sets $\xi^+(N), \xi^-(N)$ are closed. To see this let $\{y_n\}$ be a sequence of points in $\xi^+(N)$ converging to $y \in I_N^+(\eta, \xi)$. By definition of $\xi^+(N)$, for each y_n there exists a geodesic $\gamma_n \in (\eta, \xi)$ (not necessarily unique) such that $\gamma_n\left(t_{\gamma_n,N}^+\right) = y_n$. By passing to a subsequence, if necessary, $\{\gamma_n\}$ converges to a geodesic γ'. Clearly, $\gamma' \in (\eta, \xi)$ and y must belong to $\operatorname{Im}\gamma' \cap I_N^+(\eta, \xi)$. In order to complete the proof that $\xi^+(N)$ is closed we need to show that $y = \gamma'\left(t_{\gamma',N}^+\right)$.

We may assume that N is large enough so that

$$\left|t_{\gamma_n,N}^- - t_{\gamma_n,N}^+\right| > 2A$$

for all n. Observe that for every $\varepsilon > 0$ which belongs to the interval $\left(0, \left|t^-_{\gamma_n,N} - t^+_{\gamma_n,N}\right|\right)$ for all n, the sequence $\left\{\gamma_n\left(t^+_{\gamma_n,N} - \varepsilon\right)\right\}_{n\in\mathbb{N}}$ converges to the interior of the disk $D\left(\tilde{x}_0, N\right)$. Therefore, all points on $\operatorname{Im}\gamma'|_{(\eta,y]}$ of distance $\varepsilon < 2A$ from y belong to the interior of the disk $D\left(\tilde{x}_0, N\right)$. It follows that $y = \gamma'\left(t^+_{\gamma',N}\right)$ which shows that $\xi^+(N)$ is closed. Similarly we show that $\xi^-(N)$ is closed.

Denote by y^+_L (resp. y^+_R) be the leftmost (resp. rightmost) point of $\xi^+(N)$ in $I^+_N(\eta,\xi)$ and y^-_L (resp. y^-_R) be the leftmost (resp. rightmost) point of $\xi^-(N)$ in $I^-_N(\eta,\xi)$ all for which exist by compactness. We may construct a rightmost geodesic segment $\sigma_{R,N} = \left[y^-_R, y^+_R\right]$ in $D\left(\tilde{x}_0, N\right)$ with endpoints y^-_R, y^+_R and a leftmost geodesic segment $\sigma_{L,N} = \left[y^-_L, y^+_L\right]$ in $D\left(\tilde{x}_0, N\right)$ with endpoints y^-_L, y^+_L so that the following property holds:

- the convex subset of $D\left(\tilde{x}_0, N\right)$ bounded by the union

$$\sigma_{L,N} \cup \left[y^+_L, y^+_R\right] \cup \sigma_{R,N} \cup \left[y^-_R, y^+_R\right]$$

  where $\left[y^+_L, y^+_R\right]$ (resp. $\left[y^-_L, y^-_R\right]$) indicates the subinterval of $C\left(\tilde{x}_0, N\right)$ containing $\xi^+(N)$ (resp. $\xi^-(N)$), contains all segments $\operatorname{Im}\gamma \cap D\left(\tilde{x}_0, N\right)$ for all $\gamma \in (\eta,\xi)$.

The segment $\sigma_{L,N}$ (and similarly for $\sigma_{R,N}$) can be obtained by starting with a geodesic segment $\sigma'_{L,N}$ with endpoints y^-_L, y^+_L and then if a geodesic line intersects the segment $\sigma'_{L,N}$, it must do so at pairs of (conical) points (otherwise, the property of y^+_L, y^-_L being leftmost would be violated). As the intersection points are conical points, they are finitely many so we may replace (see Lemma 2) finitely many parts of the segment $\sigma'_{L,N}$ to obtain $\sigma_{L,N}$.

As in the proof of the previous proposition we obtain the desired geodesic lines as limits of the sequences $\{\sigma_{L,N}\}_{N\in\mathbb{N}}$ and $\{\sigma_{R,N}\}_{N\in\mathbb{N}}$. ■

Terminology: For each $(\eta,\xi) \in \partial^2\tilde{S}$, the geodesic lines posited in the above proposition will be called leftmost and rightmost geodesic lines in the class of (η,ξ) and will be denoted by $\gamma_{L,(\eta,\xi)}$ and $\gamma_{R,(\eta,\xi)}$ respectively.

Theorem 14 *Closed geodesics are dense in GS in the following sense: for each pair $(\eta,\xi) \in \partial^2\tilde{S}$ there exists a sequence of geodesics $\{c_n\}$ such that $c_n \to \gamma_{L,(\eta,\xi)}$ in the usual uniform sense on compact sets and $p(c_n)$ is a closed geodesic in S for all n. Similarly for $\gamma_{R,(\eta,\xi)}$.*

Proof. For arbitrary $(\eta,\xi) \in \partial^2\tilde{S}$, we orient as positive the direction from η to ξ and name left and right the components of $\partial\tilde{S} \setminus \{\eta,\xi\}$. We may choose, by Proposition 8, a sequence $\{(\phi_n(-\infty), \phi_n(+\infty))\}_{n\in\mathbb{N}}$ where each ϕ_n is a (hyperbolic) element of $\pi_1(S)$ such that $\phi_n(-\infty) \to \eta$, $\phi_n(\infty) \to \xi$ with the additional property that for all n both $\phi_n(-\infty)$, $\phi_n(\infty)$ belong to the same (say, right) component of $\partial\widetilde{S}\setminus\{\eta,\xi\}$. In particular we have that $\phi_n(-\infty)\neq\eta$ and $\phi_n(\infty)\neq\xi$.

We claim that for each $n\in\mathbb{N}$, there exists a geodesic $c''_n\in(\phi_n(-\infty),\phi_n(+\infty))$, that is, $c''_n(-\infty)=\phi_n(-\infty)$ and $c''_n(\infty)=\phi_n(+\infty)$ whose projection to S is closed. To see this, pick arbitrary $y\in\widetilde{S}$ and consider the geodesic segment $[y,\phi_n(y)]$ which, clearly, projects to a closed curve, say σ_n in S. There exists a length minimizing closed curve in the (free) homotopy class of σ_n (see [5, Ch. 1, Remark 1.13(b)]). By choosing an appropriate lift to $\widetilde{S}$ of this length minimizing closed curve we obtain a geodesic line c''_n such that the set

$$\left\{\phi_n^i(y)\mid i\in\mathbb{Z}\right\}$$

is at bounded distance from Im c''_n. Thus, $c''_n(-\infty)=\phi_n(-\infty)$ and $c''_n(\infty)=\phi_n(+\infty)$ as desired.

Since by construction the projection of c''_n to S is a closed curve, we can speak of the period of c''_n. Let $\gamma_{R,(\eta,\xi)}\in(\eta,\xi)$ be the rightmost geodesic posited in Proposition 13. As $c''_n(-\infty)\neq\eta$ and $c''_n(\infty)\neq\xi$, the intersection

$$\operatorname{Im}c''_n\cap\operatorname{Im}\gamma_{R,(\eta,\xi)}$$

has finitely many components. For each $n\in\mathbb{N}$, consider the geodesic line c'_n having the same image as c''_n and its period is a multiple of the period of c''_n so that

$$\operatorname{Im}c'_n\cap\operatorname{Im}\gamma_{R,(\eta,\xi)}$$

is contained in a single period of c'_n. We may alter c'_n in its (enlarged) period so that it does not intersect the interior of the convex subset $\widetilde{S}(\eta,\xi)$ of $\widetilde{S}$ bounded by Im $\gamma_{L,(\eta,\xi)}$ and Im $\gamma_{R,(\eta,\xi)}$. For such an alteration we only need to modify c'_n in subintervals, say $[z,w]$, of its image contained in $\widetilde{S}(\eta,\xi)$. Namely, we have to replace $c'_n|_{[z,w]}$ by $\gamma_{R,(\eta,\xi)}|_{[z,w]}$. Then, by repeating this alteration we obtain a geodesic line, denoted by c_n, whose projection to S *is* a closed geodesic. Clearly by construction

$c_n(-\infty)=c''_n(-\infty)=\phi_n(-\infty)$, $c_n(\infty)=c''_n(+\infty)=\phi_n(+\infty)$ and as $\operatorname{Im}c_n\cap\widetilde{S}(\eta,\xi)\subset\operatorname{Im}\gamma_{R,(\eta,\xi)}$, it follows that $c_n\to\gamma_{L,(\eta,\xi)}$ uniformly on compact sets. ■

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