PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

ON THE GEODESIC FLOW OF EUCLIDEAN SURFACES WITH CONICAL SINGULARITIES

CHARALAMPOS CHARITOS, IOANNIS PAPADOPERAKIS, AND GEORGIOS TSAPOGAS

(Communicated by)

ABSTRACT. Dynamical properties of closed surfaces equipped with a Euclidean metric with finitely many conical points of angle $> 2\pi$ are studied. It is shown that the geodesic flow is topologically transitive and mixing.

1. Introduction

Topological transitivity and topological mixing of the geodesic flow are two dynamical properties extensively studied for Riemannian manifolds. Anosov in [1] first proved topological transitivity of the geodesic flow for compact manifolds of negative curvature. Eberlein in [12] proved topological mixing for a large class of compact manifolds. In particular, he established topological mixing for compact manifolds of negative curvature as well as for compact manifolds of non-positive curvature not admitting isometric, totally geodesic embedding of $\mathbb{R}^2$. The latter is the class of the so called visibility manifolds (see [14] and [12]) and, in modern terminology, it can equivalently be described as the class of compact $CAT(0)$ manifolds which are hyperbolic in the sense of Gromov (see [3, Ch. II, Theorem 9.33]). For certain classes of quotients of $CAT(-1)$ spaces by discrete groups of isometries, topological mixing was shown in [6].

In all the above mentioned results the following two properties of the universal covering were essential:

- **(U):** uniqueness of geodesic lines joining two boundary points at infinity and
- **(C):** the distance of asymptotic geodesics tends, up to re-parametrization, to zero.

In this work we prove topological mixing for closed surfaces of genus ≥ 2 equipped with a Euclidean metric with finitely many conical singularities of angle $> 2\pi$. For this class of spaces, the universal cover is a $CAT(0)$ space which contains flat strips (which correspond to closed geodesics not containing conical singularities) and, thus, property (U) above does not hold. For the same reason property (C) does not hold either. However, we use the fact (shown in [7]) that in these surfaces, a non-closed geodesic has 0 distance from the set of conical points, in order to show that properties (U) and (C) above do hold when the geodesics involved are not lifts of closed geodesics.

Our approach is along the lines of the original proof in Eberlein's work (cf [12]). However, since we need to generalize from manifolds to more general settings, we

2010 *Mathematics Subject Classification.* Primary 53C22, 57M50.

will use recent terminology adopted in the line of proof in [6] and make adjustments, when needed, to deal with the implications arising from closed geodesics.

2. Definitions and Preliminaries

Let S be a closed surface of genus ≥ 2 equipped with a Euclidean metric with finitely many *conical singularities* (or *conical points*), say $s_1, ..., s_n$. Every point which is not conical will be called a *regular point* of S. Denote by $\theta(s_i)$ the angle at each s_i and we assume throughout this paper that $\theta(s_i) \in (2\pi, +\infty)$.

We write $C(v, \theta)$ for the standard cone with vertex v and angle θ, namely, $C(v, \theta)$ is the set $\{(r, t) : 0 \leq r, t \in \mathbb{R}/\theta\mathbb{Z}\}$ equipped with the metric $ds^2 = dr^2 + r^2 dt^2$.

Definition 2.1. A Euclidean surface with conical singularities $s_1, ..., s_n$ is a closed surface S equipped with a metric $d(\cdot, \cdot)$ such that at

- Every point $p \in S \setminus \{s_1, ..., s_n\}$ has a neighborhood isometric to a disk in the Euclidean plane
- Each $s_i \in \{s_1, ..., s_n\}$ has a neighborhood isometric to a neighborhood of the vertex v of the standard cone $C(v, \theta(s_i))$.

Clearly, the metric on S is a length metric and the surface S will be written *e.s.c.s.* for brevity. Note that for genus $g \geq 2$, such Euclidean structures exist, see [21].

Definition 2.2. A geodesic segment is an isometric map $h : [a, b] \to S$. If $x = h(a)$ and $y = h(b)$ then a geodesic segment joining x and y will be denoted by $[x, y]$ and by (x, y) its interior.
Let $I = [0, +\infty)$ or $I = (-\infty, +\infty)$. A geodesic line (resp. geodesic ray) in S is a local isometric map $h : I \to S$ where $I = (-\infty, +\infty)$ (resp. $I = [0, +\infty)$).
A closed geodesic is a local isometric map $h : I \to S$ which is a periodic map.
A metric space is called geodesic if every two points can be joined by a geodesic segment.

As every locally compact, complete length space is geodesic (see Th. 1.10 in [17]) we immediately have

Proposition 2.3. If S is a *e.s.c.s.* then S is a geodesic space.

Let $\widetilde{S}$ be the universal covering of S and let $p : \widetilde{S} \to S$ be the covering projection. Obviously, the universal covering $\widetilde{S}$ is homeomorphic to $\mathbb{R}^2$ and by requiring p to be a local isometric map we may lift d to a metric $\widetilde{d}$ on $\widetilde{S}$ so that $(\widetilde{S}, \widetilde{d})$ becomes a *e.s.c.s.* There is a discrete group of isometries Γ of $\widetilde{S}$ which is isomorphic to $\pi_1(S)$, acting freely on $\widetilde{S}$ so that $S = \widetilde{S}/\Gamma$. From now on we will make no distinction between Γ and $\pi_1(S)$.

The group Γ with the word metric is *hyperbolic in the sense of Gromov.* On the other hand, Γ acts co-compactly on $\widetilde{S}$; this implies that $\widetilde{S}$ is itself a hyperbolic space in the sense of Gromov (see for example Th. 4.1 in [8]) which is complete and locally compact. Hence, $\widetilde{S}$ is a proper space i.e. each closed ball in $\widetilde{S}$ is compact (see [17] Th. 1.10). Therefore, the visual boundary $\partial\widetilde{S}$ of $\widetilde{S}$ is defined by means of geodesic rays and is homeomorphic to $\mathbb{S}^1$ (see [8], p.19). Furthermore, as all conical singularities in S are assumed to have angle $> 2\pi$, we deduce that S has curvature ≤ 0, that is, S satisfies CAT(0) inequality locally (see for example [18, Theorem 3.15]). Since $\widetilde{S}$ is simply connected it follows that $\widetilde{S}$ satisfies CAT(0) globally, i.e.,

$\widetilde{S}$ is a Hadamard space (for the definition and properties of CAT(0) spaces see [2] and [3]). Note that since $\widetilde{S}$ is a CAT(0) space, geodesic lines and geodesic rays in $\widetilde{S}$ are global isometric maps.

If g is a geodesic, we will denote by $g(+\infty)$ the boundary point determined by the geodesic ray $g|_{[0,+\infty)}$ and similarly for $g(-\infty)$.

In the next proposition we state the following important property of $\partial\widetilde{S}$. For a proof see Proposition 2.1 in [8]. Uniqueness of geodesic rays follows from the fact that $\widetilde{S}$ is a CAT(0) space (see [3, Ch.II, Proposition 8.2]).

Proposition 2.4. For every pair of points $x \in \widetilde{S}$, $\xi \in \partial\widetilde{S}$ (resp. $\eta, \xi \in \partial\widetilde{S}$) there is a geodesic ray $r : [0, \infty) \to \widetilde{S} \cup \partial\widetilde{S}$ (resp. a geodesic line $l : (-\infty, \infty) \to \widetilde{S} \cup \partial\widetilde{S}$) such that, $r(0) = x$, $r(\infty) = \xi$ (resp. $l(-\infty) = \eta$, $l(\infty) = \xi$). Moreover, geodesic rays are unique.

Since $\widetilde{S}$ is a hyperbolic space in the sense of Gromov, the isometries of $\widetilde{S}$ are classified as elliptic, parabolic and hyperbolic [16]. On the other hand, Γ is a hyperbolic group, thus Γ does not contain parabolic elements with respect to its action on its Cayley graph (see Th. 3.4 in [8]). From this, it follows that all elements of Γ are hyperbolic isometries of $\widetilde{S}$. Therefore, for each $\varphi \in \Gamma$ and each $x \in \widetilde{S}$ the sequence $\varphi^n(x)$ (resp. $\varphi^{-n}(x)$) has a limit point $\varphi(+\infty)$ (resp. $\varphi(-\infty)$) when $n \to +\infty$ and $\varphi(+\infty) \neq \varphi(-\infty)$. The point $\varphi(+\infty)$ is called *attractive* and the point $\varphi(-\infty)$ *repulsive* point of φ.

The *limit set* $\Lambda(\Gamma)$ of Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial\widetilde{S}$, where x is an arbitrary point in $\widetilde{S}$. Since the action of Γ on $\widetilde{S}$ is co-compact, it is a well known fact that $\Lambda(\Gamma) = \partial\widetilde{S}$, and hence $\Lambda(\Gamma) = \mathbb{S}^1$. Note that the action of Γ on $\widetilde{S}$ can be extended to $\partial\widetilde{S}$ and that the action of Γ on $\partial\widetilde{S} \times \partial\widetilde{S}$ is given by the product action.

Denote by F_h the set of points in $\partial\widetilde{S}$ which are fixed by (hyperbolic) elements of Γ. Since $\Lambda(\Gamma) = \partial\widetilde{S}$, the following three results can be derived from [9].

Proposition 2.5. The set F_h is $\Gamma-$invariant and dense in $\partial\widetilde{S}$.

Proposition 2.6. There exists an orbit of Γ dense in $\partial\widetilde{S} \times \partial\widetilde{S}$.

Proposition 2.7. The set $\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma\}$ is dense in $\partial\widetilde{S} \times \partial\widetilde{S}$.

The following is straightforward (see [7, Lemma 8])

Lemma 2.8. *Let φ be an element of Γ and let $\eta = \varphi(-\infty)$ and $\xi = \varphi(\infty)$ be the repulsive and attractive points of φ in $\partial\widetilde{S}$. Then any geodesic line c joining η and ξ projects to a closed geodesic in S.*

Clearly, the set of points

$$F_h^c = \{\xi | \xi \neq c(+\infty) \text{ for all geodesics } c \text{ such that } p(c) \text{ is closed}\} \tag{2.1}$$

is uncountable and dense in $\partial\widetilde{S}$. Similarly the set

$$\{(\xi, \eta) | \xi, \eta \in F_h^c, \xi \neq \eta\} \text{ is dense in } \partial\widetilde{S} \times \partial\widetilde{S}. \tag{2.2}$$

We will also use the Gauss-Bonnet formula [15] in the following special case: let P be a simply connected sub-surface of $\widetilde{S}$ with piece-wise geodesic boundary. For $\widetilde{s} \in \partial P$ denote by $\theta_P(\widetilde{s})$ the angle at $\widetilde{s}$ inside P. Then

$$2\pi = \sum\nolimits_{\widetilde{s} \in P \backslash \partial P} (2\pi - \theta(\widetilde{s})) + \sum\nolimits_{\widetilde{s} \in \partial P} (\pi - \theta_P(\widetilde{s})) \tag{2.3}$$

As mentioned above, $\widetilde{S}$ is a CAT(0) space hence the following Theorem holds (see [2, Cor. 5.8.ii])

Theorem 2.9 (Flat Strip Theorem)**.** *If $f, g : \mathbb{R} \to \widetilde{S}$ are two geodesics with $f(-\infty) = g(-\infty)$ and $f(\infty) = g(\infty)$ then f and g bound a flat strip, that is, a convex region isometric to the convex hull of two parallel lines in the flat plane.*

3. Properties of geodesics

Let GS be the space of all local isometric maps $g : \mathbb{R} \to S$. The image of such a g will be referred to as a geodesic in S. Consider also the space $G\widetilde{S}$ of all isometric maps $g : \mathbb{R} \to \widetilde{S}$. Both spaces GS and $G\widetilde{S}$ are equipped with the compact-open topology. Moreover the space $G\widetilde{S}$ with the compact-open topology is metrizable and its metric is given by the formula (see 8.3.B in [16])

$$d_{G\widetilde{S}}(g_1, g_2) = \int_{-\infty}^{+\infty} d(g_1(t), g_2(t))\, e^{-|t|} dt.$$

We will denote by p both projections $\widetilde{S} \to S$ and $G\widetilde{S} \to GS$. The image of any geodesic in $\widetilde{S}$ splits $\widetilde{S}$ into two convex (open) sets, say $\widetilde{S}_L$ and $\widetilde{S}_R$, and we may speak of the distance to the left of $\operatorname{Im}\widetilde{g}$ from the set $\left\{\widetilde{s} \in \widetilde{S}_L \mid \widetilde{s} \text{ conical point}\right\}$, namely,

$$d_L = \inf\left\{d(x, \widetilde{s}) \mid x \in \operatorname{Im}\widetilde{g}, \widetilde{s} \in \widetilde{S}_L, \widetilde{s} \text{ conical point}\right\}$$

and similarly for the distance to the right d_R. All the above make sense when dealing with geodesic rays. We will use the following Theorem which follows from the main result in [7, Remark 18].

Theorem 3.1. *If $\widetilde{g}$ is a geodesic or geodesic ray in $\widetilde{S}$ which*
(a) does not contain any conical point and
(b) projects to a non-closed geodesic in S
then $d_L = d_R = 0$.

We will be saying that a geodesic f in $\widetilde{S}$ is *unique* if every geodesic g with the property $g(\pm\infty) = f(\pm\infty)$ is a re-parametrization of f. The above Theorem implies that uniqueness of geodesics in $\widetilde{S}$ does hold for all geodesics which do not project to closed geodesics in S. To see this, let f, g be two distinct geodesics with $f(-\infty) = g(-\infty)$ and $f(\infty) = g(\infty)$. By the Flat Strip Theorem f and g bound a flat strip and, thus, we may find a geodesic h in the interior of the flat strip whose existence violates Theorem 3.1. We restate Proposition 2.4 in the form we will use.

Proposition 3.2. For every pair of points $x \in \widetilde{S}$, $\xi \in \partial\widetilde{S}$ there is a unique geodesic ray $r : [0, \infty) \to \widetilde{S}$ such that, $r(0) = x$, $r(\infty) = \xi$.
For every pair of distinct points $\eta, \xi \in \partial\widetilde{S}$ there is a geodesic line $l : (-\infty, \infty) \to \widetilde{S}\cup$ such that $l(-\infty) = \eta$, $l(\infty) = \xi$. Moreover, if the geodesic l does not project to a closed geodesic in S, then it is unique.

By abuse of language, a geodesic $g \in G\widetilde{S}$ will be called *closed* (resp. *non-closed*) if $p(g)$ is (resp. is not) a closed geodesic in GS. Observe that if a closed geodesic $g \in G\widetilde{S}$ does not contain any conical point then g determines a flat strip in $\widetilde{S}$, that is, a closed convex subset of $\widetilde{S}$ isometric to $I \times \mathbb{R}$ for some closed interval I, whose boundary consists of parallel translates of g which contain conical points provided that the flat strip is maximal. Observe also that a closed geodesic $g \in G\widetilde{S}$ containing conical points may or may not be the boundary line of a flat strip, the latter being equivalent to $p(g)$ being the unique closed geodesic in its (free) homotopy class.

Lemma 3.3. *For any e.s.c.s. S there exists a closed geodesic which is unique in its free homotopy class.*

Proof. Let $\widetilde{s}$ be a conical point in $\widetilde{S}$ with angle $\theta(\widetilde{s})$. Pick $\theta_0 < \frac{\theta(\widetilde{s})-2\pi}{2}$ and let $r_i, i = 1,2,3,4$ be geodesic rays enumerated clockwise with $r_i(0) = \widetilde{s}$ and the (clockwise) angles formed by r_i at $\widetilde{s}$ satisfy the following relations

$$\sphericalangle_{\widetilde{s}}(r_1,r_2) = \theta_0 = \sphericalangle_{\widetilde{s}}(r_3,r_4) \text{ and } \sphericalangle_{\widetilde{s}}(r_2,r_3) > \pi, \sphericalangle_{\widetilde{s}}(r_4,r_1) > \pi. \tag{3.1}$$

By Proposition 2.7 there exist boundary points

$$\eta \in (r_1(+\infty), r_2(+\infty)), \zeta \in (r_3(+\infty), r_4(+\infty))$$

such that $\eta = g(-\infty), \zeta = g(+\infty)$ for some closed geodesic g in $G\widetilde{S}$. It is easy to see that $\operatorname{Im} g$ must contain $\widetilde{s}$. For, if $\widetilde{s} \notin \operatorname{Im} g$ then $\operatorname{Im} g$ must intersect $\operatorname{Im} r_i$ for at least one i. Without loss of generality, assume that $y = g(t_y) \in \operatorname{Im} r_1 \cap \operatorname{Im} g$. Let r_ζ be the unique geodesic ray with $r_\zeta(0) = \widetilde{s}$ and $r_\zeta(+\infty) = \zeta$. Then, by the inequalities in (3.1) the union

$$[y, \widetilde{s}] \cup \operatorname{Im} r_\zeta$$

is a geodesic ray from y to ζ which is different from $g|_{[t_y,+\infty)}$ because $g([t_y,+\infty))$ does not contain $\widetilde{s}$. This is a contradiction by Proposition 3.2. Thus, $\operatorname{Im} g$ contains $\widetilde{s}$ and so does any (closed) geodesic with boundary points η, ξ. Therefor, g is not the boundary line of a flat strip and, hence, its projection to S is the unique geodesic in its homotopy class. □

We now turn to asymptotic geodesics. Recall that two geodesic rays g_1, g_2 (or geodesics) are called *asymptotic* if $d\big(g_1(t), g_2(t)\big)$ is bounded for all $t \in \mathbb{R}^+$. Equivalently, two geodesic rays g_1, g_2 (or geodesics) in $\widetilde{S}$ are asymptotic if $g_1(+\infty) = g_2(+\infty)$.

Proposition 3.4. Let g_1, g_2 be two (non-closed) asymptotic geodesics with

$$\xi = g_1(+\infty) = g_2(+\infty) \in \partial\widetilde{S} \setminus F_h.$$

Then for appropriate parametrizations of g_1, g_2 we have

$$\lim_{t\to\infty} d\big(g_1(t), g_2(t)\big) = 0.$$

We first show the following

Lemma 3.5. *Let g_1, g_2 be two geodesics as in the above Theorem. Assume that*

$$\operatorname{dist}(\operatorname{Im} g_1, \operatorname{Im} g_2) := \inf\left\{d(x,y) \mid x \in \operatorname{Im} g_1, y \in \operatorname{Im} g_2\right\} = 0.$$

Then there exists a unique re-parametrization $\overline{g}_1$ of g_1 such that

$$\lim_{t\to\infty} d\big(\overline{g}_1(t), g_2(t)\big) = 0.$$

Moreover, for the generalized Busemann function $\alpha : \left(\partial\widetilde{S} \cup \widetilde{S}\right) \times \widetilde{S} \times \widetilde{S} \to \mathbb{R}$ (for its definition, see Section 4.1 below and [2, Ch. II, Sec. 1]*) we have $\alpha(\xi, \overline{g}_1(t), g_2(t)) = 0$.*

Proof. As g_1, g_2 are asymptotic, the distance function $t \to d(g_1(t), g_2(t))$, $t \geq 0$ is convex (see [2, Ch.I, Proposition 5.4]) and bounded. Therefore, it is decreasing with a global infimum, say, $C \geq 0$. Clearly, if $C = 0$ we have nothing to show. Assume $C > 0$. Then,

$$\forall n \in \mathbb{N}, \exists t'_n > 0 \text{ such that } \forall t \geq t'_n, d(g_1(t), g_2(t)) < \frac{1}{n} + C. \tag{3.2}$$

Since $\operatorname{dist}(\operatorname{Im} g_1, \operatorname{Im} g_2) = 0$,

$$\forall n \in \mathbb{N}, \exists x_n \in \operatorname{Im} g_1 \text{ and } \exists y_n \in \operatorname{Im} g_2 \text{ such that } d(x_n, y_n) < \frac{1}{n}. \tag{3.3}$$

Let t''_n be the real number such that $g_2(t''_n) = y_n$. For each point $g_2(t)$ on the geodesic g_2, denote by $x(t)$ the unique point on $\operatorname{Im} g_1$ realizing the distance $dist(g_2(t), \operatorname{Im} g_1)$ (see [2, Ch.I, Corollary 5.6]). By (3.3) we have

$$d(x(t''_n), g_2(t''_n)) < 1/n$$

and, thus,

$$\forall t \geq t''_n, \exists x(t) \in \operatorname{Im} g_1 \text{ such that } d(x(t), g_2(t)) \leq d(x(t''_n), g_2(t''_n)) < 1/n. \tag{3.4}$$

Set $t_n = \max\{t'_n, t''_n\}$. If for some large $n \in mathbbN$ the inequality

$$d(x(t_n), g_1(0)) > d(g_1(t_n), g_1(0))$$

holds for the time t_n, it follows that the same inequality holds for all $T > t_n$. In this case set $\overline{C} = C$. Otherwise, set $\overline{C} = -C$. We will show that

$$\forall t \geq t_n, d\left(g_1\left(t + \overline{C}\right), g_2(t)\right) < \frac{3}{n}.$$

Let $t > t_n$ be arbitrary. Then, by (3.2) and (3.4), for the points $g_1(t), g_1\left(t + \overline{C}\right)$, $g_2(t)$ and $x(t)$ we have

$$d(g_1(t), g_2(t)) < \frac{1}{n} + C$$
$$d(x(t), g_2(t)) < \frac{1}{n}$$
$$d\left(g_1\left(t + \overline{C}\right), g_1(t)\right) = C.$$

By elementary triangle inequalities we obtain

$$d\left(g_1\left(t + \overline{C}\right), x(t)\right) < \frac{2}{n}$$

which implies that $d\left(g_1\left(t + \overline{C}\right), g_2(t)\right) < \frac{3}{n}$ as desired. Define the geodesic $\overline{g}_1$ by $\overline{g}_1(t) := g_1(t + \overline{C})$.

We now show that $\alpha(\xi, \overline{g}_1(0), g_2(0)) = 0$. For each $n \in \mathbb{N}$, pick $t_n > 0$ such that $d(\overline{g}_1(t_n), g_2(t_n)) < 1/n$. Then,

$$\begin{aligned} &|d(g_2(0), \overline{g}_1(t_n)) - d(\overline{g}_1(0), \overline{g}_1(t_n))| = \\ &|d(g_2(0), \overline{g}_1(t_n)) - d(g_2(0), g_2(t_n))| < \\ &\qquad\qquad d(\overline{g}_1(t_n), g_2(t_n)) < 1/n. \end{aligned}$$

By letting $n \to \infty$ we obtain a sequence $\{\overline{g}_1(t_n)\}_{n \in \mathbb{N}}$ converging to ξ such that

$$\alpha(\xi, \overline{g}_1(0), g_2(0)) = \lim_{n \to \infty} |d(g_2(0), \overline{g}_1(t_n)) - d(\overline{g}_1(0), \overline{g}_1(t_n))| = 0.$$

□

Proof of Proposition 3.4 If $\operatorname{Im} g_1 \cap \operatorname{Im} g_2 \neq \emptyset$ then there must exist a $K \in \mathbb{R}$ such that

$$\operatorname{Im} g_1|_{[K,+\infty)} \subset \operatorname{Im} g_2 \tag{3.5}$$

otherwise uniqueness of geodesic rays would be violated. Clearly, if property (3.5) holds the result follows trivially, so we may assume that $\operatorname{Im} g_1 \cap \operatorname{Im} g_2 \neq \emptyset$.

Consider the convex region P bounded by

$$\operatorname{Im} g_1|_{[0,+\infty)} \cup \operatorname{Im} g_2|_{[0,+\infty)} \cup [g_1(0), g_2(0)] \equiv \partial P.$$

All terms $(2\pi - \theta(\widetilde{s}))$ in the first summand on the RHS of formula (2.3) are negative and bounded away from 0. The terms $(\pi - \theta_P(\widetilde{s}))$ are also negative for all $\widetilde{s} \neq g_1(0), g_2(0)$. It follows that there exist finitely many conical points in the interior of P. Let M_1 be a bound for the distance of the conical points in P from $[g_1(0), g_2(0)]$ and M_2 a bound for

$$\left\{ d\big(g_1(t), g_2(t)\big) \mid t \in [0, +\infty) \right\}.$$

Then the geodesic segment $[g_1(2M_1 + 2M_2), g_2(2M_1 + 2M_2)]$ splits P into two subsurfaces; a bounded one containing all conical points of P and an unbounded one not containing conical points. Hence, up to re-parametrization, we may assume that $P \setminus \partial P$ does not contain any conical points.

Assume that $\operatorname{Im} g_1|_{[0,+\infty)}$ contains finitely many conical points. Then, the distance $\operatorname{dist}(\operatorname{Im} g_1, \operatorname{Im} g_2)$ must be 0, otherwise one of the distances d_L, d_R for $\operatorname{Im} g_1$ would be positive, contradicting Theorem 3.1. The result in this case follows from Lemma 3.5. Similarly for $\operatorname{Im} g_2|_{[0,+\infty)}$. We proceed now in the case both $\operatorname{Im} g_1|_{[0,+\infty)}$ and $\operatorname{Im} g_2|_{[0,+\infty)}$ contain infinitely many conical points. Enumerate them by

$$\widetilde{s}_0^1 = g_1(0), \widetilde{s}_1^1, \widetilde{s}_2^1, \widetilde{s}_3^1, \ldots, \widetilde{s}_j^1, \ldots$$

according to their distance from $g_1(0)$, that is, $\widetilde{s}_j^1 \in \left[\widetilde{s}_0^1, \widetilde{s}_{j'}^1\right]$ for all $j < j'$. Similarly for $\operatorname{Im} g_2|_{[0,+\infty)}$.

As the angle at each $\widetilde{s}_j^1$ is $> \pi$ we may extend each geodesic segment $\left[\widetilde{s}_{j+1}^1, \widetilde{s}_j^1\right]$, $j = 1, 2, \ldots$ to a geodesic segment $\left[\widetilde{s}_{j+1}^1, x_j^1\right] \ni \widetilde{s}_j^1$ so that $x_j^1 \in \partial P$ and the angle at $\widetilde{s}_j^1$ inside P is $= \pi$. By uniqueness of geodesic rays (see Proposition 3.2) x_j^1 cannot belong to $\operatorname{Im} g_2|_{[0,+\infty)}$, thus, $x_j^1 \in [g_1(0), g_2(0)]$. For the same reason

$$x_j^1 \prec x_{j'}^1 \text{ for all } j < j'. \tag{3.6}$$

where the inequality relation $\prec$ means that $d\left(g_1(0), x_j^1\right) < d\left(g_1(0), x_{j'}^1\right)$. In particular, $x_j^1 \neq x_{j'}^1$. Do the same with the segments $\left[\widetilde{s}_{j+1}^2, \widetilde{s}_j^2\right]$, $j = 1, 2, \ldots$ and observe (again by uniqueness of geodesic rays) that

$$x_j^1 \prec x_{j'}^2 \text{ for all } j, j'. \tag{3.7}$$

Let x^1 (resp. x^2) be the unique accumulation point of the set $\left\{x_j^1 | j = 1, 2, \ldots\right\}$ $\big($resp. $\left\{x_j^2 | j = 1, 2, \ldots\right\}\big)$. Observe that x^1 may be equal to x^2 so that $\left[x^1, x^2\right]$ is a singleton. Moreover, by (3.7),

$$x_j^1 \prec x^1 \text{ and } x^2 \prec x_j^2 \text{ for all } j. \tag{3.8}$$

Let g be a geodesic ray with $g(0) \in \left[x^1, x^2\right]$ and $g(+\infty) = g_1(+\infty) = g_2(+\infty)$. By uniqueness of geodesic segments we have that $\operatorname{Im} g \subset P$. We claim that $\operatorname{Im} g \setminus \{g(0)\} \subset P \setminus \partial P$ or, equivalently,

$$\operatorname{Im} g \cap \left\{\widetilde{s}_j^1 | j = 1, 2, \ldots\right\} = \emptyset = \operatorname{Im} g \cap \left\{\widetilde{s}_j^2 | j = 1, 2, \ldots\right\}. \tag{3.9}$$

For, if $\widetilde{s}^1_{j_0} \in \operatorname{Im} g$ for some j_0, then the angle formed by $\operatorname{Im} g|_{[\widetilde{s}^1_{j_0},+\infty)}$ and $\operatorname{Im} g|_{[g(0),\widetilde{s}^1_{j_0}]}$ at $\widetilde{s}^1_{j_0}$ is (by relation (3.8) above) smaller than the angle formed by $\operatorname{Im} g_1|_{[\widetilde{s}^1_{j_0},+\infty)}$ and $\operatorname{Im} g_1|_{[g_1(0),\widetilde{s}^1_{j_0}]}$ which is, by construction, $=\pi$. This contradicts the fact that g is a geodesic. This proves property (3.9).

As $\operatorname{Im} g$ does not contain conical points, by Theorem 3.1, the distances $\operatorname{dist}(\operatorname{Im} g_1, \operatorname{Im} g)$ and $\operatorname{dist}(\operatorname{Im} g_2, \operatorname{Im} g)$ must be 0, otherwise one of the distances d_L, d_R for $\operatorname{Im} g$ would be positive. By Lemma 3.5, there exist re-parameterizations $\overline{g}_1, \overline{g}_2$ of g_1, g_2 respectively such that

$$\lim_{t\to\infty} d\big(\overline{g}_1(t), g(t)\big) = 0 \text{ and } \lim_{t\to\infty} d\big(\overline{g}_2(t), g(t)\big) = 0$$

and the result follows. ■

We now examine special properties of the boundary points $c(+\infty), c(-\infty)$ of a closed geodesic $c \in G\widetilde{S}$.

Lemma 3.6. *Let $c \in G\widetilde{S}$ be a closed geodesic and $f \in G\widetilde{S}$ any geodesic with $f(+\infty) = c(+\infty)$ and $f(-\infty) \neq c(-\infty)$. Then,*

- *if $p(c)$ is unique in its homotopy class, $f|_{[K,+\infty)} = c|_{[K',+\infty)}$ for some $K, K' \in \mathbb{R}$.*
- *if $p(c)$ is not unique in its homotopy class, either $f|_{[K,+\infty)} = c|_{[K',+\infty)}$ for some $K, K' \in \mathbb{R}$ or, $f|_{[K,+\infty)} = c'|_{[K',+\infty)}$ where c' is a parallel translate of c.*

In particular, f is a non closed geodesic and unique with points at infinity $f(+\infty), f(-\infty)$. Moreover, by altering the parametrization of c we may achieve $K = K'$.

Proof. First assume that $p(c)$ is unique in its homotopy class. Then $\operatorname{Im} c$ contains conical points, otherwise c would determine a flat strip contradicting uniqueness of $p(c)$. If the result does not hold then, by uniqueness of geodesic rays (see Proposition 3.2), $\operatorname{Im} f \cap \operatorname{Im} c = \emptyset$. Let P be the convex simply connected region bounded by

$$\operatorname{Im} f|_{[0,+\infty)} \cup \operatorname{Im} c|_{[0,+\infty)} \cup [f(0), c(0)] \equiv \partial P.$$

Using the formula (2.3) we may assume, as in the proof of Proposition 3.4 above, that $P \setminus \partial P$ does not contain any conical points. Since c is closed there exists a singular point $s_{i_0} \in S$ such that $p^{-1}(s_{i_0})$ is an infinite set

$$p^{-1}(s_{i_0}) = \left\{\widetilde{s}_{i_0,j} \mid j = 1, 2, \ldots\right\}$$

contained in $\operatorname{Im} c|_{[0,+\infty)}$. Thus, there are infinitely many terms $(\pi - \theta_P(\widetilde{s}_{i_0,j}))$ in the second summand on the RHS of formula (2.3). By periodicity, the angle $\pi - \theta_P(\widetilde{s}_{i_0,j})$ is constant for all j, which is a contradiction.

The case where $p(c)$ is not unique in its homotopy class is treated similarly by using the (maximal) flat strip determined by c which is bounded by geodesics parallel to c.

If $p(f)$ were closed then, since f, c coincide on an unbounded interval $[K, \infty)$, we would have $p(f) = p(c)$ which implies that $f \equiv c$. □

4. Transitivity and Mixing of the geodesic flow

4.1. **Definitions and classical properties.** The geodesic flow is defined by the map

$$\mathbb{R} \times GS \to GS$$

where the action of $\mathbb{R}$ is given by right translation, i.e. for all $t \in \mathbb{R}$ and $g \in GS$, $(t, g) \to t{\cdot}g$ where $t{\cdot}g : \mathbb{R} \to S$ is the geodesic defined by $(t \cdot g)(s) = g(s + t), s \in \mathbb{R}$. Similarly for $G\widetilde{S}$.

Definition 4.1. The geodesic flow $\mathbb{R} \times GS \to GS$ is topologically transitive if given any open sets $\mathcal{O}$ and $\mathcal{U}$ in GS there exists a sequence $t_n \to \infty$ such that $t_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large.

Definition 4.2. The geodesic flow $\mathbb{R} \times GS \to GS$ is topologically mixing if given any open sets $\mathcal{O}$ and $\mathcal{U}$ in GS there exists a real number $t_0 > 0$ such that for all $|t| \geq t_0$, $t \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$.

A point g in GS belongs to the *non-wandering set* Ω of the geodesic flow $\mathbb{R} \times GS \to GS$ if there exist sequences $\{g_n\} \subset GS$ and $\{t_n\} \subset \mathbb{R}$, such that $t_n \to \infty$, $g_n \to g$ and $t_n \cdot g_n \to g$. A necessary condition for topological mixing is $\Omega = GS$ which, for the class of surfaces studied here, holds as it is equivalent to $\Lambda(\Gamma) = \partial\widetilde{S}$ (see [13, Cor. 3.8]).

Define a function $\alpha : \widetilde{S} \times \widetilde{S} \times \widetilde{S} \to \mathbb{R}$ by letting

$$\alpha(\xi, x, x') := d(x', \xi) - d(x, \xi) \tag{4.1}$$

for $(\xi, x, x') \in \widetilde{S} \times \widetilde{S} \times \widetilde{S}$. It is shown in [2, Ch. II, Sec. 1] that this function extends to a continuous function

$$\left(\partial\widetilde{S} \cup \widetilde{S}\right) \times \widetilde{S} \times \widetilde{S} \to \mathbb{R}$$

denoted again by α, called the *generalized Busemann function.*

This function, in fact, generalizes the classical Busemann function whose definition makes sense in our context. Conversely, for arbitrary $\xi \in \partial\widetilde{S}$ and $y \in \widetilde{S}$, the restriction $\alpha(\xi, y, \cdot) \equiv \alpha|_{\{\xi\}\times\{y\}\times\widetilde{S}}$ is simply the Busemann function associated to the unique geodesic ray from y to ξ. We will use the following facts.

Lemma 4.3. *(a) The generalized Busemann function α is Lipschitz with respect to the second and third variable with Lipschitz constant 1.*
(b) If $f, g \in G\widetilde{S}$ with $f(-\infty) = g(+\infty)$ then

$$\alpha(g(+\infty), g(0), f(t)) = t + \alpha(g(+\infty), g(0), f(0)).$$

(c) If $f, g \in G\widetilde{S}$ are asymptotic geodesics, then there exists a unique re-parametrization $\overline{f}$ of f such that $\alpha\big(f(+\infty), \overline{f}(0), g(0)\big) = 0$.

A proof of (a) can be found in [2, Ch. II, Sec. 1] and the proof given in [6, Lemma 2.3] holds verbatim for (b) and (c).

Definition 4.4. We say that a geodesic $h \in GS$ belongs to the stable set $W^s(g)$ of a geodesic g if g, h are asymptotic. Two points $x, x' \in \widetilde{S}$ are said to be equidistant from a point $\xi \in \partial\widetilde{S}$ if $\alpha(\xi, x, x') = 0$.

We say that a geodesic $h \in G\widetilde{S}$ belongs to the strong stable set $W^{ss}(g)$ of a geodesic g if $h \in W^s(g)$ and $g(0), h(0)$ are equidistant from $g(\infty) = h(\infty)$.

Similarly, if $h, g \in GS$, we say that $h \in W^{ss}(g)$ $\big($respectively $W^s(g)\big)$ if there exist lifts $\overline{h}, \overline{g} \in G\widetilde{S}$ of h, g such that $\overline{h} \in W^{ss}(\overline{g})$ $\big($respectively $W^s(\overline{g})\big)$.

The following is a restatement of Proposition 3.4 taking into account Lemma 3.6.

Proposition 4.5. Let $f, g \in G\widetilde{S}$ with $f \in W^{ss}(g)$. If neither f nor g is a closed non-unique geodesic then

$$\lim_{t\to\infty} d\big(f(t), g(t)\big) = 0.$$

4.2. **Topological transitivity.** It is apparent that topological mixing implies topological transitivity. However, in the proof of topological mixing below we will need a property equivalent to topological transitivity, namely, that $\overline{W^s(f)} = GS$ for any $f \in GS$. In this section we will establish this property.

We will use the following theorem shown in [2, page 48] in the context of Hadamard spaces which satisfy the duality condition.

Theorem 4.6. *There exists a geodesic γ in GS whose orbit $\mathbb{R}\gamma$ under the geodesic flow is dense in GS.*

Duality Condition: For each $f \in G\widetilde{S}$, there exists a sequence of isometries $\{\phi_n\}_{n\in\mathbb{N}} \subset \Gamma \equiv \pi_1(S)$ such that $\phi_n(x) \to f(+\infty)$ and $\phi_n^{-1}(x) \to f(-\infty)$ for some (hence any) $x \in \widetilde{S}$.

Lemma 4.7. *$\widetilde{S}$ satisfies the duality condition.*

Proof. If $f \in G\widetilde{S}$ is closed we may consider $\{\phi_n\}$ to be powers of the hyperbolic isometry corresponding to f. Then clearly $\phi_n(f(0)) \to f(+\infty)$ and $\phi_n^{-1}(f(0)) \to f(-\infty)$.

Suppose $f \in G\widetilde{S}$ is non closed. By Theorem 2.7, there exists a sequence of closed geodesics c_n such that $c_n(+\infty) \to f(+\infty)$ and $c_n(-\infty) \to f(-\infty)$. Moreover, by changing appropriately the parametrizations of each c_n, we obtain $c_n \to f$. We may alter the period t_n of each c_n so that $t_n \nearrow +\infty$ as $n \to \infty$. Set ϕ_n to be the isometry which corresponds to translating c_n by t_n. Then, $\phi_n(c_n(0)) \to f(+\infty)$ and, since $f(0)$ is at bounded distance from $c_n(0)$ for all n, it follows that $\phi_n(f(0)) \to f(+\infty)$. Similarly we show that $\phi_n^{-1}(f(0)) \to f(-\infty)$. □

Proposition 4.8. For any $f \in GS$, $\overline{W^s(f)} = GS$.

Proof. Let $g \in GS$ be arbitrary and pick lifts $\widetilde{g} \in G\widetilde{S}$ of g and $\widetilde{f} \in G\widetilde{S}$ of f. Since by the previous Lemma $\widetilde{S}$ satisfies the duality condition, we may use Theorem 4.6, to obtain a geodesic $\widetilde{\gamma} \in G\widetilde{S}$ and sequences $\{t_n\}$ in $\mathbb{R}$ and $\{\phi_n\}$ in Γ such that $\phi_n(t_n \cdot \widetilde{\gamma}) \to \widetilde{g}$. Set $\widetilde{\gamma}_n := \phi_n(t_n \cdot \widetilde{\gamma})$.

Since the orbit $\Gamma \cdot \widetilde{f}(+\infty)$ is dense in $\partial\widetilde{S}$ we may pick, for each fixed n, a sequence $\{\phi_{n,k}\}_{k=1}^{\infty} \subset \Gamma$ such that $\phi_{n,k}\left(\widetilde{f}(+\infty)\right) \to \widetilde{\gamma}_n(+\infty)$. For all k, consider geodesics $\widetilde{g}_{n,k}$ with $\widetilde{g}_{n,k}(+\infty) = \phi_{n,k}\left(\widetilde{f}(+\infty)\right)$ and $\widetilde{g}_{n,k}(-\infty) = \widetilde{\gamma}_n(-\infty)$. Clearly $g_{n,k} = p(\widetilde{g}_{n,k}) \in W^s(f)$ for all $k, n \in \mathbb{N}$ and by a diagonal argument we obtain a sequence $g_{n,k(n)} = p\left(\widetilde{g}_{n,k(n)}\right)$ which converges to $g = p(\widetilde{g})$. □

4.3. **Proof of topological mixing.** For the proof of topological mixing for the geodesic flow on S, we will closely follow the notation and the line of the analogous proof for CAT(-1) spaces in [6] which, in fact, follows the steps of Eberlein's work (cf [12]). The main difficulty in the present set up is the fact that $G\widetilde{S}$ is only a CAT-(0) space and, in turn, Proposition 4.5 does not hold for certain classes of closed geodesics whereas for CAT(-1) spaces holds for arbitrary geodesics. For the proofs of the following Lemmata, we will refer to the corresponding proofs in [6] and deal only with the implications arising from the closed geodesics.

Lemma 4.9. *(a) For any $g \in GS$ and $t \in \mathbb{R}$, $\overline{W^{ss}(t \cdot g)} = t \cdot \left(\overline{W^{ss}(g)}\right)$.*

(b) Let $g, h \in GS$ with g closed and unique, $h \in W^{ss}(g)$ and $\mathcal{O} \subset GS$ an open neighborhood of h. Then there exists an open neighborhood $\mathcal{A}$ containing g such that for any $g' \in \mathcal{A}$, $W^{ss}(g') \cap \mathcal{O} \neq \emptyset$.

(c) If $h \in \overline{W^{ss}(g)}$ and g is closed and unique, then $\overline{W^{ss}(h)} \subset \overline{W^{ss}(g)}$.

Proof. (a) If $h \in \overline{W^{ss}(t \cdot g)}$, there exist a sequence $\{h_n\}_{n \in \mathbb{N}} \subset W^{ss}(t \cdot g)$ with $h_n \to h$. It is clear from the definitions that $(-t) \cdot h_n \to (-t) \cdot h$ and $\{(-t) \cdot h_n\}_{n \in \mathbb{N}} \subset W^{ss}(g)$. This shows that $(-t) \cdot h \in \overline{W^{ss}(g)}$ and, hence, $h = t \cdot ((-t) \cdot h) \in t \cdot \left(\overline{W^{ss}(g)}\right)$. Similarly, we show the converse inclusion.

Part (c) follows easily from part (b).

(b) Each conical point $\widetilde{s} \in \operatorname{Im} g$ splits g into two geodesic rays which subtend two angles (left and right according to the positive orientation of g) at $\widetilde{s}$ denoted by $\theta_L(\widetilde{s}), \theta_R(\widetilde{s})$. As g is a geodesic, both $\theta_L(\widetilde{s}), \theta_R(\widetilde{s})$ are $\geq \pi$. If $\theta_L(\widetilde{s}) = \pi$ for all $\widetilde{s} \in \operatorname{Im} g$ then we may parallel translate g to the left, thus g is not unique, a contradiction. Thus, $\theta_L(\widetilde{s}) > \pi$ for at least one $\widetilde{s} \in \operatorname{Im} g$. Similarly for $\theta_R(\widetilde{s})$. Observe that, as g is a closed geodesic, both sets

$$\left\{\theta_L(\widetilde{s}) \mid \widetilde{s} \in \operatorname{Im} g\right\}, \left\{\theta_R(\widetilde{s}) \mid \widetilde{s} \in \operatorname{Im} g\right\}$$

are finite. Set

$$\theta_L = \min\left\{\theta_L(\widetilde{s}) - \pi \mid \widetilde{s} \in \operatorname{Im} g, \theta_L(\widetilde{s}) > \pi\right\}$$

As explained above, $\theta_L > 0$ and, similarly, θ_R is defined and it is > 0. Choose $\theta_0 < \min\{\pi/2, \theta_L, \theta_R\}$ and set

$$T = \frac{1}{2}\min\left\{d(\widetilde{s}_1, \widetilde{s}_2) \mid \widetilde{s}_1, \widetilde{s}_2 \text{ conical points in } \widetilde{S}\right\}.$$

<u>Claim 1</u>: There exists $\varepsilon_0 > 0$ such that for any conical point $\widetilde{s} = g(t_{\widetilde{s}}) \in \operatorname{Im} g$ with $\theta_L(\widetilde{s}) > \pi$ (resp. $\theta_R(\widetilde{s}) > \pi$) the following holds: if x is a point with $d(x, g(t_{\widetilde{s}} + T)) < \varepsilon_0$ then the geodesic segments $[\widetilde{s}, x]$ and $[\widetilde{s}, g(t_{\widetilde{s}} + T)]$ subtend at $\widetilde{s}$ an angle $\theta < \frac{\theta_0}{2}$, provided that x lies on the left (resp. right) of $\operatorname{Im} g$. The same holds for points y such that $d(x, g(t_{\widetilde{s}} - T)) < \varepsilon_0$.

The proof is elementary as, by definition of T, the geodesic triangle formed by the points $\widetilde{s}, g(t_{\widetilde{s}} + T)$ and x for sufficiently small ε_0 is Euclidean and acute.

<u>Claim 2</u> : For the ε_0 posited in Claim 1, the following holds: for any conical point $\widetilde{s} = g(t_{\widetilde{s}})$ and any points x, y with $d(x, g(t_{\widetilde{s}} + T)) < \varepsilon_0$ and $d(y, g(t_{\widetilde{s}} - T)) < \varepsilon_0$, the geodesic segment $[x, y]$ necessarily contains $\widetilde{s}$, provided that $\theta_L(\widetilde{s}) > \pi$ (resp. $\theta_R(\widetilde{s}) > \pi$) and both x, y lie on the left (resp. right) of $\operatorname{Im} g$.

Proof of Claim 2: We may assume that $\theta_L(\widetilde{s}) > \pi$ and x, y lie on the left of $\operatorname{Im} g$. Then, by Claim 1,

$$\measuredangle_{\widetilde{s}}([\widetilde{s}, x], [\widetilde{s}, g(t_{\widetilde{s}} + T)]) < \frac{\theta_0}{2} \text{ and } \measuredangle_{\widetilde{s}}([\widetilde{s}, y], [\widetilde{s}, g(t_{\widetilde{s}} - T)]) < \frac{\theta_0}{2}.$$

Thus, the angle $\measuredangle_{\widetilde{s}}([\widetilde{s}, x], [\widetilde{s}, y])$ subtended on the left of $\operatorname{Im} g$ is at least

$$\theta_L(\widetilde{s}) - 2\frac{\theta_0}{2} > \pi + \theta_L - 2\frac{\theta_0}{2} > \pi.$$

Clearly, the angle subtended by $[\widetilde{s}, x], [\widetilde{s}, y]$ at $\widetilde{s}$ on the right of $\operatorname{Im} g$ is $> \pi$ because g is a geodesic. It follows that the union $[\widetilde{s}, x] \cup [\widetilde{s}, y]$ is a geodesic at $\widetilde{s}$ and the

Claim follows by uniqueness of geodesic segments (see Proposition 3.2).

Notation: Denote by T_g the period of g and by $\mathcal{A}(g,K,\varepsilon)$ for $K \in \mathbb{R}, K>0$ and $\varepsilon>0$ the open neighborhood in the compact open topology around g of the form

$$\mathcal{A}(g,K,\varepsilon)=\left\{f \in G\widetilde{S} \mid d(f(t),g(t))<\varepsilon \text{ for all } t\in[-K,K]\right\}$$

<u>Claim 3</u>: For the ε_0 posited in Claim 1 and 2, we have

$$\forall g' \in \mathcal{A}(g,3T_g,\varepsilon_0) \Longrightarrow g([-T_g,T_g]) \subset \operatorname{Im} g'.$$

Proof: We first show that for arbitrary $g' \in \mathcal{A}(g,3T_g,\varepsilon_0)$, $g'([T_g,2T_g])$ intersects $g([T_g,3T_g])$.

If we $g'(T_g), g'(2T_g)$ lie on opposite sides of $\operatorname{Im} g$, we have nothing to show. If $g'(T_g), g'(2T_g)$ lie on the same side, say, left, of $\operatorname{Im} g$ we may find a conical point $\widetilde{s}_1 = g(t_{\widetilde{s}_1}) \in g([T_g,2T_g])$ with $\theta_L(\widetilde{s}_1)>\pi$. By Claim 2, the geodesic segment $[g'(T_g), g'(2T_g)]$ contains $\widetilde{s}_1$, thus, $g'([T_g,2T_g])$ intersects $g([T_g,2T_g]) \subset g([T_g,3T_g])$. Similarly we show that $\operatorname{Im} g'$ intersects $g([-3T_g,-T_g])$. By Proposition 3.2, it follows that $\operatorname{Im} g'$ contains $g([-T_g,T_g])$. This completes the proof of Claim 3.

We return to the proof of part (b) of the Lemma. Let $h \in W^{ss}(g) \cap \mathcal{O}$, for an open neighborhood $\mathcal{O}$ of h. By Lemma 3.6, there exists T_0 such that $h(t)=g(t)$ for all $t \geq T_0$. It suffices to show the result for sufficiently small neighborhoods $\mathcal{O}$ of h, namely, for neighborhoods of the form

$$\mathcal{O}=\mathcal{O}(h,K,\varepsilon_h)=\left\{f \in G\widetilde{S} \mid d(f(t),h(t))<\varepsilon_0 \text{ for all } t\in[-K,K]\right\}$$

for $K>|T_0|$ and $\varepsilon_h>0$.

For any neighborhood $\mathcal{O}$ of h as above, let $\mathcal{A}=\mathcal{A}(g,T_1,\varepsilon_g)$ be the neighborhood of g in the compact open topology determined by $T_1>0$ and $\varepsilon_g>0$ satisfying

$$T_1-|T_0|>6T_g \text{ and } \varepsilon_g<\min\left\{\frac{1}{2}\varepsilon_h,\varepsilon_0\right\}.$$

By Claim 3 we have that for every $g' \in \mathcal{A}$, there exists an interval $I=[r_1,r_2]\subset[T_0,T_1]$ such that $\operatorname{Im} g|_I = \operatorname{Im} g'|_I$. We already have, in addition, that $\operatorname{Im} g|_I = \operatorname{Im} h|_I$ and since $h \in W^{ss}(g)$ it follows $g(t)=h(t)$ for all $t \in I$. Since $g' \in \mathcal{A}(g,T_1,\varepsilon_g)$, we have

$$d(g'(r_2),g(r_2))=\delta<\frac{1}{2}\varepsilon_h.$$

or, equivalently $g'\left(r_2+\overline{\delta}\right)=g(r_2)$ where $\overline{\delta}=+\delta$ or, $-\delta$. Then, for the geodesic h' defined by

$$h'(t):=\begin{cases} h(t), & \text{if } t \leq r_2 \\ g'\left(t+\overline{\delta}\right), & \text{if } t \geq r_2 \end{cases}$$

we clearly have $h' \in \mathcal{A}$ and, by construction, $h' \in W^{ss}(g')$ as required. □

Proposition 4.10. There exists a geodesic $g \in GS$ such that $\overline{W^{ss}(g)}=GS$.

Proof. The proof is identical to the analogous Proposition 4.1 in [6]. The first step in the proof is to show that

$$\forall\, \mathcal{O},\mathcal{U} \subseteq GS \textit{ open}, \exists\, g \in \mathcal{O} : W^{ss}(g) \cap \mathcal{U} \neq \emptyset. \tag{4.2}$$

In that proof, a choice of $f \in p^{-1}(\mathcal{O})$ and $h \in p^{-1}(\mathcal{U})$ is made so that

$$\left(f(+\infty),h(+\infty)\right)=(\phi(+\infty),\phi(-\infty)) : \phi \in \Gamma \textit{ is hyperbolic}.$$

The additional observation needed in our setup is that, by Lemma 3.6, f and h can be chosen to be non-closed and unique and the same holds for all translates $\phi(f), \phi(h)$. All approximations in $\widetilde{S}$ and $G\widetilde{S}$ performed in the proof of Proposition 4.1 in [6] are valid verbatim in our setup. □

Proposition 4.11. Suppose $c \in GS$ is a closed geodesic with the property of being unique in its free homotopy class. Then $\overline{W^{ss}(c)} = GS$.

Proof. Let g be the geodesic posited in Proposition 4.10, that is, $\overline{W^{ss}(g)} = GS$ and let c be a closed geodesic unique in its homotopy class with period ω. Observe that such a geodesic c exists by Lemma 3.3. By Proposition 4.8, $\overline{W^s(c)} = GS$ so that $g \in \overline{W^s(c)} = GS$. Thus, there exists a sequence $\{g_n\} \subset W^s(c)$ such that $g_n \to g$. For each $n \in \mathbb{N}$, consider lifts $\widetilde{g}_n, \widetilde{c}$ of g_n, c respectively satisfying $\widetilde{g}_n \in W^s(\widetilde{c})$ and use Lemma 4.3(c) to obtain a real number $\widetilde{t}_n$ such that $\widetilde{t}_n \cdot g_n \in W^{ss}(c)$. Each $\widetilde{t}_n$ may be written as

$$\widetilde{t}_n = k\omega + t_n$$

where $k \in \mathbb{Z}$ and $t_n \in [0, \omega)$. By choosing, if necessary a subsequence, $t_n \to t$ for some $t \in [0, \omega]$. Then $t_n \cdot g_n \to t \cdot g$ and $t_n \cdot g_n \in W^{ss}(c)$ which simply means that $t \cdot g \in \overline{W^{ss}(c)}$ and by Lemma 4.9(c) we have $\overline{W^{ss}(c)} \supset \overline{W^{ss}(t \cdot g)} = \overline{W^{ss}(g)} = GS$. □

We will need a point-wise version of topological mixing and a criterion for such property.

Definition 4.12. Let h, f be in GS and let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence converging to $+\infty$ or $-\infty$. We say that h is s_n-mixing with f (notation, $h \sim_{s_n} f$) if for every neighborhood $\mathcal{O}, \mathcal{U}$ in GS of h, f respectively, $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large.

If $h \sim_{s_n} f$ for some $h, f \in GS$, then using decreasing sequences of open neighborhoods of h and f it is easily shown that for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GY$ such that $h_n \to h$ and $r_n \cdot h_n \to f$. Moreover, by property (2.2), we may choose $\{h_n\}$ so that each h_n is non-closed for all n. The proof of the converse statement is elementary, hence, the following criterion for the s_n-mixing of h, f holds.

Criterion 4.13. If $h, f \in GS$, then $h \sim_{s_n} f$ if and only if for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence of non-closed geodesics $\{h_n\} \subset GS$ such that $h_n \to h$ and $r_n \cdot h_n \to f$.

We next show the following lemma which asserts that point-wise topological mixing is transferred via the strong stable relation of geodesics.

Lemma 4.14. *Let $f, g, g' \in GS$ so that $f \in \overline{W^{ss}(g)}, f$ is non-closed, g, g' are closed and unique. Then if $g \sim_{s_n} g'$ for some sequence $s_n \to \infty$, then $f \sim_{s_n} g'$.*

Proof. Fix a sequence $\{s_n\}$ with $s_n \to \infty$. It suffices to prove the assertion of the lemma for f non-closed and $f \in W^{ss}(g)$. To see this, let $\{h_k\}_{k \in \mathbb{N}} \subset W^{ss}(g)$ be a sequence with $h_k \to f$. By Lemma 3.6 each h_k is non-closed. Let $\mathcal{O}, \mathcal{U}$ be neighborhoods of f, g' respectively. Since $h_k \to h$, $\mathcal{O}$ is also a neighborhood of h_{k_0}, for some k_0 large enough. As we assumed the result for non-closed geodesics in $W^{ss}(g)$ we have $h_{k_0} \sim_{s_n} g'$, which implies that $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for n sufficiently large. The latter simply means that $f \sim_{s_n} g'$. Therefore, it suffices to prove the assertion of the lemma for f non-closed and $f \in W^{ss}(g)$.

Observe now that:

- g is unique in its homotopy class and f non-closed thus Proposition 4.5 asserts that
$$\lim_{t\to\infty} d\big(f(t), g(t)\big) = 0.$$
- Criterion 4.13 above which gives an equivalent statement for the s_n-mixing relation is slightly different than Criterion 4.3 in [6], namely,in our setup it refers to sequences of non-closed geodesics. This asserts that the sequences $\{\overline{g_n}\}_{n\in\mathbb{N}}, \{\overline{f_n}\}_{n\in\mathbb{N}}$ constructed in the proof of [6, Lemma 4.4] are non-closed and unique, thus,
$$\lim_{t\to\infty} d\big(\overline{f_n}(t), \overline{g_n}(t)\big) = 0$$
for all $n \in \mathbb{N}$.

The above two properties along with the standard properties of the generalized Busemann function stated in Lemma 4.3 above is all one needs to conclude the proof verbatim with the one given in [6, Lemma 4.4]. □

Theorem 4.15. *The geodesic flow $\mathbb{R} \times GS \to GS$ is topologically mixing.*

Proof. It suffices to show that

$$\forall\, h, f \in GS \text{ and } \forall\ \{t_n\} \text{ with } t_n \to \infty, \exists \text{ subsequence } \{s_n\} \subset \{t_n\} \text{ such that } h \sim_{s_n} f. \tag{4.3}$$

Since the notion of s_n-mixing is defined via neighborhoods it suffices to show the above property only for geodesics h, f which are not closed. This will allow the use of Lemma 4.14.

Let c be a closed geodesic posited in Proposition 4.11 and Lemma 3.3, that is, $\overline{W^{ss}(c)} = GS$. Clearly, for all $t \in \mathbb{R}$, $\overline{W^{ss}(t\cdot c)} = GS$. Let $\{s_n\}$ be a subsequence of $\{t_n\}$ such that $s_n \cdot c \to t\cdot c$ for some $t \in [0, period(c)]$. Then $f \in \overline{W^{ss}(c)} = GS$ and $c \sim_{s_n} t\cdot c$, thus, by Lemma 4.14, $f \sim_{s_n} t\cdot c$. The latter is equivalent to $-t\cdot c \sim_{s_n} -f$. As $-h \in \overline{W^{ss}(-t\cdot c)} = GS$ we have, by Lemma 4.14 again, $-h \sim_{s_n} -f$ which yields $f \sim_{s_n} h$, as required. □

References

[1] D.V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Stelkov Instit. Math. 90, Amer. Math.Soc., Providence, Rhode Island, 1969.

[2] W. Ballmann, *Lectures on spaces of non positive curvature*, Birkhäuser, 1995.

[3] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag, Berlin, 1999.

[4] W. Ballman, E. Ghys, A. Haefliger, P. de la Harpe, E. Salem, R. Strebel et M. Troyanov, *Sur les groups hyperboliques d'aprés Gromov* (Seminaire de Berne), édité par E. Ghys et P. de la Harpe, (a paraitre chez Birkhäuser), 1990.

[5] Ch. Champetier, *Petite simplification dans les groupes hyperboliques*, Ann. Fac. Sci. Toulouse, VI. Ser., Math. 3, No.2, 161-221 (1994).

[6] C. Charitos, G. Tsapogas, *Topological mixing in CAT (-1)-spaces* Trans. Amer. Math. Soc. 178 (2001), pp. 235–264.

[7] Ch. Charitos, I. Papadoperakis, G. Tsapogas, *The geometry of Euclidean surfaces with conical singularities,* http://arxiv.org/abs/1306.1759 to appear in Math. Z.

[8] M. Coornaert, T. Delzant, A. Papadopoulos, *Géometrie et théorie des groupes,* Lecture Notes in Mathematics, 1441, Sringer-Verlag, 1980.

[9] M. Coornaert, *Sur les gropes proprement discontinus d'isometries des espaces hyperboliques au sens de Gromov*, Thèse de U.L.P., Publication de I.R.M.A., 1990.

[10] F. Dal'bo, *Topologie du Feuilletage fortement stable*, Ann. Inst. Fourier, Grenoble 50, 3 (2000), pp. 981–993.

[11] K. Dankwartk, *Typical geodesics on flat surfaces*,

[12] P. Eberlein, *Geodesic flows on negatively curved manifolds II*, Trans. Amer. Math. Soc. 178 (1973), pp.57-82.

[13] P. Eberlein, *Geodesic flows on negatively curved manifolds I*, Annals of Math. 95(1972), 492-510.

[14] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math., 46 (1973), 45-109.

[15] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics.* Vol. 1., 2006.

[16] M. Gromov, *Hyperbolic groups* in "Essays in group theory", M.S.R.I. Publ. 8 (Springer 1987), 75-263.

[17] M. Gromov, J. Lafontaine, P. Pansu, *Structures metriques pour les variétés Riemanniennes*, Fernand Nathan, Paris, 1981.

[18] F. Paulin, *Constructions of hyperbolic groups via hyperbolization of polyhedra,* In: Group Theory from a Geometrical Viewpoint, ICTP, Trieste, Italy, March 26–April 6, 1990, (eds. E. Ghys and A. Haefliger), 1991.

[19] E. Pervova, C. Petronio, *On the existence of branched coverings between surfaces with prescribed branch data I,* Algebraic & Geometric Topology 6, 1957-1985, 2006.

[20] J. Ratcliffe, *Foundations of hyperbolic geometry*, GTM Springer-Verlag, 1994.

[21] M. Troyanov, *Les surfaces euclidiennes a singularites coniques*, Enseign. Math. (2) 32 (1986), no. 1-2, 79-94.

[22] A. Zorich, *Flat surfaces*, Frontiers in Number Theory, Physics, and Geometry Vol.I, P. Cartier, B. Julia, P. Moussa, P. Vanhove (Editors), Springer Verlag, 2006, 439–586.

Agricultural University of Athens, Athens, Greece
E-mail address: bakis@aua.gr

Agricultural University of Athens, Athens, Greece
E-mail address: papadoperakis@aua.gr

University of the Aegean, Samos, Greece
E-mail address: get@aegean.gr