

Choice Principles for Special Subsets of the Real Line

Kyriakos KEREMEDIS and Eleftherios TACHTSIS

Dedicated to the memory of Jean E. Rubin

November 17, 2002

Abstract

We study the role the axiom of choice plays in the existence of some special subsets of \mathbb{R} and its power set $\wp(\mathbb{R})$.

1 Introduction

Clearly, the family \mathcal{V} of all non-empty open subsets of \mathbb{R} has a choice set and, it is not hard to verify, see [5] and [6], that the family \mathcal{F} of all non-empty closed subsets of \mathbb{R} has a choice function. However, one cannot generalize to arbitrary families of subsets of \mathbb{R} sharing the same property as the following straightforward proposition indicates:

Proposition 1 “ $\mathcal{F}_C = \{\cup A : A \in [\mathcal{F}]^{\leq \omega}\}$ has a choice set” implies $AC_\omega(\mathbb{R})$ (see Section 2 for the definition of $AC_\omega(\mathbb{R})$).

The traditional definition, see [8], of the Lebesgue measure μ of the real line requires first the definition of the set function $\mu^* : \wp(\mathbb{R}) \rightarrow [0, +\infty]$, $\mu^*(E) = \inf\{\sum_{n \in \omega} (q_n - p_n) : E \subset \cup\{(p_n, q_n) : p_n, q_n \in \mathbb{Q}, p_n < q_n\}\}$. Then one verifies that μ^* is an outer measure, i.e., it satisfies:

1. $\mu^*(\emptyset) = 0$,
2. if $A \subset B \subset \mathbb{R}$, then $\mu^*(A) \leq \mu^*(B)$ (monotonicity),
3. if $\mathcal{A} = \{A_n : n \in \omega\} \subset \wp(\mathbb{R})$, then $\mu^*(\cup \mathcal{A}) \leq \sum_{n \in \omega} \mu^*(A_n)$ (countable subadditivity)

When one tries to establish 3. he will soon realize that he needs the axiom $CAC(\mathbb{R})$, i.e., the axiom of choice for countable families of non-empty subsets of \mathbb{R} .

After the definition of μ^* one defines the notion of the (**Carathéodory**) **measurable** set and proves, using the countable subadditivity of μ^* , that the collection \mathcal{L} of all measurable sets forms a σ -algebra **including** the σ -algebra \mathcal{B} of all **Borel** sets. Thus, the proof that \mathcal{L} is a σ -algebra including \mathcal{B} requires some (weak) form of the axiom of choice.

Finally, one defines the Lebesgue measure $\mu : \mathcal{L} \rightarrow [0, +\infty]$ by requiring: $\mu(A) = \mu^*(A)$ for every $A \in \mathcal{L}$. It follows that the definition of μ requires the countable subadditivity of μ^* . Hence, the Lebesgue measure μ cannot be defined in **ZF**, the Zermelo - Fraenkel set theory without the axiom of choice. Hereafter, we will follow the traditional definition of the Lebesgue measure and whenever we refer to the **countable additivity** of μ we shall always mean the **countable subadditivity** of μ^* .

The countable additivity of the Lebesgue measure, Form 37 in [2], clearly implies that the real line \mathbb{R} can not be expressed as a countable union of countable sets as otherwise \mathbb{R} would have zero measure. However, in the Feferman-Lévy forcing model, Model $\mathcal{M}9$ in [2], \mathbb{R} is written as a countable union of countable sets. Thus 37 fails in $\mathcal{M}9$ and the Lebesgue measure cannot be defined in that model.

It is well-known, see [8], that “*there exists a subset of the reals which is not Lebesgue measurable*” and “*there exists a subset of the reals which is not Borel*” (Forms 93 and 273 respectively in [2]). These statements are theorems of **ZFC** (= ZF + the axiom of choice **AC**) and it is impossible to prove either of them, without using some form of choice. Indeed, in 1970 Robert Solovay [9] constructed the forcing model $\mathcal{M}5(\aleph)$ of [2]. In $\mathcal{M}5(\aleph)$, μ is defined (in the traditional way), every subset of reals is measurable and Form 273 is true. Consequently, in ZF, 273 does not imply 93. In 1974, John Truss [11] constructed the forcing model $\mathcal{M}12(\aleph)$ in [2]. In $\mathcal{M}12(\aleph)$ all subsets of reals are Borel. Hence, 273 fails in this model. Yet, it is still unknown whether 93 implies 273. In Proposition 14 and Corollary 15 however, we observe that $37 + 93$ implies 273 and $37 + \text{AC}_\omega(\mathbb{R})$ implies 273.

In Section 4 we study certain ideals of the reals, such as the ideal $\mathcal{M}_\mathbb{R}^\omega$ of all countable unions of countable subsets of \mathbb{R} , the ideal $\mathcal{M}_\mathbb{R}$ of all meager subsets of \mathbb{R} and the ideal \mathcal{L}_N of outer Lebesgue measure zero subsets of \mathbb{R} . In connection with Proposition 1, we supply in Theorem 5 a characterization of the axiom $\text{AC}(\mathbb{R})$ in terms of the ideals $\mathcal{M}_\mathbb{R}$ and \mathcal{L}_N . Furthermore, in Theorem 10 we give conditions under which $\mathcal{M}_\mathbb{R}^\omega \subset \mathcal{M}_\mathbb{R}$ and $\mathcal{M}_\mathbb{R}^\omega \subset \mathcal{L}_N$.

In Section 6 we study a couple of ZFC properties of the σ -algebra \mathcal{B} of the Borel subsets of \mathbb{R} . In particular, it is well-known, see Jech [3], Example 2.4.6. p.22, that the axiom of countable choice implies that the Borel sets lie in a hierarchy. We observe, in Lemma 19, that the regularity of the cardinal \aleph_1 suffices to deduce this property of \mathcal{B} . In [2] it is indicated as unknown

whether the axiom of choice for partitions of \mathbb{R} , Form 203 in [2], implies $|\mathcal{B}| = |\mathbb{R}|$. We prove in Theorem 20 that if “the union of 2^{\aleph_0} sets each of power 2^{\aleph_0} has power 2^{\aleph_0} ”, then 203 implies $|\mathcal{B}| = |\mathbb{R}|$.

In Theorems 16, 18 and Corollary 17 we introduce some weak forms of choice related to \mathcal{B} and study their set-theoretic strength.

In Section 5 we are dealing with the axiom 203 and the question of its possible equivalence with $\text{AC}(\mathbb{R})$. We provide in Corollary 13 a condition under which 203 implies $\text{AC}(\mathbb{R})$.

2 Definitions and some preliminary results

Below we list the statements from set theory and analysis we will be dealing with in the sequel. Any statement of the form “Form x” has been considered in [2] where all known implications between these forms are given in Table I, see <http://www.math.purdue.edu/~jer/Papers/conseq.html>.

Form 5, $\text{CC}_\omega(\mathbb{R})$: For every family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty countable subsets of \mathbb{R} there exists a set $c = \{c_i : i \in \omega\}$ such that for all $i \in \omega$, $c_i \in A_i$.

Form 6, $\text{CUC}(\mathbb{R})$: A countable union of countable subsets of \mathbb{R} is countable.

Form 13 : Every infinite subset of \mathbb{R} has a countably infinite subset.

Form 22 : If every member of an infinite set of cardinality 2^{\aleph_0} has power 2^{\aleph_0} , then the union has power 2^{\aleph_0} .

Form 34 : \aleph_1 is a regular cardinal.

Form 35 : The union of countably many meager subsets of \mathbb{R} is meager.

Form 37 : The Lebesgue measure is countably additive.

Form 38 : \mathbb{R} can not be written as a countable union of countable sets.

Form 79, $\text{AC}(\mathbb{R})$: AC restricted to subsets of \mathbb{R} .

$\text{AC}_\omega(\mathbb{R})$: $\text{AC}(\mathbb{R})$ restricted to countable subsets of \mathbb{R} .

Form 93 : There is a non-measurable subset of reals.

Form 94, $\text{CAC}(\mathbb{R})$: $\text{AC}(\mathbb{R})$ restricted to countable families.

Form 170 : $\aleph_1 \leq 2^{\aleph_0}$.

Form 203 : Every partition of $\wp(\omega)$ into non-empty subsets has a choice function.

Form 212 : $\text{AC}(\mathbb{R})$ restricted to families of size $|\mathbb{R}|$.

Form 273 : There is a subset of \mathbb{R} which is not Borel.

Form 363 : There are exactly 2^{\aleph_0} Borel sets in \mathbb{R} .

Form 368 : The set of all denumerable subsets of \mathbb{R} has power 2^{\aleph_0} .

Lemma 2 (i) ([1]) 203 implies 368.

(ii) ([4]) 203 implies 212.

3 Topological characterizations of the axioms 212 and 203

Theorem 3 *The following are equivalent:*

- (i) 212.
- (ii) *The Tychonoff product of any family $\mathcal{A} = \{A_i : i \in 2^{\aleph_0}\}$ of non-empty subsets of \mathbb{R} is separable.*

Proof. (i) \rightarrow (ii). Fix a family $\mathcal{A} = \{A_i : i \in 2^{\aleph_0}\}$ of non-empty subsets of \mathbb{R} . Let $X = \prod_{i \in k} A_i$ and $B = \{I_n : n \in \omega\}$ be the set of all open intervals with rational endpoints. Put $A = \{I_n \cap A_i : (n, i) \in \omega \times 2^{\aleph_0}\} \setminus \{\emptyset\}$. As $|2^{\aleph_0}| \leq |\omega \times 2^{\aleph_0}| \leq |2^{\aleph_0} \times 2^{\aleph_0}| = |2^{\aleph_0}|$, it follows that A is of power 2^{\aleph_0} . Fix, by 212, a choice set for A $D = \{d_{in} \in A_i \cap I_n : (n, i) \in \omega \times 2^{\aleph_0}\}$. It follows that for each $i \in 2^{\aleph_0}$, $D_i = \{d_{in} : n \in \omega\}$ is an enumeration of a dense set in A_i (taken with the subspace topology). Complete now the proof of (i) \rightarrow (ii) as in theorem 16.4 from [12].

(ii) \rightarrow (i). Fix a family $\mathcal{A} = \{A_i : i \in 2^{\aleph_0}\}$ of non-empty subsets of \mathbb{R} and let $D = \{d_n : n \in \omega\}$ be a dense subset of the Tychonoff product $X = \prod_{i \in 2^{\aleph_0}} B_i, B_i = A_i \cup \{1\}$. For every $i \in 2^{\aleph_0}$, let $n_i = \min\{n \in \omega : d_n(i) \in A_i\}$. Then $c = \{d_{n_i}(i) : i \in 2^{\aleph_0}\}$ is a choice set for \mathcal{A} finishing the proof of the theorem. ■

By mimicking the proof of the last theorem one can readily verify the following

Corollary 4 *The following are equivalent:*

- (i) 203.
- (ii) *The Tychonoff product of any family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non-empty subsets of \mathbb{R} is a non-empty separable space.*
- (iii) *The Tychonoff product of any family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non-empty subsets of \mathbb{R} is separable and $|k| \leq |\mathbb{R}|$.*

4 Meager and null subsets of \mathbb{R}

Let (X, T) be a topological space. A subset G of X is called *meager* or of the *first category* if and only if G can be written as a countable union of *nowhere dense* sets, i.e., sets whose closures have no interior. Since the union of finitely many meager sets is again meager it follows that $\mathcal{M}_X = \{G \in \wp(X) : G = \cup Q, Q \in [\mathcal{N}_X]^{\leq \omega}\}$, where \mathcal{N}_X is the ideal of all nowhere dense sets of X , is an ideal of $\wp(X)$. In particular, we shall be concerned with

the ideal $\mathcal{M}_{\mathbb{R}}$ of all meager subsets of \mathbb{R} and besides $\mathcal{M}_{\mathbb{R}}$ with the following ideals of subsets of \mathbb{R} : $\mathcal{M}_{\mathbb{R}}^c = \{G \in \wp(\mathbb{R}) : G = \cup Q, Q \in [\mathcal{N}_{\mathbb{R}}^c]^{\leq \omega}\}$, where $\mathcal{N}_{\mathbb{R}}^c$ is the ideal of all **closed** nowhere dense sets of \mathbb{R} and, $\mathcal{M}_{\mathbb{R}}^\omega = \{G \in \wp(\mathbb{R}) : G = \cup Q, Q \in [[\mathbb{R}]^{\leq \omega}]^{\leq \omega}\}$. Clearly, the ideal $\mathcal{M}_{\mathbb{R}}^c$ satisfies $\mathcal{M}_{\mathbb{R}}^c \subseteq \mathcal{M}_{\mathbb{R}}$ but in the absence of AC $\mathcal{M}_{\mathbb{R}}^\omega$ may contain non-meager sets or sets of positive outer Lebesgue measure.

Let $\mathcal{L}_N = \{X \subset \mathbb{R} : \mu^*(X) = 0\}$. Clearly \mathcal{L}_N is always an ideal of \mathbb{R} and whenever μ^* is countably subadditive it coincides with the ideal of all Lebesgue null sets.

Theorem 5 *The following are equivalent:*

- (i) $AC(\mathbb{R})$.
- (ii) $\mathcal{M}_{\mathbb{R}} \setminus \{\emptyset\}$ has a choice set.
- (iii) $\mathcal{L}_N \setminus \{\emptyset\}$ has a choice set.

Proof. (i) \longleftrightarrow (ii). We show (\leftarrow) as the other direction is straightforward. Assume that $\mathcal{M}_{\mathbb{R}} \setminus \{\emptyset\}$ has a choice set. Fix $C \subseteq \mathbb{R}$ a closed nowhere dense set of size 2^{\aleph_0} (e.g., Cantor's ternary set; it is constructed by deleting the open middle third of the interval $[0, 1]$, then deleting the open middle thirds of each of the intervals $[0, 1/3]$ and $[2/3, 1]$ and so on. If C_n denotes the union of the 2^n closed intervals of length $1/3^n$ which remain at the n -th stage, then $C = \cap \{C_n : n \in \omega\}$). As $\wp(C) \setminus \{\emptyset\}$ is a family of non-empty meager sets, it follows that C can be well ordered. Thus, \mathbb{R} can be well ordered and consequently $AC(\mathbb{R})$ holds.

(i) \longleftrightarrow (iii). Note that Cantor's ternary set C has Lebesgue outer measure 0 ($\mu^*(C) \leq (2/3)^n$, for all $n \in \omega$). ■

Corollary 6 *The following are equivalent:*

- (i) $CAC(\mathbb{R})$.
- (ii) $CAC(\mathbb{R})$ restricted to the family of all nowhere dense sets.
- (iii) $CAC(\mathbb{R})$ restricted to the family of all Lebesgue measure zero sets.

Theorem 7 (i) 203 implies that $\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ has a choice set.

(ii) The statement " $\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ has a choice set" is equivalent to the proposition "every family $\mathcal{A} = \{A_i : i \in k\}$ of subsets of \mathbb{R} such that each A_i can be expressed as a countable union of closed nowhere dense sets, has a choice set". In particular, " $\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ has a choice set" implies $AC_\omega(\mathbb{R})$.

Proof. (i) The set of all closed subsets of \mathbb{R} has size 2^{\aleph_0} , thus, $|\mathcal{N}_{\mathbb{R}}^c| \leq |\mathbb{R}|$, and by Lemma 1(i), $|\mathcal{M}_{\mathbb{R}}^c|^{\leq \omega} \leq |\mathbb{R}|$. Define a relation \sim on $[\mathcal{N}_{\mathbb{R}}^c]^{\leq \omega}$ by requiring for all $x, y \in [\mathcal{N}_{\mathbb{R}}^c]^{\leq \omega}$, $x \sim y$ iff $\cup x = \cup y$. It is evident that \sim is an equivalence relation on $[\mathcal{N}_{\mathbb{R}}^c]^{\leq \omega}$. By 203 fix a choice set $\{x_i : i \in I\}$ for

$[\mathcal{M}_{\mathbb{R}}^c]^{<\omega} / \sim$. Then $|I| \leq |\mathbb{R}|$. Define a function $f : I \rightarrow \mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ by letting $f(i) = \cup x_i$ for all $i \in I$. It is straightforward to verify that f is a bijection, and so $|\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}| = |I| \leq |\mathbb{R}|$. By Lemma 1(ii) $\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ has a choice set as required.

(ii) The first assertion is straightforward. To see the second, fix $\mathcal{A} = \{A_i : i \in k\}$ a family of non-empty countable subsets of \mathbb{R} . Since each $A_i \in \mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ the conclusion follows immediately. \blacksquare

Definition. We say that a family of functions $\mathcal{F} \subset {}^\omega\omega$ is *dominated* iff there exists a function $g \in {}^\omega\omega$ such that $\forall f \in \mathcal{F}, \forall^\infty n \in \omega, f(n) \leq g(n)$, where $\forall^\infty n$ abbreviates the statement: for all but finitely many $n \in \omega$.

Let $\mathcal{BF} = \{\mathcal{F} \subset {}^\omega\omega : \mathcal{F} \text{ is dominated}\}$. In view of the forthcoming theorem 8 (i) it is easy to see that every countable $\mathcal{F} \subset {}^\omega\omega$ is dominated. Thus, \mathcal{BF} includes $[{}^\omega\omega]^\omega$, i.e., the set all countable subsets of ${}^\omega\omega$. Clearly, “ \mathcal{BF} is σ -closed” implies “the subideal $[{}^\omega\omega]^\omega$ of \mathcal{BF} is σ -closed under domination”. We shall show in theorem 10 that “[${}^\omega\omega$] is σ -closed under domination” implies that $\mathcal{M}_{\mathbb{R}}^\omega \subseteq \mathcal{M}_{\mathbb{R}}$.

Theorem 8 (i) *CUC*(\mathbb{R}) implies *CBF* (= For every family $\mathcal{F} = \{F_i \subset {}^\omega\omega : i \in \omega\}$ of non-empty countable sets there exists a set $H = \{h_i \in {}^\omega\omega : i \in \omega\}$ such that: $\forall i \in \omega, \forall f \in F_i, \forall^\infty n \in \omega, f(n) \leq h_i(n)$).
(ii) *CBF* iff $[{}^\omega\omega]^\omega$ is σ -closed under domination.
(iii) *CBF* implies 38.

Proof. (i) Fix $\mathcal{F} = \{F_i \subset {}^\omega\omega : i \in \omega\}$ a family of countable sets. By *CUC*(\mathbb{R}) $\cup\mathcal{F}$ is countable. Let $\{f_n : n \in \omega\}$ be an enumeration of $\cup\mathcal{F}$. It can be readily verified that the function $h : \omega \rightarrow \omega$ given by $h(n) = \max\{f_m(n) : m \leq n\}$ satisfies $\forall i \in \omega, \forall f \in F_i, \forall^\infty n \in \omega, f(n) \leq h(n)$ as required.

(ii) Fix $\mathcal{A} = \{A_i \subset {}^\omega\omega : i \in \omega\}$ a family of non-empty countable sets. Let by *CBF* $H = \{h_i \in {}^\omega\omega : i \in \omega\}$ be a family of functions satisfying: $\forall i \in \omega, \forall f \in F_i, \forall^\infty n \in \omega, f(n) \leq h_i(n)$. Continue now as in (i) in order to define a function $h : \omega \rightarrow \omega$ dominating $\cup\mathcal{A}$.

(iii) If ${}^\omega\omega$ can be expressed as $\cup\{A_i \subset {}^\omega\omega : i \in \omega\}$ of countable sets then ${}^\omega\omega$ is dominated by some $g \in {}^\omega\omega$ which is impossible. \blacksquare

Clearly, *CUC*(\mathbb{R}) implies $\mathcal{M}_{\mathbb{R}}^\omega \subseteq \mathcal{M}_{\mathbb{R}}$. Furthermore, in Cohen’s original model, model $\mathcal{M}1$ in [2], $\mathcal{M}_{\mathbb{R}}^\omega \subseteq \mathcal{M}_{\mathbb{R}}$ is true, since *CUC*(\mathbb{R}) is true in $\mathcal{M}1$, but *CAC*(\mathbb{R}) fails. Thus, *CBF* does not imply *CAC*(\mathbb{R}).

ω^ω will denote the **Baire space**, i.e., the Tychonoff product of ω copies of the discrete space ω . All the properties we consider regarding category and measure are equivalent whether stated for \mathbb{R} or the Baire space ω^ω .

By mimicking the proof of Theorem 1.2 from [7] one can readily verify the following result.

Theorem 9 *35 implies CBF.*

Theorem 10 (i) *CBF implies “a countable union of countable sets is meager”.*

(ii) *“The real line cannot be covered by countably many meager sets” implies “the union of countably many countable subsets of \mathbb{R} has Lebesgue outer measure 0”. Consequently, Form 35 implies the latter statement.*

Proof. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of countable sets.

(i) Since the set \mathbb{Q} of rational numbers is meager, it suffices to show that $\cup\mathcal{A}\setminus\mathbb{Q}$ is meager. For every $x \in \cup\mathcal{A}\setminus\mathbb{Q}$ let $D_x = \mathbb{R}\setminus\{x\}$. Clearly $\mathcal{D} = \{D_x : x \in \cup\mathcal{A}\setminus\mathbb{Q}\}$ is a family of dense open sets of \mathbb{R} such that each D_x includes \mathbb{Q} . Let $\{q_n : n \in \omega\}$ be an enumeration of \mathbb{Q} . For every $x \in \cup\mathcal{A}\setminus\mathbb{Q}$ define a function $f_x : \omega \rightarrow \omega$ by requiring: $f_x(n) = \min\{m \in \mathbb{N} : x \notin (q_n - \frac{1}{2m}, q_n + \frac{1}{2m})\}$. Put $\mathcal{F} = \{\{f_x : x \in A_i\setminus\mathbb{Q}\} : i \in \omega\}$ and let, by CBF, $h : \omega \rightarrow \omega$ be a function dominating $\cup\mathcal{F}$. On the basis of h define for every $n \in \omega$ a subset \mathbf{D}_n of \mathbb{R} by requiring: $\mathbf{D}_n = \cup\{(q_m - \frac{1}{2h(m)}, q_m + \frac{1}{2h(m)}) : m \geq n\}$. Since $Q_n = \{q_m : m \geq n\}$ differs from \mathbb{Q} in a finite set, it follows that each \mathbf{D}_n is a dense open set of \mathbb{R} . Furthermore, for every $x \in \cup\mathcal{A}\setminus\mathbb{Q}$ there exists $n \in \omega$ such that $\mathbf{D}_n \subset D_x$. Indeed, fix $x \in \cup\mathcal{A}\setminus\mathbb{Q}$ and $n_0 \in \omega$ satisfy: $(\forall n \geq n_0)(f_x(n) \leq h(n))$. It follows that for every $n \geq n_0$ $(q_n - \frac{1}{2h(n)}, q_n + \frac{1}{2h(n)}) \subset (q_n - \frac{1}{2f_x(n)}, q_n + \frac{1}{2f_x(n)})$. Thus, for every $n \geq n_0$, $x \notin (q_n - \frac{1}{2h(n)}, q_n + \frac{1}{2h(n)})$ and consequently $x \notin \mathbf{D}_{n_0}$. Hence, $\mathbf{D}_{n_0} \subset D_x$ as required.

Now, it follows that $\cup\mathcal{A}\setminus\mathbb{Q} \subset \cup\{\mathbb{R}\setminus D_x : x \in \cup\mathcal{A}\setminus\mathbb{Q}\} \subset \cup\{\mathbb{R}\setminus \mathbf{D}_n : n \in \omega\}$. Since $K = \cup\{\mathbb{R}\setminus \mathbf{D}_n : n \in \omega\}$ is a countable union of closed nowhere dense sets, it follows that K is a meager set and consequently $\cup\mathcal{A}\setminus\mathbb{Q}$, being a subset of a meager set, is a meager set as required.

(ii) In order to show that $\mu^*(\cup\mathcal{A}) = 0$, it suffices to show that for every $\varepsilon > 0$ there exists an open set O_ε with $\cup\mathcal{A} \subset O_\varepsilon$ and $\mu^*(O_\varepsilon) < \varepsilon$. Let $\{q_n : n \in \omega\}$ be an enumeration of \mathbb{Q} . For every $n, m \in \omega$ put $B_m^n = (q_m - \varepsilon/2^{n+1}, q_m - \varepsilon/2^{n+1})$ and for every $x \in \cup\mathcal{A}$, let $N_x = \{f \in {}^\omega\omega : \exists^\infty n \in \omega, x \in B_{f(n)}^n\}$, where $\exists^\infty n \in \omega$ abbreviates the statement: There are infinitely many $n \in \omega$. It can be readily verified that $N_x = \cap\{D_{x,m} = \{f \in {}^\omega\omega : \exists n > m \in \omega, x \in B_{f(n)}^n\} : m \in \omega\}$, and each $D_{x,m}$ is a dense open subset of the Baire space ${}^\omega\omega$. By our hypothesis it follows that $G = \cap\{D_{x,m} : x \in \cup\mathcal{A}, m \in \omega\} \neq \emptyset$. Fix $g \in G$. It is straightforward to see that $\cup\mathcal{A} \subset O_\varepsilon = \cup\{B_{g(n)}^n : n \in \omega\}$ and $\mu^*(O_\varepsilon) < \varepsilon$ as required. \blacksquare

Remark. In $\mathcal{M}12(\aleph)$ $\text{CUC}(\mathbb{R})$, hence CBF , holds but both 34 and 170 fail, see [2] and [11]. Thus, CBF implies neither 34 nor 170 in ZF .

5 When 203 is equivalent to $\text{AC}(\mathbb{R})$

In this section we provide a condition under which 203 is equivalent to $\text{AC}(\mathbb{R})$.

Theorem 11 *The following are equivalent:*

- (i) 203.
- (ii) Every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty subsets of \mathbb{R} such that for all $i, j \in k, i \neq j$, $A_i, A_j \not\subseteq \overline{A_i \cap A_j}$, has a choice set.
- (iii) Every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty subsets of \mathbb{R} for which there is a family $\mathcal{B} = \{B_n : n \in \omega\}$ of subsets of \mathbb{R} such that for all $i, j \in k, i \neq j$ if $A_i \cap A_j \neq \emptyset$, then there exists $n \in \omega$ with $A_i \cap B_n \neq \emptyset$ and $A_j \cap B_n = \emptyset$ or $A_i \cap B_n = \emptyset$ and $A_j \cap B_n \neq \emptyset$, has a choice set.
- (iv) Every family $\mathcal{A} = \{A_i : i \in m\}$ of non-empty subsets of \mathbb{R} such that \mathcal{A} is disjoint or $\mathcal{A} = \cup\{B_n : n \in k\}, |k| \leq |\mathbb{R}|$, for all $\overline{n, m} \in k, n \neq m, B_n \cap B_m = \emptyset$, and for all $n \in k$ and $a, b \in B_n$, $a \not\subseteq \overline{a \setminus b}$, $b \not\subseteq \overline{b \setminus a}$, has a choice set.

Proof. (i) \rightarrow (ii). Fix a family \mathcal{A} as in (ii). For every $i \in k$, let $B_i = \{X \in [A_i]^\omega : \overline{X} = A_i\}$.

Claim. 203 implies that each B_i is a non-empty set.

Proof of the claim. Fix an $i \in k$. It is known that 203 implies $\text{CAC}(\mathbb{R})$ (see [2], Note 67) and since every subspace of \mathbb{R} has a countable base, $\text{CAC}(\mathbb{R})$ implies that A_i is separable. This completes the proof of the claim. \blacksquare (claim)

By our hypothesis we also have that $B_i \cap B_j = \emptyset$ for all $i, j \in k, i \neq j$. As $[A_i]^\omega \subseteq [\mathbb{R}]^\omega$ and $|[\mathbb{R}]^\omega| = |\mathbb{R}|$ (Lemma 1(i)), we may consider the B_i 's as subsets of \mathbb{R} and by 203 we may choose for each $i \in k$, a $b_i \in B_i$. By 203 again, $\{b_i : i \in k\}$ has a choice set c . Clearly c is a choice set for \mathcal{A} as well.

(ii) \rightarrow (i). This is straightforward.

(i) \rightarrow (iii). Fix \mathcal{A} and \mathcal{B} as in (iii). For each $i \in k$, let

$$X_i = \prod_{n \in \omega} Y_{in}, Y_{in} = \begin{cases} A_i \cap B_n & \text{if } A_i \cap B_n \neq \emptyset \\ A_i & \text{otherwise} \end{cases}.$$

Claim 1. For each $i \in k$, X_i is non-empty.

Proof of claim 1. Fix an $i \in k$ and let $M = \{n \in \omega : A_i \cap B_n \neq \emptyset\}$. Without loss of generality we may assume that M is infinite. By $\text{CAC}(\mathbb{R})$, $\{A_i \cap B_n : n \in M\}$ has a choice set $\{c_{in} : n \in M\}$. Then if $x \in A_i$, the function f defined by

$$f(n) = \begin{cases} c_{in} & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}, \text{ for all } n \in \omega$$

belongs to X_i and the proof of the claim is complete. ■ (claim 1)

Claim 2. For all $i, j \in k, i \neq j, X_i \cap X_j = \emptyset$.

Proof of claim 2. Fix $i, j \in k, i \neq j$. We consider the following two cases:

(i) $A_i \cap A_j = \emptyset$. By the definition of the X_i 's it is immediate that $Y_{in} \cap Y_{jn} = \emptyset$ for all $n \in \omega$. Thus, $X_i \cap X_j = \emptyset$.

(ii) $A_i \cap A_j \neq \emptyset$. Fix $n \in \omega$ such that $A_i \cap B_n \neq \emptyset$ and $A_j \cap B_n = \emptyset$. Then $Y_{in} = A_i \cap B_n$ and $Y_{jn} = A_j$. Hence $Y_{in} \cap Y_{jn} = \emptyset$ and consequently $X_i \cap X_j = \emptyset$ finishing the proof of the claim. ■ (claim 2)

Put $\mathcal{F} = \{X_i : i \in k\}$. Then \mathcal{F} is a disjoint family of non-empty subsets of \mathbb{R}^ω . Since $|\mathbb{R}^\omega| = |\mathbb{R}|$ we may view \mathcal{F} as a family of subsets of \mathbb{R} . Thus, by 203 we may fix a choice set $\{x_i : i \in k\}$ of \mathcal{F} . Then $\{x_i(0) : i \in k\}$ is clearly a choice set for the family \mathcal{A} .

(iii) \rightarrow (i). This is straightforward.

(i) \rightarrow (iv). Fix a family \mathcal{A} of subsets of \mathbb{R} as in (iv). For every $n \in k$, let $C_n = \{X \in [\mathbb{R}]^\omega : \forall b \in B_n, X \cap b \neq \emptyset\}$.

Claim. $C_n \neq \emptyset$.

Proof of the claim. Fix $b \in B_n$, and let by 203, X be a countable dense subset of b . If $a \in B_n$ then $X \cap a \neq \emptyset$ as otherwise $X \subset b \setminus a$ and $b \subseteq \overline{X} \subseteq \overline{b \setminus a}$, a contradiction, finishing the proof of the claim. ■ (claim)

Put $C = \{C_n : n \in k\}$. By Lemma 1(i) we may view C as a family of subsets of \mathbb{R} . Since $|k| \leq |\mathbb{R}|$, let by 203, $\{X_n : n \in k\}$ be a choice set of the family C and fix, by 203 again, a well ordering $\{x_{nv} : v \in \omega\}$ of each X_n . It can be readily verified that the set $c = \{x_{n_i v_i} : i \in m\}$, where n_i is the unique $n \in k$ with $A_i \in B_n$ and v_i is the least $v \in \omega$ with $x_{n_i v} \in A_i$, is a choice set for the family \mathcal{A} finishing the proof.

(iv) \rightarrow (i). This is straightforward. ■

Corollary 12 (i) 203 implies that the power of every family \mathcal{A} as in (iii) of Theorem 11 is $\leq |\mathbb{R}|$.

(ii) The statement: "every family $\mathcal{A} = \{A_i : i \in m\}$ of non-empty subsets of \mathbb{R} such that \mathcal{A} is disjoint or $\mathcal{A} = \cup\{B_n : n \in k\}, |k| \leq |\mathbb{R}|$, and for all $n \in k$ and $a, b \in B_n, a \setminus b$ is nowhere dense in a with the subspace topology and $b \setminus a$ is nowhere dense in b , has a choice set" can be added to the list of Theorem 11.

Proof. (i) Since the sets X_i defined in the proof of Theorem 11 (i) \rightarrow (iii) are pairwise disjoint and indexed by the same set k , we conclude by 203 that $|k| \leq |\mathbb{R}|$.

(ii) Assume 203 and let \mathcal{A} be a family of subsets of \mathbb{R} as in (ii). For each $n \in k$, define:

$$C_n = \{X \in [\mathbb{R}]^\omega : \forall b \in B_n, b \subseteq \overline{X \cap b}\}. \quad (1)$$

Claim. $C_n \neq \emptyset$.

Proof of the claim. Fix $b \in B_n$, and let by 203, X be a countable dense subset of b . Let $a \in B_n$. We have $X = [X \cap (b \setminus a)] \cup [X \cap (b \cap a)]$ and $b = \overline{X} = \overline{[X \cap (b \setminus a)]} \cup \overline{[X \cap (b \cap a)]}$ (the closures are taken in the subspace b). As the interior of $[X \cap (b \setminus a)]$ in b is empty it follows that every non-empty open set O in b meets $c = X \cap (b \cap a)$ meaning that c is dense in b , hence, in $b \cap a$ also. We show that c is dense in a . Fix Q a non-empty open set in a . Since $a = (a \setminus b) \cup (a \cap b)$ and $a \setminus b$ is nowhere dense in a , it follows that Q meets $a \cap b$, and as c is dense in $a \cap b$, we have that Q meets c . Therefore, $c = X \cap a$ is dense in a and the proof of the claim is complete. \blacksquare (claim)

By mimicking the proof of the claim we can also show that for all $n, m \in k$, if $B_n \cap B_m \neq \emptyset$, then $C_n = C_m$, where C_n, C_m are given by (1). Define now a relation \sim on k by requiring for all $n, m \in k$, $n \sim m$ iff $C_n = C_m$. Clearly \sim is an equivalence relation on k and since $|k| \leq |\mathbb{R}|$, by 203 we conclude that $|k/\sim| \leq |\mathbb{R}|$, where k/\sim is the set of all equivalence classes. Let by 203, $G = \{n : n \in I \subseteq k\}$ be a choice set for k/\sim and put $C' = \{C_n : n \in I\}$. As in the proof of (i) \rightarrow (iv) of Theorem 11, let $\{X_n : n \in I\}$ be a choice set of the family C' and for each $n \in I$, let $\{x_{nv} : v \in \omega\}$ be a well ordering of X_n . Then $c = \{x_{n_i v_i} : i \in m\}$, where n_i is the unique $n \in I$ with $A_i \in B_n$ and v_i is the least $v \in \omega$ with $x_{n_i v} \in A_i$, is a choice set for the family \mathcal{A} finishing the proof.

On the other hand it is straightforward that the statement given in (ii) implies 203. This completes the proof of the corollary. \blacksquare

Remark. We define a relation \leq on $\wp(\mathbb{R}) \setminus \{\emptyset\}$ by requiring: $x \leq y \iff (\forall g)(\overline{x \cap g} \supseteq x \rightarrow \overline{y \cap g} \supseteq y)$. It is easy to see that the following hold:

- (a). If $x \leq y$, then x must intersect y (otherwise for $g = x$ we have $\overline{y \cap g} = \emptyset$).
- (b). If $x \leq y$, then $x \cap y$ is dense in y .
- (c). \leq is reflexive and transitive but not, in general, antisymmetric (take for example, $x = [0, 1)$ and $y = (0, 1]$).

As \leq is reflexive and transitive, it follows that the relation \sim on $\wp(\mathbb{R}) \setminus \{\emptyset\}$ given by: $x \sim y$ iff $x \leq y$ and $y \leq x$, i.e., $x \sim y \iff (\forall g)(\overline{x \cap g} \supseteq x \leftrightarrow \overline{y \cap g} \supseteq y)$ is an equivalence relation. The following are easily verified.

- (a). If x is finite, then the only element of $\wp(\mathbb{R}) \setminus \{\emptyset\}$ equivalent to x is x itself.
- (b). If $x \sim y$, then $x \cap y$ is dense in both x and y .
- (c). If $x \sim y$, then $x \cap y$ is non nowhere dense in both x and y .

(d). Each equivalence class $[x]$ of \sim satisfies: For all $a, b \in [x]$, $a \setminus b$ is not dense in the subspace a and $b \setminus a$ is not dense in b .

In view of the above Remark and Theorem 11 we immediately have the following

Corollary 13 *If $|(\wp(\mathbb{R}) \setminus \{\emptyset\}) / \sim| = |\mathbb{R}|$, then 203 implies $AC(\mathbb{R})$.*

6 Choice and the σ -algebra of Borel subsets of \mathbb{R}

Proposition 14 (i) $AC_\omega(\mathbb{R}) + \text{Form 37}$ implies Form 273.

(ii) Form 273 implies Form 38.

Proof. (i) The proof of the existence of a non-measurable set c as given in [3] p.2, goes through in case AC is replaced by $AC_\omega(\mathbb{R})$ because the equivalence classes are countable subsets of \mathbb{R} . Since 37 implies $\mathcal{B} \subset \mathcal{L}$, it follows that $c \notin \mathcal{B}$ and this completes the proof of (i).

(ii) Assume on the contrary that $\mathbb{R} = \cup\{A_n : n \in \omega\}$, where $|A_n| = \omega$. Then every subset of reals, being a countable union of countable, hence Borel, sets is Borel. This contradicts 273 and completes the proof of the proposition.

■

Corollary 15 *Form 37 + Form 93 implies Form 273.*

Next we introduce some weak forms of choice regarding \mathcal{B} . Let $\mathbf{BC}(\mathbb{R})$ stand for the statement “ $\mathcal{B} \setminus \{\emptyset\}$ has a choice set”, and let $\mathbf{CBC}(\mathbb{R})$ stand for $\mathbf{BC}(\mathbb{R})$ restricted to countable subfamilies of \mathcal{B} .

Theorem 16 (i) $BC(\mathbb{R})$ implies $\mathcal{M}_{\mathbb{R}}^c \setminus \{\emptyset\}$ has a choice set. In particular, $\mathbf{CBC}(\mathbb{R})$ implies $CC_\omega(\mathbb{R})$.

(ii) $\mathbf{CBC}(\mathbb{R})$ implies Form 38.

(iii) $BC(\mathbb{R})$ implies every uncountable Borel subset of the reals has an uncountable well-ordered subset. In particular, $BC(\mathbb{R})$ implies 170.

(iv) Form 170 + every uncountable Borel subset of the reals has a perfect (= closed without isolated points) subset implies $CUC(\mathbb{R})$.

(v) $BC_\omega(\mathbb{R}) = \{X \in \mathcal{B} : |X| \leq \aleph_0\} \setminus \{\emptyset\}$ has a choice set implies $UT_{\mathbb{R}}(WO, \aleph_0, WO)$ (= the well-ordered union of countable subsets of \mathbb{R} can be well-ordered).

Proof. (i) The first assertion of (i) follows from the observation that every set which is expressible as a countable union of closed nowhere dense sets is Borel. To see the second, fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ consisting of non-empty countable subsets of \mathbb{R} . As singletons of \mathbb{R} are closed and \mathcal{B} is a σ -algebra, we have that each $A_i \in \mathcal{B}$. Thus, \mathcal{A} has a choice set as required.

(ii) This follows from (i) and the fact that $CC_\omega(\mathbb{R})$ implies Form 38, see [1].

(iii) Let f be a choice function of $\mathcal{B} \setminus \{\emptyset\}$. We define via a straightforward induction on \aleph_1 a subset $A = \{a_i : i \in \aleph_1\}$ of \mathbb{R} . For $i = 0$ let $a_0 = f(\mathbb{R})$, for $i = v + 1$ let $a_i = f(\mathbb{R} \setminus \{a_j : j \leq v\})$ if $\mathbb{R} \setminus \{a_j : j \leq v\} \neq \emptyset$ and for i a limit ordinal of \aleph_1 , we let $a_i = f(\mathbb{R} \setminus \{a_j : j \in i\})$ if $\mathbb{R} \setminus \{a_j : j \in i\} \neq \emptyset$.

(iv) Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family of countable subsets of \mathbb{R} . Assume on the contrary that $\cup \mathcal{A}$ is not countable. Clearly, $\cup \mathcal{A} \in \mathcal{B}$. Since in ZF every perfect set has power 2^{\aleph_0} , see [11], our hypothesis yields that $|\cup \mathcal{A}| = |\mathbb{R}|$. Furthermore, by 170 we may conclude that \aleph_1 can be expressed as a countable union of countable subsets of \mathbb{R} . Thus, \aleph_1 is an uncountable Borel subset of \mathbb{R} . Consequently, $|2^{\aleph_0}| = |\aleph_1|$. But then \mathbb{R} is well ordered, hence, it can not be expressed as a countable union of countable sets. A contradiction.

(v) Let $\mathcal{A} = \{A_i : i \in \aleph\}$ be a family of countable subsets of \mathbb{R} . Clearly, $\mathcal{M} = \cup\{\emptyset(A_i) \setminus \{\emptyset\} : i \in \aleph\} \subset \{X \in \mathcal{B} : |X| \leq \aleph_0\} \setminus \{\emptyset\}$. Fix, by $BC_\omega(\mathbb{R})$, f a choice function of the family \mathcal{M} . On the basis of f and similarly to the proof of (iii) one can easily construct via a transfinite induction on \aleph_1 a well-ordering $\{a_{ij} : j \in v_i\}$ of each A_i of type $v_i \in \aleph_1$. It follows that $\cup \mathcal{A} = \cup\{\{a_{ij} : j \in v_i\} : i \in \aleph\}$ being a well-ordered union of well-ordered sets is well-ordered as required finishing the proof of (v) and of the theorem.

■

Corollary 17 (i) *The statement “every uncountable Borel subset of \mathbb{R} has a perfect subset” is unprovable in ZF.*

(ii) $BC_\omega(\mathbb{R})$ iff $AC_\omega(\mathbb{R})$.

(iii) $AC_\omega(\mathbb{R}) + 34$ implies $CUC(\mathbb{R})$.

(iv) $BC_\omega(\mathbb{R})$ does not imply 13.

(v) $BC(\mathbb{R}) + 22$ implies 368.

(vi) $BC(\mathbb{R})$ iff for every family $\mathcal{A} = \{A_i : i \in k\}$ of subsets of \mathbb{R} such that each A_i can be written as a countable union of Borel sets has a choice set.

Proof. (i) It is shown in [1] that in Figura’s model $\mathcal{M}36$ in [2], $CUC(\mathbb{R})$ is false. On the other hand Figura proves that 170 is true in his model. Thus, by Theorem 16 it follows that in $\mathcal{M}36$ there exists an uncountable Borel subset of reals with no perfect subset.

(iii) Fix a countable family A consisting of countable subsets of \mathbb{R} . By Theorem 16 (v) we have that $\cup A$ is well-orderable. If $\cup A$ is uncountable, then $\aleph_1 \leq |\cup A|$ contradicting the regularity of \aleph_1 .

(iv) $\text{AC}_\omega(\mathbb{R})$ holds in $\mathcal{M}1$ but Form 13 fails.

(v) Following the ideas of the proof of Theorem 16 (v), one can readily verify that $\text{BC}(\mathbb{R})$ implies $|\mathbb{R}^\omega| \leq |\mathbb{R}^{<\aleph_1}| = |\cup \{v\mathbb{R} : v \in \aleph_1\}|$. Since $\text{BC}(\mathbb{R})$ implies 170, we may view \aleph_1 as a subset of the reals. On the other hand, $|v\mathbb{R}| = |\omega\mathbb{R}| = |\mathbb{R}|$, recall that $\omega\mathbb{R}$ with the Tychonoff product topology is a separable metrizable space. By 22 we conclude that $|\cup \{v\mathbb{R} : v \in \aleph_1\}| \leq |\mathbb{R}|$ and the proof is complete. \blacksquare

$\text{BC}(X, \mathbb{R})$ stands for the statement “if $X \subset \mathbb{R}$, then the Borel algebra \mathcal{B}_X of X with the relative topology has a choice set” and $\text{CBC}(X, \mathbb{R})$ denotes its countable version.

Theorem 18 (i) $\text{BC}(X, \mathbb{R})$ implies “every uncountable subset of the reals has an uncountable well-ordered set”.

(ii) $\text{CBC}(X, \mathbb{R})$ iff $\text{CAC}(\mathbb{R})$.

Proof. (i) Fix X an uncountable subset of reals and let f be a choice function on $\mathcal{B}_X \setminus \{\emptyset\}$. Working as in the proof of Theorem 16 (iii) one can readily verify that X has a subset of size \aleph_1 .

(ii) It suffices to show (\rightarrow) as the other implication is obvious. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of non-empty subsets of \mathbb{R} . Without loss of generality we may assume that for every $i \in \omega$, $A_i \subset (i, i+1)$. Clearly, $\mathcal{A} \subset \mathcal{B}_X$, $X = \cup \mathcal{A}$ and by $\text{CBC}(X, \mathbb{R})$ \mathcal{A} has a choice as required. \blacksquare

Let B_0 be the set of all open sets of \mathbb{R} .

For $\alpha = \lambda + 1$, a successor ordinal of \aleph_1 , let $B_\alpha = \{X \setminus Y : (X, Y) \in B_\lambda \times B_\lambda\} \cup \{\cup X : X \in [B_\lambda]^\omega\} \cup \{\cap X : X \in [B_\lambda]^\omega\}$.

For α a limit ordinal of \aleph_1 , put $B_\alpha = \cup_{\beta < \alpha} B_\beta$. Put

$$\mathfrak{B} = \cup \{B_\alpha : \alpha \in \aleph_1\}. \quad (2)$$

In view of the Example 2.4.6 in [3] p.22 and the subsequent remarks one can readily verify that:

Lemma 19 If Form 34 is true, then $\mathcal{B} = \mathfrak{B}$, where $\mathfrak{B} \subseteq \wp(\mathbb{R})$ is the family given in (2).

It is unknown whether 203 implies 363, see Table I in [2]. We show next that 203 + 22 implies 363.

Theorem 20 (i) *203 + 22 implies 363.*

(ii) *$BC(\mathbb{R}) + 22$ implies 363.*

(iii) *203 + 22 implies $BC(\mathbb{R})$.*

(iv) *203 + 363 implies $BC(\mathbb{R})$.*

Proof. (i) It is known, see [1], that 203 implies both 34 and 170. Thus, we may consider \aleph_1 as a subset of \mathbb{R} . Moreover, by Lemma 19 we have that $\mathcal{B} = \cup\{B_\alpha : \alpha \in \aleph_1\}$. Since the set T of all open subsets of \mathbb{R} has power 2^{\aleph_0} and $T \subset \mathcal{B}$ we have $|\mathbb{R}| \leq |\mathcal{B}|$. It remains to show that $|\mathcal{B}| \leq |\mathbb{R}|$. By transfinite induction on \aleph_1 we show first that $|B_\alpha| \leq |\mathbb{R}|$, for all $\alpha \in \aleph_1$.

For $\alpha = 0$, $|B_0| = |T| = |\mathbb{R}|$.

For $\alpha = \lambda + 1$ we have $|B_\lambda \times B_\lambda| \leq |\mathbb{R}|$ and $|[B_\lambda]^\omega| \leq |\mathbb{R}|$ (This follows from the induction hypothesis and the fact that 203 implies 368, see Lemma 1). Consequently, $|B_\alpha| \leq |\mathbb{R}|$.

For α a limit ordinal, it follows by 22 that $|B_\alpha| = |\cup\{B_\beta : \beta < \alpha\}| \leq |\mathbb{R}|$. The induction terminates.

One more application of 22 yields $|\mathcal{B}| = |\cup\{B_\alpha : \alpha \in \aleph_1\}| \leq |\mathbb{R}|$ and completes the proof of (i).

(ii) This can be proved as in (i) using the result of Corollary 17 that the conjunction of $BC(\mathbb{R})$ and 22 implies 368.

(iii) This follows from (i) and the fact that 203 implies 212, see Lemma 1.

(iv) As in (iii). ▀

Remark. It is clear that 363 implies both 273 and 368. Alfred Tarski [10] has shown that in ZF 368 implies 170. Thus, 363 implies 170. On the other hand, the validity of the implications $170 \implies 363$, $273 \implies 363$, $273 \implies 170$ is indicated as unknown in Table 1 of [2]. We point out here that none of the latter implications holds true in ZF. Indeed, in [4] we showed that in Feferman's model, called $\mathcal{M}2$ in [2], 170 is true, but 368 is false. In addition, 273 holds in $\mathcal{M}2$, see [2]. Thus, neither 170 nor 273 implies 363 in ZF. Finally, 273 does not imply 170 since in Solovay's model, called $\mathcal{M}5(\aleph)$ in [2], 273 holds but 170 fails, see [2].

7 Summary

The following diagram summarizes the interrelations between weak forms of choice studied in this paper.

- [9] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. Math. Ser. 2, **92** (1970), 1-56.
- [10] A. Tarski, *On well-ordered subsets of any set*, Fund. Math. **32** (1939), 176-183.
- [11] J. Truss, *Models of set theory containing many perfect sets*, Ann. Math. Logic **7** (1974), 197-219.
- [12] S. Willard, *General Topology*, Addison-Wesley Publ. Co., 1968.

Kyriakos Keremedis
Department of Mathematics,
University of the Aegean,
Karlovasi, 83200, Samos, GREECE
e-mail: kker@aegean.gr

Eleftherios Tachtsis
Department of Statistics and Actuarial Science,
University of the Aegean,
Karlovasi, 83200, Samos, GREECE
e-mail: ltah@aegean.gr