Independent Families and Some Notions of Finteness

Eric Hall, Kyriakos Keremedis

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Abstract

In **ZF**, the well known Fichtenholz-Kantorovich-Hausdorff theorem concerning the existence of independent families of X of size $|\mathcal{P}(X)|$ is equivalent to the following portion of the equally well known Hewitt-Marczewski-Pondiczery theorem concerning the the density of product spaces: "The product $\mathbf{2}^{\mathcal{P}(X)}$ has a dense subset of size $|X|$". However, the latter statement turns out to be strictly weaker than **AC** but the full Hewitt-Marczewski-Pondiczery theorem is equivalent to **AC**.

We study the relative strengths in **ZF** between the statement "X has no independent family of size $|\mathcal{P}(X)|$" and some of the definitions of "X is finite" studied in Levy's classic paper, observing that the former statement implies one such definition, is implied by another, and incomparable with some others.

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1 Notation and Prerequisites

$\mathbf{2}^X$ denotes the Tychonoff product of the discrete space $\mathbf{2}$ ($2 = \{0, 1\}$ is taken with the discrete topology). The canonical (clopen) base for the product topology on $\mathbf{2}^X$ will be denoted by

$$\mathcal{B}(\mathbf{2}^X) = \{[p] : p \in \mathrm{Fn}(X, 2, \omega)\},$$

where $\mathrm{Fn}(X, 2, \omega)$ is the set of all finite partial functions from X into 2 and

$$[p] = \{f \in 2^{\mathbf{X}} : p \subset f\}.$$

A family $\mathcal{F}$ of subsets of X is *independent* if for any two non-empty finite and disjoint subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ the set $\bigcap \mathcal{A} \cap (\bigcap \{B^c : B \in \mathcal{B}\}$ is infinite. An independent family of subsets of X is called *large* if it has cardinality $|\mathcal{P}(X)|$.

Below we define the choice principles we are going to deal with in this paper. Let X be an infinite set:

1. $\mathbf{LIF}(X)$: *X has a Large Independent Family—that is, an independent family of size $|\mathcal{P}(X)|$.*

2. **FKHT** (Fichtenholz-Kantorovich-Hausdorff theorem) : *For every infinite set X, $\mathbf{LIF}(X)$.*

3. $\mathbf{HM}(X)$: *The product $\mathbf{2}^{\mathcal{P}(X)}$ has a dense set of size $|X|$.*

4. **HMPT** (Hewitt-Marczewski-Pondiczery theorem) : *For every infinite set X, $\mathbf{HM}(X)$.*

5. **SHMPT** (Strong Hewitt-Marczewski-Pondiczery theorem) : *For every set k and every family $\{\mathbf{X}_i : i \in I\}$ of topological spaces such that $|I| \leq |2^k|$ and each $\mathbf{X}_i$ has a dense subset of size $\leq |k|$, the Tychonoff product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ has a dense set of size $\leq |k|$.*

6. $\mathbf{T}(X)$: $|X \times X| = |X|$.

7. $\mathbf{T}^+(X)$: $|X| = |[X]^{<\omega}|$. ($[X]^{<\omega}$ denotes the set of finite subsets of X.)

8. $\mathbf{BPI}(X)$: *Every filterbase of X is included in an ultrafilter of X.*

9. **BPI** : $(\forall Y)\mathbf{BPI}(Y)$.

10. **AC** : *Every family of non-empty sets has a choice function.*

In this paper, the intended context for reasoning and statements of theorems will be **ZF**, unless otherwise noted. The theory **ZFA** is a weakening of **ZF** which allows atoms. Some independence results will be obtained by using permutation models, which are models of **ZFA**. However, unless otherwise noted, these results transfer to yield independence results in **ZF** by the embedding theorems of Jech and Sochor, and of Pincus. We will also use symmetric models, which are models of **ZF**. For more about **ZFA**, permutation models, symmetric models, embedding theorems, and further references, please see [6], and see Note 103 of [5] for a summary of further embedding (transfer) theorems. We will freely use terminology and techniques for working with permutation models and symmetric models as found in [6]. In particular, we will refer to some well-known models of **ZF** and **ZFA** which are described well in [6], namely, the *Basic Fraenkel Model*, the *Ordered Mostowski Model*, and the *Basic Cohen Model*.

2 Introduction and some preliminary results

The remarkable Hewitt-Marczewski-Pondiczery theorem **SHMPT** is concerned with the density of product spaces and has a plethora of applications in general topology. (This is no surprise as we will see in the forthcoming Theorem 1 that **SHMPT** is yet another disguised form of the axiom of choice **AC**.) Likewise, the Fichtenholz-Kantorovich-Hausdorff theorem **FKHT** has numerous applications in general topology, as well as in set theory. (Statements of these theorems are in the previous section.) In ongoing investigations of topology assuming only weak choice principles, we have found it helpful to understand more about the strength of these theorems in **ZF**.

The theorem **HMPT** is a special case of the strong form **SHMPT**. Part (i) of the next theorem is well known and it states that **FKHT** is a consequence of **HMPT**.

Theorem 1. *(i) Let X be an infinite set.* $\mathbf{HM}(X)$ *implies* $\mathbf{LIF}(X)$*. Hence,* **HMPT** *implies* **FKHT***.*
(ii) **SHMPT** *implies* **AC***.*

Proof. (i) Fix X an infinite set and let, by our hypothesis, D be a dense subset of $\mathbf{2}^{\mathcal{P}(X)}$ of size $|X|$. It is easy to see that the family $\mathcal{A} = \{A_x = \{d \in D : d(x) = 1\} : x \in \mathcal{P}(X)\}$ is independent.

(ii) Tarski proved in [9] that **AC** is equivalent to the statement "for all infinite X, $|X \times X| = |X|$," and so for this paper, we use $\mathbf{T}(X)$ as an abbreviation for $|X \times X| = |X|$. We will show that **SHMPT** implies $\mathbf{T}(X)$ for every infinite set X.

Let X be infinite and carry the discrete topology. By **SHMPT**, the product space $X \times X$ has a dense set D with $|D| \leq |X|$. But $X \times X$ is discrete, so $D = X \times X$, and thus $|X \times X| = |X|$. $\square$

In view of Theorem 1 one may ask the following questions:

Question 1. Does **FKHT** imply **HMPT** in **ZF**?

Question 2. Does **HMPT** imply **AC** in **ZF**?

We will answer affirmatively for Question 1 in Section 3, and find that the answer to Question 2 is no, in Section 4.

Looking at a "textbook" proof of **FKHT** (for example, [7]) quickly yields the following lemma about independent families, provable in **ZF**.

Lemma 2. *Let X be an infinite set, and let*

$$\mathcal{H}(X) = \{\langle F, \mathcal{A}\rangle \,:\, F \in [X]^{<\omega} \text{ and } \mathcal{A} \subseteq \mathcal{P}(X)\}.$$

Then $\mathcal{H}(X)$ has an independent family whose cardinality is $|\mathcal{P}(X)|$.

In the case where X is well-orderable, it is apparent that $|\mathcal{H}(X)| = |X|$, and so in **ZFC**, the lemma yields **FKHT** as an immediate corollary. In **ZF**, we observe that the condition $|X| = |\mathcal{H}(X)|$ is equivalent to the condition $|X| = |[X]^{<\omega}|$, which we call $\mathbf{T}^+(X)$ in this paper (see Proposition 5 below for a proof of this observation), so we have the theorem: $\mathbf{T}^+(X)$ implies $\mathbf{LIF}(X)$.

On the other hand, we can quickly see that $\mathbf{LIF}(X)$ is not a theorem of **ZF**. If X has an independent family of size $|\mathcal{P}(X)|$, then this independent family witnesses that the set $\mathcal{P}(X)$ has a proper subset with the same cardinality as itself. This trivial observation has a non-trivial consequence: $\mathcal{P}(X)$ is Dedekind infinite (equivalently, $\mathcal{P}(X)$ has an infinite well-orderable subset). Thus:

Proposition 3. *If* **ZF** *is consistent, then* **FKHT** *is not a theorem of* **ZF** + **BPI**.

Proof. The Ordered Mostowski Model of **ZFA** is a model of **ZFA** in which **BPI** holds ([6]). It has an infinite set whose power set is Dedekind finite ([4]), and so **FKHT** is false by the discussion just above.

This shows that the sentence "**BPI**, and $\exists X$ such that $\neg\,\mathbf{LIF}(X)$" is consistent with **ZFA**, and this result transfers from **ZFA** to **ZF**, as mentioned in the previous section, by theorems of Pincus in [8]. $\square$

To summarize, we have these preliminary bounds for the strength of $\mathbf{LIF}(X)$:

Proposition 4. *For any infinite set X, $\mathbf{T}^+(X) \Rightarrow \mathbf{LIF}(X) \Rightarrow$ "$\mathcal{P}(X)$ is Dedekind infinite."*

The condition of a set X having a Dedekind *finite* power set is a well-studied finiteness condition (e.g. called III-finiteness by Levy in [4] and Δ_4-finiteness by Truss in [10]), clearly stronger than Dedekind finiteness of X itself. (It is worth mentioning that III-finiteness of X is equivalent to the existence of a surjection from X onto ω.) Proposition 4 says that III-finiteness of X implies the *lack* of an independent family of size $|\mathcal{P}(X)|$, which in turn implies that $|X|$ is strictly smaller than $|[X]^{<\omega}|$. This last condition can also be considered a notion of finiteness (a rather weak one), since in **ZFC**, $|X| < |[X]^{<\omega}|$ is true only for finite sets. We are lead investigate the following question (Section 5):

Question 3. How does the strength of $\mathbf{LIF}(X)$ (or $\neg\,\mathbf{LIF}(X)$) in **ZF** relate to other well-understood notions of finiteness (Dedekind finiteness of X, and others)?

We next establish some equivalents of $\mathbf{T}(X)$ and of the stronger arithmetical principle $\mathbf{T}^+(X)$. (As mentioned in the proof of Theorem 1, the statement "For all infinite X, $\mathbf{T}(X)$" implies **AC**, so it will follow from part (ii) below that "For all infinite X, $\mathbf{T}^+(X)$" implies **AC**.)

Proposition 5. *Let X be any infinite set.*
(i) $\mathbf{T}(X)$ iff $|X| = |X^{<\omega}|$, where $X^{<\omega}$ is the set of finite sequences of elements of X.
(ii) $\mathbf{T}^+(X)$ implies $\mathbf{T}(X)$.
(iii) $\mathbf{T}^+(X)$ iff $|\,\mathrm{Fn}(X, 2, \omega)| = |X|$.
(iv) $\mathbf{T}^+(X)$ iff $|\mathcal{H}(X)| = |X|$, where $\mathcal{H}(X)$ is as in Lemma 2.

Note: We will show in Corollary 18 below that the converse of (ii) may fail in **ZF**; that is, $\mathbf{T}^+(X)$ is strictly stronger than $\mathbf{T}(X)$.

Proof. (i) Clearly, $\mathbf{T}(X)$ implies $|X \times 2| = |X|$. This weaker arithmetic statement is well known to be equivalent to $|\omega \times X| = |X|$, and implies that X is Dedekind infinite.

Furthermore, $\mathbf{T}(X)$ implies that one can define effectively for all $n \in \omega$ a one-to-one and onto function $f_n : X^n \to X$ (an injection from X^2 to X

can be used to construct an injection from X^4 to X^2, etc.). From the family $\{f_n : n \in \omega\}$ and a bijection from $\omega \times X$ to X, it is easy to show that $|X| = |X^{<\omega}|$.

(ii) Assume $\mathbf{T}^+(X)$. As a first step, we will show that $|X \times 2| = |X|$. Fix distinct elements $c, d \in X$, and then define $A = \{\{x\} : x \in X\}$ and $B = \{\{c, d, x\} : x \in X\} \cup \{\varnothing\}$. Clearly A and B are disjoint subsets of $[X]^{<\omega}$, each with cardinality $|X|$, and we have $|X| \leq |X \times 2| = |A \cup B| \leq |[X]^{<\omega}| = |X|$.

Now it follows from $\mathbf{T}^+(X)$ that $|[X \times 2]^{<\omega}| = |X|$. The function $H \colon X \times X \to [X \times 2]^{<\omega}$ given by $H(\langle x, y\rangle) = \{\langle x, 0\rangle, \langle y, 1\rangle\}$ is clearly one-to-one. Therefore, $|X| \leq |X \times X| \leq |[X \times 2]^{<\omega}| = |X|$.

(iii) and (iv) An injection $[X]^{<\omega} \to \mathrm{Fn}(X, 2, \omega)$ is given by $B \mapsto B \times \{1\}$. An injection $\mathrm{Fn}(X, 2, \omega) \to \mathcal{H}(X)$ is given by $g \mapsto \langle \mathrm{Dom}(g), \{g^{-1}(1), g^{-1}(0)\}\rangle$. (It is easy to see these are both one-to-one.) Thus

$$|X| \leq |[X]^{<\omega}| \leq |\mathrm{Fn}(X, 2, \omega)| \leq |\mathcal{H}(X)|.$$

So if either of the two larger sets has cardinality $|X|$, then $\mathbf{T}^+(X)$.

Conversely, assume $\mathbf{T}^+(X)$. Then by part (ii) and repeated applications of $\mathbf{T}^+(X)$, we have $|X| = |[X]^{<\omega} \times [[X]^{<\omega}]^{<\omega}|$. Since $\mathcal{H}(X)$ is a subset of $[X]^{<\omega} \times [[X]^{<\omega}]^{<\omega}$, all the cardinalities displayed above must be equal. $\square$

3 Equivalence of the FKH and HMP Theorems

We first show that $\mathbf{LIF}(X)$ is equivalent to a stronger looking formulation of $\mathbf{LIF}(X)$.

Lemma 6. *Let X be an infinite set.*
(i) $\mathbf{LIF}(X)$ implies $\mathbf{T}^+(\mathcal{P}(X))$ (that is, $|[\mathcal{P}(X)]^{<\omega}| = |\mathcal{P}(X)|$.)
(ii) $\mathbf{LIF}(X)$ is equivalent to the statement "there exists an independent family $\mathcal{A}$ of subsets of X of size $|\mathcal{P}(X)|$ such that for every $x, y \in X, x \neq y$ there exists $A \in \mathcal{A}$ with $x \in A$ and $y \notin A$".

Proof. (i) Fix X an infinite set and let, by our hypothesis, $\mathcal{A} = \{A_i : i \in \mathcal{P}(X)\}$ be an independent family of subsets of X. Define a function $H : [\mathcal{P}(X)]^{<\omega} \to \mathcal{P}(X)$ by requiring:

$$H(F) = \bigcap\{A_i : i \in F\}.$$

We show that H is one-to-one. Fix $F, K \in [\mathcal{P}(X)]^{<\omega}, F \neq K$ with $H(F) = H(K)$ and let $i_0 \in F \backslash K$. Clearly, $H(F) \cap A_{i_0}^c = \varnothing$, while $H(K) \cap A_{i_0}^c \neq \varnothing$. Contradiction! Thus, $H(F) \neq H(K)$ and H is as required. Hence, $|[\mathcal{P}(X)]^{<\omega}| = |\mathcal{P}(X)|$ finishing the proof of the claim.

(ii) It suffices to show $(\rightarrow)$ as the other implication is obvious.

From (i), one can readily verify that $|X \times X| \leq |\mathcal{P}(X)|$. Fix $f : X \times X \rightarrow \mathcal{P}(X)$ a one-to-one function. For every $(x, y) \in X \times X, x \neq y$ replace the element $f((x, y))$ of $\mathcal{A}$ by the subset $\{x\} \cup (f((x, y)) \backslash \{y\})$ of X. Let $\mathcal{B} = (\mathcal{A} \backslash \{f((x, y)) : (x, y) \in X \times X, x \neq y\}) \cup \{\{x\} \cup (f((x, y)) \backslash \{y\}) : (x, y) \in X \times X, x \neq y\}$. It can be readily verified that $\mathcal{B}$ is an independent family of subsets of X of size $|\mathcal{P}(X)|$ such that for every $x, y \in X, x \neq y$ there exists $B \in \mathcal{B}$ with $x \in B$ and $y \notin B$. $\square$

Now we are equipped to answer Question 1.

Theorem 7. *For every infinite set X,* **LIF**(X) *iff* **HM**(X)*. In particular,* **HMPT** *iff* **FKHT***.*

Proof. It suffices, in view of the proof of Theorem 1, to show that **LIF**(X) implies **HM**(X). Fix, by Lemma 6, an independent family $\mathcal{A}$ of X of size $|\mathcal{P}(X)|$ such that for all $x, y \in X, x \neq y$ there exist $A \in \mathcal{A}$ with $x \in A$ and $y \notin A$. Let $f : X \rightarrow \mathbf{2}^{\mathcal{A}}$ be the function given by $f(x) = (\chi_A(x))_{A \in \mathcal{A}}$. We show $\overline{\{f(x) : x \in X\}} = \mathbf{2}^{\mathcal{A}}$. Fix $[p]$ a basic open set of $\mathbf{2}^{\mathcal{A}}$ and let

$$\mathcal{W}_p = p^{-1}(1) \cup \{A^c : A \in p^{-1}(0)\}.$$

Since p is finite and $\mathcal{A}$ is an independent family, it follows that $\bigcap \mathcal{W}_p \neq \varnothing$. Hence, for every $x \in \bigcap \mathcal{W}_p, f(x) \in [p]$ and $\overline{\{f(x) : x \in X\}} = \mathbf{2}^{\mathcal{A}}$. Since for all $x, y \in X, x \neq y, f(x) \neq f(y)$ (if $A \in \mathcal{A}$ is such that $x \in A$ and $y \in A^c$ then $f(x)(A) = 1$ and $f(y)(A) = 0$), it follows that $D = \{f(x) : x \in X\}$ is a dense subset of $\mathbf{2}^{\mathcal{A}}$ of size $|X|$. Since $|\mathcal{A}| = |\mathcal{P}(X)|$ the conclusion follows immediately. $\square$

Theorem 8. *Let Y be an infinite set and D a dense subset of the product $\mathbf{2}^Y$. Then, $|[Y]^{<\omega}| \leq |\mathcal{P}(D)|$. In particular,* **HM**$(X)$ *implies* $\mathbf{T}^+(\mathcal{P}(X))$*.*

Proof. To see this, fix a dense set D of $\mathbf{2}^Y$ and let $H : \mathrm{Fn}(Y, 2, \omega) \rightarrow \mathcal{P}(D)$ be the function given by the rule: $H(p) = [p] \cap D$. It is straightforward to verify that H is one-to-one. Thus, $|\mathrm{Fn}(Y, 2, \omega))| \leq |\mathcal{P}(D)|$. Since the function $T : [Y]^{<\omega} \to \mathrm{Fn}(Y, 2, \omega)$ given by $T(A) = \chi_A$, is clearly one-to-one, it follows that $|[Y]^{<\omega}| \leq |\operatorname{Fn}(Y, 2, \omega))| \leq |\mathcal{P}(D)|$ as required.

The second assertion follows from the first part and the fact that the product $\mathbf{2}^{\mathcal{P}(X)}$ has, in view of $\mathbf{HM}(X)$, a dense set D of size $|X|$. $\square$

4 Independence of FKHT from AC

This section will answer Question 2 by showing that **FKHT** is consistent with $\neg$**AC** in **ZF**. Consider the following statement $(*)$ about cardinal exponentiation, which may be considered a weak choice principle:

$(*)$ For every X, there is a well-ordered cardinal $\aleph$ such that X has a partition of size $\aleph$, and $2^{|X|} = 2^{\aleph}$.

We will show that $(*)$ implies **FKHT**, and then show that $(*)$ is consistent with $\neg$**AC** in **ZF**.

Lemma 9. *(i) If Z has an independent family $\mathcal{F}$, and there is a surjection from X onto Z (or equivalently, X has a partition of cardinality $|Z|$), then X has an independent family of cardinality $|\mathcal{F}|$.*
(ii) The principle $(*)$ above implies **FKHT**.

Proof. (i) Given an onto function $f \colon X \to Z$, it is easy to see that the set $\{f^{-1}(Y) : Y \in \mathcal{F}\}$ is the required independent family of subsets of X.

(ii) Suppose X has a partition W of cardinality $\aleph$. Since W is well-orderable, W has an independent family of cardinality $2^{\aleph}$. Then by part (i), so does X. $\square$

Theorem 10. *If* **ZF** *is consistent, then* **ZF** + **BPI** + **FKHT** *is consistent with the existence of an infinite, Dedekind finite set of reals. (Hence,* **FKHT** *is not equivalent to* **AC** *in* **ZF**.*)*

Proof. Let $\mathcal{N}$ denote the Basic Cohen Model. The model $\mathcal{N}$ is a symmetric model obtained by starting with a ground model $\mathcal{M}$ of **ZFC**, then forcing to a generic extension $\mathcal{M}[G]$ by adding a countable sequence of Cohen generic reals, and then retracting to a model $\mathcal{N} \subset \mathcal{M}[G]$ which contains the countable set of generic reals, which we'll call A, but no well-ordered enumeration of any infinite subset of A. Thus A is Dedekind finite in $\mathcal{N}$, and so is the set $B = [A]^{<\omega}$. It is well-known that **BPI** holds in $\mathcal{N}$.

It remains to show that **FKHT** holds in $\mathcal{N}$. By Lemma 9 (ii), it suffices to show that the principle $(*)$ holds in $\mathcal{N}$. The following known facts about $\mathcal{N}$ are useful here.

F1. In $\mathcal{N}$, every infinite set has an infinite, countable partition. (A proof is in [10]. This is Form 82 in the reference [5].)

F2. For every $X \in \mathcal{N}$, there is an ordinal α and a function $f \in \mathcal{N}$ such that $f\colon X \to B \times \alpha$ is one-to-one. (Lemma 5.25 in [6]).

We will identify well-orderable cardinals with initial ordinals, and the variable κ will range over these. For each set $X \in \mathcal{N}$, define $K(X)$ to be the least κ such that in $\mathcal{N}$, $|X| \leq |\kappa \times B|$. (The definition makes sense by fact F2 above.) Observe that if $\kappa = K(X)$, then there exists a one-to-one function $j\colon X \to \kappa \times B$ in $\mathcal{N}$ such that for each $\alpha < \kappa$, the set $j^{-1}(\{\alpha\} \times B)$ is nonempty, and hence the sets of this form compose a partition of X of cardinality κ.

Let $X \in \mathcal{N}$ be an infinite set, and fix $\kappa = \max\{K(X), \aleph_0\}$. By the previous paragraph and fact F1 above, X has a partition of cardinality κ in $\mathcal{N}$. It follows that $\mathcal{N} \models 2^\kappa \leq 2^{|X|}$. It now remains to show that $\mathcal{N} \models 2^{|X|} \leq 2^\kappa$.

Case 1: $K(X) \leq \kappa = \aleph_0$.

In this case, let $P = (\mathcal{P}(X))^{\mathcal{N}}$, the power set of X in $\mathcal{N}$; we'll show $|P| \leq 2^{\aleph_0}$. The hypothesis implies there exists in $\mathcal{N}$ a one-to-one $j\colon X \to \aleph_0 \times B$. Let $\mathfrak{c}$ denote the (well-orderable) cardinality of the continuum in $\mathcal{M}$ (and in $\mathcal{M}[G]$), and $\mathfrak{c}^+$ its successor. In $\mathcal{M}[G]$, the set B is countable, and so X is countable and $|P| \leq |\mathcal{P}(X)| = \mathfrak{c}$. It follows there is no partition of P of cardinality $\mathfrak{c}^+$ in $\mathcal{M}[G]$, or in $\mathcal{N}$.

Thus $\mathcal{N} \models K(P) \leq \mathfrak{c}$, and so $|P| \leq |\mathfrak{c} \times B|$. Clearly $\mathfrak{c} < |\mathbb{R}|$ in $\mathcal{N}$, and since $A \subseteq \mathbb{R}$, we have $|B| = |[A]^{<\omega}| \leq |[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$. So $|P| \leq |\mathfrak{c} \times B| \leq |\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$ in $\mathcal{N}$.

Case 2: $K(X) = \kappa \geq \aleph_1$.

Work in $\mathcal{N}$. Since $K(B) = 1$, it follows from Case 1 that $2^{|B|} = 2^{\aleph_0}$. From $\kappa = K(X)$, it follows that $|X| \leq |\kappa \times B|$. Thus

$$2^{|X|} \leq 2^{|\kappa \times B|} = (2^{|B|})^\kappa = (2^{\aleph_0})^\kappa = 2^\kappa.$$

This finishes Case 2 and the proof of the theorem. $\square$

5 Some Notions of Finiteness

In [4], Levy studied several definitions of finiteness, including the following:

- X is *III-finite* if $\mathcal{P}(X)$ is Dedekind finite.
- X is *IV-finite* if X is Dedekind finite. (That is, X has no infinite, well-orderable subset.)
- X is *V-finite* if $|X| < 2|X|$ or $|X| = 0$.
- X is *VI-finite* if $|X| < |X|^2$ or $|X| \leq 1$. (Equivalently, in our terminology, $\neg\mathbf{T}(X)$.)

In [10], Truss studied several definitions of finiteness, (some of which were equivalent to some of Levy's), including this strengthening of Dedekind finiteness:

- X is Δ_3*-finite* if X has no infinite, linearly orderable subset.

Remark 11. It is established in [4] that each of Levy's notions listed is stronger than the notions following in the list (III $\to$ IV $\to$ V $\to$ VI), and indeed strictly stronger, finding in the Ordered Mostowski Model examples to show that the implications are not reversible. In [10], the independence of III and Δ_3 from each other is established.

We observed in the introduction that if X is III-finite, then X has no independent family of size $|\mathcal{P}(X)|$. The next theorem will show that $\neg\,\mathbf{LIF}(X)$ implies that X is VI-finite, bounding the strength of $\neg\,\mathbf{LIF}(X)$ on both ends by some of Lévy's notions of finiteness.

Theorem 12. *For any infinite set X, $\mathbf{T}(X)$ implies $\mathbf{LIF}(X)$. (In other words, $\neg\,\mathbf{LIF}(X)$ implies that X is VI-finite.)*

Proof. Let X be an infinite set, and assume $\mathbf{T}(X)$. From Proposition 5, $\mathbf{T}(X)$ implies $|X| = |X^{<\omega}|$ and $|X| = |X \times \omega|$, whence $|X| = |Y|$ where $Y = X^{<\omega} \times [[\omega]^{<\omega}]^{<\omega}$. Recall the set $\mathcal{H}(X)$ from Lemma 2, and define a function $f\colon Y \to \mathcal{H}(X)$ as follows. For a pair $\langle \mathbf{s}, T\rangle \in Y$, with $\mathbf{s} = (x_1, \dots x_n)$, define

$$f(\mathbf{s}, T) = \langle \{x_1, \dots, x_n\}, \{\, \{x_i : i \in t\} \,:\, t \in T\,\}.$$

It is easy to see that f is surjective.

By Lemma 2, $\mathcal{H}(X)$ has an independent family $\mathcal{I}$ of size $|\mathcal{P}(X)|$. Since $|X| = |Y|$ and Y can be mapped surjectively onto $\mathcal{H}(X)$, it follows from Lemma 9(i) that $|X|$ also has an independent family of size $|\mathcal{P}(X)|$. $\square$

Theorem 12 has given an improvement over Proposition 4. The following diagram now summarizes implications provable in **ZF** for a given set X, as given by Propositions 4 and Proposition 5, Remark 11, and Theorem 12, In the diagram, "III(X)" is an abbreviation for "X is III-finite," etc.

$$\begin{tikzcd}
\Delta_3(X) \arrow[dr] & & & & \\
\mathrm{III}(X) \arrow[r] \arrow[drr] & \mathrm{IV}(X) \arrow[r] & \mathrm{V}(X) \arrow[r] & \mathrm{VI}(X) \arrow[r] & \neg\mathbf{T}^+(X) \\
& & \neg\,\mathbf{LIF}(X) \arrow[ur] & &
\end{tikzcd}$$

We will show that none of these implications is reversible in **ZF**, and that the diagram is the best possible between these particular statements, with no implications provable in **ZF** beyond those following from the given arrows and transitivity. Given the known independence results in Remark 11, it just needs to be shown that $\Delta_3(X) \not\to \neg\,\mathbf{LIF}(X)$, $\neg\,\mathbf{LIF}(X) \not\to \mathrm{V}(X)$, and $\neg\mathbf{T}^+(X) \not\to \mathrm{VI}(X)$; these statements can be found below in Corollaries 14, 17, and 18, respectively.

The proof of Theorem 10 showed that a certain model of **ZF** has Dedekind finite sets with large independent families. We will now show the relative consistency of Δ_3-finite sets with large independent families, a stronger result. Thus, although $\mathbf{LIF}(X)$ requires a strong "infiniteness" condition on $\mathcal{P}(X)$ (part (i) of Lemma 6), $\mathbf{LIF}(X)$ turns out to be consistent with fairly strong finiteness conditions on X itself.

Theorem 13. *Let L be an infinite set, and suppose there exists a choice function which assigns, to each finite subset F of L, a linear ordering of F. (For example, if the Axiom of Choice for families of finite sets is also assumed, then such a choice function would exist for every set L.) Let $X = [L]^{<\omega} \times [L]^{<\omega}$. Then*
(i) $\mathbf{LIF}(X)$ *holds, and*
(ii) if moreover L is Δ_3-finite, then the sets $[L]^{<\omega}$ and X are also Δ_3-finite.

Proof. Let L be an infinite set, and let $F \mapsto \preccurlyeq_F$ be an assignment giving a linear ordering $\preccurlyeq_F$ to every finite $F \subset L$.

The proof of part (ii) is short. Suppose L is Δ_3-finite. Using the assignment $F \mapsto \preccurlyeq_F$, we can easily define an injection from $[L]^{<\omega}$ to the set of finite sequences of members of L with no repeating entries. This latter set is Δ_3-finite by [10, Lemma 6], and hence $[L]^{<\omega}$ and X are also Δ_3-finite.

To prove part (i), let $X = [L]^{<\omega} \times [L]^{<\omega}$. To show that $\mathbf{LIF}(X)$ holds, it will suffice to show that there is a surjective function $f\colon X \to \mathcal{H}(X)$ (by Lemmas 2 and 9(i)). Observe that the function $t\colon X = [L]^{<\omega} \times [L]^{<\omega} \to [L]^{<\omega} \times \omega$ defined by $t(\langle E, F\rangle) = \langle E, |F|\rangle$ is surjective, so it remains to show that there exists a surjection from $[L]^{<\omega} \times \omega$ to $\mathcal{H}(X)$.

Let E be a finite subset of L. From the order $\preccurlyeq_E$, we can easily define a linear order on the finite set $\mathcal{P}(E)$ (lexicographically). In turn, we can define a linear order on $\mathcal{P}^(E) = \mathcal{P}(\mathcal{P}(E))$, and so on hierarchically. In this way, for each $E \in [L]^{<\omega}$, we can effectively define an order $\leq_E$, in order type ω, of the set $\mathcal{P}^\omega(E) = \bigcup_{n\in\omega} \mathcal{P}^n(E)$.

Observe, from the definitions of X and $\mathcal{H}(X)$, that $\mathcal{H}(X) \subset \mathcal{P}^\omega(L)$. (See that $[L]^{<\omega} \subset \mathcal{P}(L)$, so $X = ([L]^{<\omega})^2 \subseteq \mathcal{P}^3(L)$ if Kuratowski ordered pairs are used, and so on.) Moreover, if $\langle S, \mathcal{T}\rangle \in \mathcal{H}(X)$, then S is a finite set $\{\langle E_1, F_1\rangle, \dots \langle E_m, F_m\rangle\}$ with each E_j and F_j in $[L]^{<\omega}$, and so there is a finite E (take $E = \bigcup_{j=1}^m (E_j \cup F_j)$) such that S and $\langle S, \mathcal{T}\rangle$ are members of $\mathcal{P}^\omega(E)$.

Define a function $h\colon [L]^{<\omega} \times \omega \to \mathcal{P}^\omega(L)$ by declaring $h(E, n)$ to be the n^{th} element in the order $\leq_E$ on $\mathcal{P}^\omega(E)$. We have just argued that in fact every element of $\mathcal{H}(X)$ is in the range of h. Thus, h maps $[L]^{<\omega} \times \omega$ onto $\mathcal{H}(X)$, and this completes the proof. □

Corollary 14. *If* **ZF** *is consistent, then $\Delta_3(X)$ does not imply $\neg\,\mathbf{LIF}(X)$ in* **ZF** *(and hence none of VI(X), V(X), or VI(X) imply $\neg\,\mathbf{LIF}(X)$.)*

Proof. Consider the permutation model used by Läuchli [3] to establish the independence of **AC** for Finite Sets ("every collection of non-empty finite sets has a choice function") from the Ordering Principle ("every set is linearly orderable"). (The model is denoted $\mathcal{N}7$ in [5], and it is described in section 7.3 of [6].) In this model, the set A of atoms is Δ_3-finite (the proof in [6] that A is not linearly orderable shows just as well that no infinite subset is linearly orderable), but **AC** for Finite Sets holds. Thus, by Theorem 13, the set $X = [A]^{<\omega} \times [A]^{<\omega}$ satisfies $\mathbf{LIF}(X)$ and $\Delta_3(X)$ in this model. □

Remark 15. The argument of Theorem 13 can be modified to show, for example, that whenever L is an infinite, linearly orderable Dedekind finite set, the Dedekind finite set $[L]^{<\omega} \times [L]^{<\omega}$ has a large independent family.

The next result, together with Lemma 6, will yield that $2|X| = |X|$ does not imply $\mathbf{LIF}(X)$ in **ZF**; or in other words, $\neg\,\mathbf{LIF}(X)$ does not imply $\mathrm{V}(X)$.

Theorem 16. **ZF** *is consistent with the existence of a set X such that* $2|X| = |X|$ *and* $\neg\mathbf{T}^+(\mathcal{P}(X))$.

Proof. Let $\mathcal{N}$ be the basic Fraenkel model, and let A denote the set of atoms in $\mathcal{N}$. In $\mathcal{N}$, A is amorphous, meaning that it is infinite but has no partition into two infinite parts. Let $X = A \times \omega$. Clearly, $2|X| = |A \times \omega \times 2| = |A \times \omega| = |X|$.

Next, suppose that $\mathbf{T}^+(\mathcal{P}(X))$ holds. First observe that $|\mathcal{P}(A \times \omega)| = 2^{|\omega \times A|} = |\mathbb{R}^A|$. ($\mathbb{R}^A$ denotes the set of functions from A to $\mathbb{R}$.) So the assumption $\mathbf{T}^+(\mathbb{R}^A)$ yields $|[\mathbb{R}^A]^{<\omega}| = |\mathbb{R}^A|$. Let $\Phi\colon [\mathbb{R}^A]^{<\omega} \to \mathbb{R}^A$ be a bijection, with $\Phi \in \mathcal{N}$. We will derive a contradiction from this.

Let $E \subset A$ be a finite support for Φ. Let $a_1, a_2, a_3, a_4 \in A \setminus E$, and let π be the permutation of A which cycles those four atoms, mapping a_1 to a_2 and so on, leaving all other atoms fixed. Observe that $\pi(\Phi) = \Phi$ (since π fixes all elements of E). Let g_1 be the characteristic function $\chi_{\{a_1,a_2\}} \in \mathbb{R}^A$; then let $g_2 = \pi(g_1)$, $g_3 = \pi(g_2)$, and $g_4 = \pi(g_3)$. Then clearly we also have $g_1 = \pi(g_4)$. It follows that π fixes the function $f := \Phi(\{g_1, g_2, g_3, g_4\}) \in \mathbb{R}^A$. Thus $f(a_1) = f(\pi(a_1)) = f(a_2)$. (Repeating this reasoning, it follows that f is constant on the set $\{a_1, a_2, a_3, a_4\}$.)

Next, let ρ be the permutation of A which swaps a_1 with a_2, and leaves all other atoms fixed. Since $f(a_1) = f(a_2)$, in fact ρ fixes f. And ρ also fixes all elements of E, so $\rho(\Phi) = \Phi$. Thus $f = \Phi(\rho(\{g_1, g_2, g_3, g_4\}))$. But this is a contradiction, since Φ is supposed to be one-to-one, but $\rho(\{g_1, g_2, g_3, g_4\}) \neq \{g_1, g_2, g_3, g_4\}$. □

Corollary 17. *In* **ZF**, *it is not provable that for every infinite set X,* $2|X| = |X|$ *implies* $\mathbf{LIF}(X)$.

Proof. Follows from Theorem 16 just above and Lemma 6, which says that $\mathbf{LIF}(X)$ implies $\mathbf{T}^+(\mathcal{P}(X))$. □

Corollary 18. *In* **ZF**, *it is not provable that for every infinite set Y,* $\mathbf{T}(Y)$ *implies* $\mathbf{T}^+(Y)$.

Proof. First observe that for any X such that $2|X| = |X|$, we have $\mathbf{T}(\mathcal{P}(X))$, since $(2^{|X|})^2 = 2^{2|X|} = 2^{|X|}$. The result now follows from Theorem 16, taking $Y = \mathcal{P}(X)$. $\square$

The models of **ZF** and **ZFA** used in this paper so far all have Dedekind sets. (The proof of Theorem 16 exhibited a set X which is Dedekind infinite for which $\mathbf{LIF}(X)$ is false, but this X has subsets which are amorphous, hence infinite but Dedekind finite.) Here is one more independence result, showing that $\mathbf{LIF}(X)$ can fail even when there are no Dedekind sets.

Theorem 19. **ZF** + *"Every Dedekind finite set is finite" is consistent with* $\neg$**FKHT**.

Proof. We use the permutation model $\mathcal{N}$ which is called $\mathcal{N}2(\aleph_1)$, "Jech's model," in [5], in which every Dedekind finite set is finite (Form 9 in [5]). The model is obtained as follows. Let A be the set of atoms in a model of **ZFA**, and let $\{P_i : i \in \aleph_1\}$ be a partition of A into pairs. Then $\mathcal{N}$ is the permutation submodel formed from the group $G = \{g \in \mathrm{Sym}(A) : \forall i \in \aleph_i \; g(P_i) = P_i\}$ (identifying permutations of A with automorphisms of the model), and countable supports.

We will show that $\mathbf{LIF}(A)$ is false in this model. Let $\mathcal{I}$ be an independent family of subsets of A, and let $C \subset A$ be a countable support for $\mathcal{I}$. Since C is countable, there is some $\alpha < \aleph_1$ such that $C \subseteq \bigcup_{i \leq \alpha} P_i$. We claim that every member of $\mathcal{I}$ is a union of sets in the pairwise disjoint family $F = \{\{a\} : a \in P_i \text{ and } i \leq \alpha\} \cup \{P_i : \alpha < i < \aleph_1\}$. Suppose the claim is false, so there exist some $S \in \mathcal{I}$ and some β such that $\alpha < \beta < \aleph_1$ and $|S \cap P_\beta| = 1$. Let $g \in G$ be the permutation which swaps the two elements of P_β and leaves all other atoms fixed. Then g fixes each element of C, so $g(\mathcal{I}) = \mathcal{I}$. It follows that $g(S) \in \mathcal{I}$. But the sets S and $g(S)$ differ only on elements of the finite set P_i, so this is a contradiction; they cannot both be members of the same independent family.

From the claim it follows that each member of $\mathcal{I}$ corresponds to a distinct subset of F, so that $|\mathcal{I}| \leq |\mathcal{P}(F)|$. Observe that $|F| = \aleph_1$, and $\mathcal{P}(\aleph_1)$ is always linearly orderable (in fact, it is well-orderable in this model). However, A is not linearly orderable, so $|A| \not\leq |\mathcal{P}(F)|$, and hence $|\mathcal{P}(A)| \not\leq |\mathcal{P}(F)|$. Thus $|\mathcal{I}| \neq |\mathcal{P}(A)|$. Since $\mathcal{I}$ was an arbitrary independent family on A, $\neg\,\mathbf{LIF}(A)$ has been shown. $\square$

Remark. (Some other notions of finiteness.) More notions of finiteness have been considered in the literature. De la Cruz [1] serves as a good overview, with summaries of known results and a few new results, and a nice diagram. We mention a couple more notions here.

- X is Δ_5*-finite* if there is no surjective function from X to $X \cup \{u\}$, with $u \notin X$ (Truss [10]).

  The notion Δ_5 lies between III and IV in strength, so it can be asked whether $\Delta_5(X)$ is strong enough to imply $\neg\,\mathbf{LIF}(X)$. The answer is no, because the Basic Cohen Model contains infinite Δ_5-finite sets (see [10] for a proof). Thus the proof of Theorem 10 shows that $\Delta_5(X)$ is consistent with $\mathbf{LIF}(X)$.

- X is *VI″-finite* if $\mathcal{P}(X)$ is VI-finite (i.e. $\neg\mathbf{T}(\mathcal{P}(X))$).

  Clearly $\mathrm{III}(X) \to \mathrm{VI}''(X)$. Part (i) of Lemma 6 immediately implies that $\mathrm{VI}''(X) \to \neg\,\mathbf{LIF}(X)$. The proofs of Theorem 16 and Corollary 18 show that this implication is not reversible in **ZF**.

From the results covered in the present paper (including the observations just above) and De la Cruz's diagram, the relative strengths of $\neg\,\mathbf{LIF}(X)$ compared with any of the notions in [1] are clear.

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Eric Hall
Department of Mathematics & Statistics
College of Arts & Sciences
University of Missouri - Kansas City
206 Haag Hall, 5100 Rockhill Rd
Kansas City, MO 64110, USA
E-mail: halle@umkc.edu

Kyriakos Keremedis
Department of Mathematics
University of the Aegean
Karlovassi, Samos 83200, Greece
E-mail: kker@aegean.gr