On Lindelöf Metric Spaces and Weak Forms of the Axiom of Choice

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Abstract

We show that the *countable axiom of choice* CAC strictly implies the statements "*Lindelöf metric spaces are second countable*" and *"Lindelöf metric spaces are separable"*. We also show that CAC is equivalent to the statement: "*If* (X, T) *is a Lindelöf topological space with respect to the base* $\mathcal{B}$*, then* (X, T) *is Lindelöf*".

1 Introduction and preliminary results

Let LMSC and LMS stand for the propositions

Lindelöf metric spaces are second countable

and

Lindelöf metric spaces are separable

respectively. C. Good and I. Tree [1] establish that the latter statements imply CAC_{fin} (the axiom of choice for countable families of non empty finite sets). Further, H. L. Bentley and H. Herrlich [2] consider the pseudometric versions of the above statements, i.e. *Lindelöf pseudometric spaces are second countable* and *Lindelöf pseudometric spaces are separable* (denoted by LPSC

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and LPS respectively). They prove that LPS is actually equivalent to CAC (the axiom of choice for countable families of non empty sets) and that LPSC implies CAC_{fin}. Yet, they ask about the exact set theoretical status of LPSC. It is still unknown whether LMSC (and consequently LPSC, see Corollary 1.8) or LMS imply a stronger choice principle than CAC_{fin}. We only know that both these statements do not imply CAC in ZF^0 (the Zermelo-Fraenkel set theory minus the axiom of regularity). This is Theorem 3.2 in our section "Models". Definitions and notation are given below.

Definition 1.1 *The countable axiom of choice CAC (Form 8 in [5]) is the assertion: For every set $\mathcal{A} = \{A_i : i \in \omega\}$ of non empty disjoint sets there exists a set C consisting of one and only one element from each element of $\mathcal{A}$.*

Definition 1.2 *(i) CAC_{fin} (Form 10 in [5]) is CAC restricted to families of disjoint finite sets.*
(ii) CAC_{ω} (Form 32 in [5]) is CAC restricted to families of disjoint countable sets.
(iii) $CAC(\mathbb{R})$ (Form 94 in [5]) is CAC restricted to families of subsets of the real line $\mathbb{R}$.

Definition 1.3 *The countable multiple axiom of choice CMC (Form 126 in [5]) is the proposition: For every set $\mathcal{A} = \{A_i : i \in \omega\}$ of non empty disjoint sets there exists a family $\mathcal{F} = \{F_i : i \in \omega\}$ of finite non empty sets such that for every $i \in \omega$, $F_i \subseteq A_i$.*

Definition 1.4 *The countable union theorem CUC (Form 31 in [5]) is the statement: The countable union of countable sets is countable.*

Definition 1.5 *PW (Form 91 in [5]) is the proposition that the power set of a well ordered set can be well ordered.*

Let (X, T) be a topological space and $\mathcal{B}$ a base for X.

Definition 1.6 *X is said to be Lindelöf with respect to $\mathcal{B}$ iff every open cover $\mathcal{U} \subset \mathcal{B}$ has a countable subcover $\mathcal{V}$.*

If every open cover $\mathcal{U} \subset \mathcal{B}$ has a finite subcover $\mathcal{V}$, then X is compact and this without any use of choice. That is the statement "if X is *compact with respect to the base* $\mathcal{B}$, then X is compact" is provable in ZF^0. If however X is Lindelöf with respect to the base $\mathcal{B}$, then we cannot in general infer that X is Lindelöf. Indeed, $\mathbb{N}$ with the discrete topology may fail to be Lindelöf even though the set of all singletons of $\mathbb{N}$ forms a countable base. (For more information on this matter see [3]). Thus, the statements:

L(T_i,$\mathcal{B}$) = *If* (X, T) *is a Lindelöf* T_i *space with respect to the base* $\mathcal{B}$, *then* X *is Lindelöf* , *i = -1, 0, 1, 2,*

LPL($\mathcal{B}$) = *If* (X, d) *is a Lindelöf pseudometric space with respect to the base* $\mathcal{B}$, *then* X *is Lindelöf,*

where a T_{-1} space is an arbitrary topological space, are not provable in ZF^0. We will show in the next section that $\mathrm{L}(T_i, \mathcal{B})$, $i = -1, 0, 1, 2$, is in fact equivalent to CAC (Theorem 2.1) and that $\mathrm{LPL}(\mathcal{B})$ implies CAC_ω (Corollary 2.2).

Now, under CAC, any Lindelöf metric space (X, d) is easily seen to be second countable, separable and of size at most 2^ω. Indeed, for each $n \in \omega$, set $\mathcal{U}_n = \{D(x, 1/n) : x \in X\}$ and $\mathcal{W}_n = \{V : V$ is a countable subcover of $\mathcal{U}_n\}$. By CAC, let $\{\mathcal{V}_n : n \in \omega\}$ be a choice set for the family $\{\mathcal{W}_n : n \in \omega\}$. For each $n \in \omega$, let $\mathcal{C}_n$ be the set of the centers of the discs in $\mathcal{V}_n$ and $\mathcal{D}_n = \{f \in (\mathcal{C}_n)^\omega : f$ is a bijection$\}$. By CAC again, $\{\mathcal{D}_n : n \in \omega\}$ has a choice set $\{f_n : n \in \omega\}$. Clearly $\mathcal{V} = \{D(f_n(m), 1/n) : n, m \in \omega\}$ is a countable base for X and $\mathcal{D} = \{f_n(m) : n \in \omega, m \in \omega\}$ is a countable dense set of X. Hence, X is second countable and separable. Since the mapping $x \to V_x$, where $V_x \subset \mathcal{V}$ is the set of all neighborhoods of x is 1:1 and $|\mathcal{V}| = \omega$, it follows that $|X| \leq |2^\omega|$. Thus, in ZF^0 a second countable metric space has size at most 2^ω. So in addition to LMSC and LMS we consider the proposition

$\mathrm{LM}(2^\omega)$ = *Lindelöf metric spaces have size at most* 2^ω.

We show in Corollary 2.12 that the latter three statements imply CAC_{fin}. Thus, they are not provable in ZF^0.

Motivated by $\mathrm{LPL}(\mathcal{B})$ we define

LPSC($\mathcal{B}$) = *If* (X, T) *is a Lindelöf pseudometric space with respect to the base* $\mathcal{B}$, *then* X *is second countable.*

As LPSC($\mathcal{B}$) implies LPSC and LPSC implies CAC_{fin}, it follows that LPSC($\mathcal{B}$) is not provable in ZF^0. Moreover, we show in Corollary 2.2 that LPSC($\mathcal{B}$) implies CUC.

If (X, d) is a pseudometric space, then the metric identification (X^*, d^*) of (X, d) is the set X^* of all equivalence classes in X of the equivalence relation $\sim$ given by

$$x \sim y \text{ iff } d(x, y) = 0$$

and $d^* : X^* \times X^* \to \mathbb{R}$ is given by $d^*([x], [y]) = d(x, y)$, where $[x]$ denotes the equivalence class of the element x. Clearly, see [8] p. 20, d^* is a metric on X^* and if $h : X \to X^*$ is the mapping given by $h(x) = [x]$, then a set A in X is closed (open) iff $h(A)$ is closed (open) in X^*. Therefore we have the following

Proposition 1.7 *Let* (X, d) *be a pseudometric space. Then* (X, d) *is compact, Lindelöf or second countable iff* (X^*, d^*) *is compact, Lindelöf or second countable respectively.*

As a corollary to the last proposition we get:

Corollary 1.8 *(i) LPSC iff LMSC.*
(ii) LPSC($\mathcal{B}$) iff LMSC($\mathcal{B}$) (= If (X, d) *is a Lindelöf metric space with respect to the base* $\mathcal{B}$, *then* X *is second countable).*
(iii) LPL($\mathcal{B}$) iff LML($\mathcal{B}$) (= If (X, d) *is a Lindelöf metric space with respect to the base* $\mathcal{B}$, *then* X *is Lindelöf).*

In [3] it has been shown that CAC($\mathbb{R}$) is equivalent to each of the assertions:

(1) $\mathbb{R}$ *is Lindelöf.*

(1a) $\mathbb{N}$ *is Lindelöf.*

(1b) $\mathbb{Q}$ *is Lindelöf.*

(2) *Every second countable* T_2 *space is Lindelöf.*

Combining (1), (2) and the last proposition we get at once the following propositions.

Proposition 1.9 *(i) LMSC($\mathcal{B}$)+CAC($\mathbb{R}$) implies LML($\mathcal{B}$).*
(ii) LMSC($\mathcal{B}$)+CAC($\mathbb{R}$) implies LMHL (= "Subspaces of Lindelöf metric spaces are Lindelöf").

Proposition 1.10 *(see [4]). CAC($\mathbb{R}$) is equivalent to the statement "Every second countable (pseudo)metric space is Lindelöf".*

Consider now the statement:

(3) *A pseudometric space (X, d) is Lindelöf iff it is second countable.*

Clearly (3) implies CAC($\mathbb{R}$). On the other hand any permutation model satisfies CAC($\mathbb{R}$) (see [5]) and since (3) implies CAC_{fin} we see that in the basic Fraenkel Model (3) fails (CAC_{fin} fails). Thus CAC($\mathbb{R}$) does not imply (3). Since CAC clearly implies (3), we see that (3) lies in strength between CAC and CAC($\mathbb{R}$).

2 Results

Our first result in this section is a characterization of CAC.

Theorem 2.1 *CAC iff L(T_i,$\mathcal{B}$), i = -1, 0, 1, 2.*

Proof. ($\rightarrow$). Since L($T_{-1}, \mathcal{B}$) implies L($T_i, \mathcal{B}$), $i = 0, 1, 2$, it suffices to show that CAC implies L($T_{-1}, \mathcal{B}$). To this end, fix (X, T) and $\mathcal{B}$ as in the statement of L($T_{-1}, \mathcal{B}$) and let $\mathcal{U}$ be any open cover of X. Since $\mathcal{B}$ is also an open cover of X and X is Lindelöf with respect to $\mathcal{B}$, it follows that some countable collection $\mathcal{Q} = \{Q_n : n \in \omega\}$ of the open cover $\mathcal{W} = \{B \in \mathcal{B} : B \subset U$ for some $U \in \mathcal{U}\}$ covers X. Use CAC to pick for each $n \in \omega$ an element $U_n \in \mathcal{U}$ with $Q_n \subset U_n$. Since $\mathcal{Q}$ covers X, it follows that $\{U_n : n \in \omega\}$ is a countable subcover of $\mathcal{U}$ as required.

($\leftarrow$). It suffices to show that L($T_2, \mathcal{B}$) implies CAC. To see this, fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of non empty disjoint sets. Without loss of generality we may assume that $\omega \cap (\bigcup \mathcal{A}) = \emptyset$. Put

$$X = \omega \cup (\cup \mathcal{A}) \cup \{\infty\}$$

and let T be the topology on X which is generated by the base

$$\mathcal{B} = \{\{x\} : x \in \omega \cup (\cup\mathcal{A})\} \cup \{F \cup \{\infty\} : F \subset \cup\mathcal{A}, |(\cup\mathcal{A})\backslash F| < \omega\}.$$

It can be readily verified that X is T_2 and Lindelöf with respect to $\mathcal{B}$. Thus, by L(T_2,$\mathcal{B}$), X is Lindelöf. Let

$$\mathcal{U} = \{\{n\} \cup \{z\} : n \in \omega, z \in A_n\} \cup \{(\cup\mathcal{A}) \cup \{\infty\}\}.$$

Clearly $\mathcal{U}$ is an open cover of X. Thus, $\mathcal{U}$ has a countable subcover $\mathcal{V} = \{V_m : m \in \omega\}$. For each $n \in \omega$, let m_n be the least m such that $n \in V_{m_n}$. Then $\{z_n : n \in \omega\}$, where z_n is the unique element of $V_{m_n}\backslash\{n\}$, is a choice set for $\mathcal{A}$. This completes the proof of the theorem. $\square$

Corollary 2.2 *(i) LML($\mathcal{B}$) implies CAC_ω.*
(ii) LMSC($\mathcal{B}$) implies CUC.
(iii) LML($\mathcal{B}$)+LMHL implies CUC.
(iv) LM($\mathcal{B}, 2^\omega$) (= "Lindelöf metric spaces with respect to a base $\mathcal{B}$ have cardinality at most 2^ω") + CAC($\mathbb{R}$) implies CAC_ω.

Proof. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of disjoint countable sets. Without loss of generality we may assume that $\omega \cap (\bigcup \mathcal{A}) = \emptyset$. Put $X = \omega \cup (\cup\mathcal{A}) \cup \{\infty\}$. Define a function $d : X \times X \to \mathbb{R}$ by requiring $d(x, x) = 0$ and

$$d(x,y) = d(y,x) = \begin{cases} max\{\frac{1}{n+1}, \frac{1}{m+1}\} \text{ if } x \in A_n \text{ and } y \in A_m \\ \\ \frac{1}{n+1} \text{ if } x = \infty \text{ and } y \in A_n \\ \\ 1 \text{ if } x \in \omega \text{ or } y \in \omega \end{cases} .$$

It can be readily verified that d is a metric on X. As before,

$$\mathcal{B} = \{\{x\} : x \in \omega \cup (\cup\mathcal{A})\} \cup \{D(\infty, \frac{1}{n+1}) : n \in \omega\}$$

is easily seen to be a base for X and X is Lindelöf with respect to $\mathcal{B}$. Working as in the last theorem and using LML($\mathcal{B}$) we can find a choice set for $\mathcal{A}$. On the other hand, if we use LMSC($\mathcal{B}$), we can easily find an enumeration of $\cup\mathcal{A}$ (a 1:1 and onto function $f : \omega \to \cup\mathcal{A}$) meaning that $\cup\mathcal{A}$ is countable as required.

By LMHL, $\cup\mathcal{A}$ is a Lindelöf subspace of X. Thus, the open cover $\{\{x\} : x \in \cup\mathcal{A}\}$ is countable and consequently $\cup\mathcal{A}$ is countable.

By LM($\mathcal{B}, 2^\omega$), there exists an injective function $f : \cup\mathcal{A} \to \mathbb{R}$. By CAC($\mathbb{R}$), let $c = \{c_i : i \in \omega\}$ be a choice set for the family $\{f[A_i] : i \in \omega\}$. Clearly, $\{f^{-1}(c_i) : i \in \omega\}$ is a choice set for $\mathcal{A}$. $\square$

Corollary 2.3 *(i) LML($\mathcal{B}$)+LMSC implies LMSC($\mathcal{B}$).*
(ii) LML($\mathcal{B}$)+LMSC implies CUC.

If X is any set, then $X(a)$ will denote the Alexandroff one-point compactification of the discrete space X. Since every base for $X(a)$ will include all singletons of X, it follows that the statement

CWOB = Every compact space (X, T) has a well ordered base $\mathcal{B}$.

implies that every set X has a well ordering. Thus we have:

Proposition 2.4 *AC iff CWOB.*

Further, since $X(a)$ is a T_2 space, we have:

Corollary 2.5 *AC iff Every compact T_2 space (X, T) has a well ordered base $\mathcal{B}$.*

Motivated by CWOB we define:

Definition 2.6 *(i) CMSC = Every compact metric space is second countable.*
(ii) CMWOB = Every compact metric space has a well ordered base.
(iii) LMWOB = Every Lindelöf metric space has a well ordered base.

As every set X endowed with the discrete topology is a metrizable space we see that

Proposition 2.7 *AC iff every metric space (X,d) has a well ordered base $\mathcal{B}$.*

For compact and Lindelöf metric spaces we have:

Theorem 2.8 *(i) CMSC iff CMWOB.*
(ii) In all permutation models, LMSC iff LMWOB.

Proof. (i) ($\rightarrow$). This is straightforward.

(i) ($\leftarrow$). Fix (X, ρ) a compact metric space and let $\mathcal{B} = \{B_i : i \in \aleph\}$ be a well ordered base for the metric topology T_ρ on X. Since for every $n \in \omega^+$,

$$|[\aleph]^n| = |\aleph^n| = \aleph \tag{1}$$

where $[\aleph]^n$ is the set of all n-element subsets of $\aleph$, it follows that each $[\mathcal{B}]^n$ is a well ordered set. For every $n \in \omega$ let

$$\mathcal{U}_n = \{B \in \mathcal{B} : diameter(B) < \frac{1}{n+1}\}.$$

As $\mathcal{U}_n$ is a cover of X, it follows that $\mathcal{U}_n$ has a finite subcover $\mathcal{W}_n$, say of size m. Let m_n be the smallest such integer m. As $[\mathcal{B}]^{m_n}$ is a well ordered set and $S = \{\mathcal{V} \in [\mathcal{B}]^{m_n} : \mathcal{V}$ is a subcover of $\mathcal{U}_n\} \neq \emptyset$, we may put $\mathcal{V}_n = \min(S)$. By (1), each $\mathcal{V}_n$ has a fixed ordering. Thus, $\mathcal{V} = \cup\{\mathcal{V}_n : n \in \omega\}$ is a countable base for T_ρ. $\square$

Proposition 2.9 *Assume the statement "$\mathcal{P}(\mathbb{R})$ is well orderable" (Form 130 in [5]). Then the following are equivalent.*
(i) LM(2^ω),
(ii) LMSC,
(iii)LMS.

Proof. (i) $\leftrightarrow$ (ii). It suffices to show that (i) $\rightarrow$ (ii). Let (X, d) be a Lindelöf metric space. By (i), we can view X as a subset of $\mathbb{R}$, therefore by our hypothesis, we have that $\mathcal{P}(X)$ is well orderable. Thus, the base for the metric topology of all open discs of X is also well orderable and consequently, by Theorem 2.8, X is a second countable space.

(ii) $\leftrightarrow$ (iii). It is known that *every separable metric space is second countable* and this without use of any choice principle. Hence, we only need to show that (ii) $\rightarrow$ (iii). Fix (X, d) a Lindelöf metric space with a countable base $\mathcal{B} = \{B_n : n \in \omega\}$. Then $|X| \leqq |\mathbb{R}|$(see page 3) and since $\mathcal{P}(X)$ is well orderable, X has a well ordering $\leqq$. Clearly $Q = \{q_n : n \in \omega\}$, $q_n = \min(B_n)$, is a dense subset of X. This completes the proof of the proposition. $\square$

Proposition 2.10 *(see also [1]). CMSC implies CAC_{fin}.*

Proof. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of non empty finite sets. Let $X = (\cup\mathcal{A}) \cup \{\infty\}$ and define a function $d : X \times X \to \mathbb{R}$ by requiring $d(x,x) = 0$ for all $x \in X$ and

$$d(x,y) = d(y,x) = \begin{cases} max\{\frac{1}{n+1}, \frac{1}{m+1}\} \text{ if } x \in A_n \text{ and } y \in A_m \\ \\ \frac{1}{n+1} \text{ if } x = \infty \text{ and } y \in A_n \end{cases} \tag{2}$$

Clearly d is a metric on X and (X,d) is a compact (hence, Lindelöf) metric space. (Any open set O including ∞, it also includes all but finitely many members of $\mathcal{A}$. Since we need finitely many open sets to cover the remaining members, it follows that every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$ and consequently (X,d) is compact.) Since all singletons of $\cup\mathcal{A}$ are open sets we see that every base for the metric topology on X must contain all singletons. Thus, any countable base (which exists by our hypothesis) induces a well ordering on $\cup\mathcal{A}$. On the basis of this well ordering we can pass easily to a choice set for $\mathcal{A}$ finishing the proof of the proposition. $\square$

Remark 2.11 *One could possibly conjecture that the proof of proposition 2.10 goes through with countable in place of finite and conclude that LMSC implies CUC. The proof of proposition 2.10 breaks down at the point where the space X is claimed to be Lindelöf. In particular, the assertion that X is Lindelöf implies CAC_ω. Indeed, fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of non empty countable sets. Without loss of generality we may assume that $A_0 = \omega$. Let X be as in proposition 2.10. If X were Lindelöf, then any countable subcover of the open cover $\mathcal{U} = \{\{n,x\} : n \in \omega, x \in A_{n+1}\} \cup \{X \backslash A_0\}$ would give rise to a choice set for $\mathcal{A}$.*

Corollary 2.12 *Each of the following statements implies CAC_{fin}.*
(i) LMSC.
(ii) CMWOB.
(iii) LMWOB.
(iv) LMS.
(v) LM(2^ω).
(vi) LMHL.

Proof. We only prove (v) and (vi).

(v). Define $\mathcal{A}, X$ and d as in the proof of proposition 2.10. By our hypothesis, there exists an injective function $f : X \to \mathbb{R}$, therefore, X has a

linear ordering $\leqq$. For each $i \in \omega$, let $c_i = \min A_i$. Clearly $C = \{c_i : i \in \omega\}$ is a choice set for $\mathcal{A}$.

(vi). Define $\mathcal{A}, X$ and d as in the proof of (v). By LMHL, $\cup\mathcal{A}$ is a Lindelöf subspace of X. Thus, the open cover $\{\{x\} : x \in \cup\mathcal{A}\}$ is countable and so $\cup\mathcal{A}$ is countable. Therefore, $\mathcal{A}$ has a choice set. This completes the proof of (vi) and of the corollary. $\square$

Next we give a result that relates LMSC and CUC.

Proposition 2.13 *The conjuction of the following statements implies CUC.*
(i) LMSC,
(ii) UT($\omega, \omega, < 2^\omega$): The countable union of countable sets has cardinality less than 2^ω,
(iii) $\mathcal{P}(\mathbb{R})$ is well orderable.

Proof. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of countable sets and let (X, d) be defined as in proposition 2.10. By (ii), there exists an injective function $f : \mathcal{P}(X) \to \mathcal{P}(\mathbb{R})$, therefore by (iii), $\mathcal{P}(X)$ is well orderable. Via a straightforward induction and using the fact that any open cover $\mathcal{U}$ of X is well ordered we can pass to a countable subcover of $\mathcal{U}$. Thus, X is Lindelöf and $\cup\mathcal{A}$ is countable as in proposition 2.10. $\square$

Remark 2.14 *In view of proposition 2.13 and remark 2.11, we see that the conjuction of UT($\omega, \omega, < 2^\omega$) and $\mathcal{P}(\mathbb{R})$* is well orderable *implies CAC_ω. Since the statement $\mathcal{P}(\mathbb{R})$* is well orderable *does not imply CAC_ω in ZF^0 (see [5]), we have that UT($\omega, \omega, < 2^\omega$) is not provable in ZF^0.*

3 Models

In this section we show that CAC is independent from the statements LMSC, LM(2^ω), LMS, CMSC, LMHL, LMSC($\mathcal{B}$) and LML($\mathcal{B}$). In particular, we prove that the latter statements hold in the basic Fraenkel model $\mathcal{N}$ ($\mathcal{N}1$ in [5]). It is known that CAC fails in $\mathcal{N}$, therefore, the independence result follows. We will use the result of Theorem 9 in [6], namely that, in the basic Fraenkel model every metric space (X, d) has an $\aleph$-discrete base $\mathcal{B}$ (i.e. $\mathcal{B} = \cup\{B_n : n \in \aleph\}$ and for each $x \in X$ and $n \in \aleph$ there exists a neighborhood of x which meets at most one element of B_n), where $\aleph$ is a well orderable cardinal. First we prove the following

Lemma 3.1 *In all permutation models,*
(i) LMSC, LMS and LM(2^ω) are equivalent,
(ii) LMSC($\mathcal{B}$) implies the statements LML($\mathcal{B}$) and LMHL,
(iii) LM($\mathcal{B}, 2^\omega$) implies CUC.

Proof. (i). Since the statement $\mathcal{P}(\mathbb{R})$ *is well orderable* is true in every permutation model (see [5]), the results follow immeadiately from proposition 2.9.
(ii). This follows immeadiately from Proposition 1.9 because CAC($\mathbb{R}$) is true in every permutation model.
(iii). In view of proposition 2.9 and of (i) of the present lemma, we deduce that LMSC($\mathcal{B}$) and LM($\mathcal{B}, 2^\omega$) are equivalent in each permutation model. The conclusion follows from Corollary 2.2. □

Theorem 3.2 *Each of the following statements holds in the basic Fraenkel Model $\mathcal{N}$.*
(i) LMSC.
(ii) LM(2^ω).
(iii) LMS.
(iv) LML($\mathcal{B}$).
(v) LMHL.
(vi) LMSC($\mathcal{B}$).
(vii) CMSC.

Proof. (i) up to (vi). By the fact that LMSC($\mathcal{B}$) implies LMSC and by Lemma 3.1, it suffices to show that LMSC($\mathcal{B}$) holds in $\mathcal{N}$. Fix (X, d) a metric space in $\mathcal{N}$ which is Lindelöf with respect to the base $\mathcal{B}$. According to Theorem 9 in [6], (X, d) has an $\aleph$-discrete base $\mathcal{C} = \cup\{\mathcal{C}_n : n \in \aleph\}$ in $\mathcal{N}$, where each $\mathcal{C}_n$ is a family of open (necessarily disjoint) discs.

Claim 1. Every non countable subset G of X has an accumulation point.

Proof of claim 1. Assume on the contrary that G is a non countable set without accumulation points. Then G is a closed set in X, therefore, any cover of G by sets in $\mathcal{B}$ has a countable subcover. Put $\mathcal{U} = \{B \in \mathcal{B} : (\exists x \in G)(x \in B \wedge B \cap G = \{x\})\}$. Since G has no accumulation points, $\mathcal{U}$ is a cover of G and consequently $\mathcal{U}$ has a countable subcover $\mathcal{V}$. It is straightforward to verify that $|G| \leq |\mathcal{V}|$, therefore, G is countable. This contradiction completes

the proof of Claim 1.

Claim 2. $\mathcal{C}_n$ is at most countable.

Proof of claim 2. It suffices to show that C_n, the set of all centers of the members of $\mathcal{C}_n$ is at most countable. Assume on the contrary that C_n is not countable. By Claim 1, C_n has an accumulation point c. Since $\mathcal{C}_n$ is discrete there exists a neighborhood V of c which meets at most one element of $\mathcal{C}_n$. Thus, $|V \cap C_n| \leq 1$. This contradicts the fact that c is an accumulation point of C_n and completes the proof of claim 2.

It follows by claim 2 that $\mathcal{C}$ is a well ordered union of countable sets. Since the statement

the well ordered union of well orderable sets is well orderable,

Form 231 in [5], is true in $\mathcal{N}$ (see [5]), it follows that $\mathcal{C}$ is well ordered in $\mathcal{N}$. Now in the proof of Theorem 2.8, in order to get a countable base for X we only needed X to be Lindelöf with respect to a well ordered base. Therefore, the proof of (vi) will be complete once we show the following Claim.

Claim 3. X is Lindelöf with respect to $\mathcal{C}$.

Proof of claim 3. Let $\mathcal{U}$ be a cover of X by sets in $\mathcal{C}$. Since $\mathcal{C}$ is well ordered, so is $\mathcal{U}$. Put

$$\mathcal{V} = \{B \in \mathcal{B} : B \subseteq U \text{ for some } U \in \mathcal{U}\}.$$

As X is Lindelöf with respect to $\mathcal{B}$ there exists a countable subcover $\{B_n : n \in \omega\}$ of the cover $\mathcal{V}$. Via an easy induction and using the fact that $\mathcal{U}$ is well ordered we can easily pass to a countable subcover of $\mathcal{U}$. This completes the proof of the Claim as well as (vi).

(vii). In the light of the proof of Claim 2 in (vi), one can easily show that if (X, d) is a compact metric space, then each $\mathcal{C}_n$ is a finite set. As in the proof of (vi), we have that $\mathcal{C}$ is well ordered in $\mathcal{N}$. The conclusion now follows from Theorem 2.8. $\square$

By Theorem 3.2, we easily deduce that the statement CMS (= *compact metric spaces are separable*) is equivalent to the conjuction of the statements:

1. Every compact metric space (X, d) has an ω-discrete base $\mathcal{B} = \cup\{B_n : n \in \omega\}$, where each B_n is a family of open discs, and

2. CAC_{fin}.

Corollary 3.3 *CMC and statement P are independent of each other, where $P \in \{$LMSC, LM(2^ω), LMS, CMSC, LMHL, LMSC($\mathcal{B}$), LML($\mathcal{B}$)$\}$.*

Proof. By Theorem 3.2, we have that every statement P holds in the basic Fraenkel model $\mathcal{N}$, whereas CMC is false in $\mathcal{N}$ ([5]). This means that CAC fails also. On the other hand, the Second Fraenkel model ($\mathcal{N}2$ in [5]) satisfies CMC but not CAC_{fin} (see [5]). Since each statement P implies CAC_{fin}, the conclusion follows. □

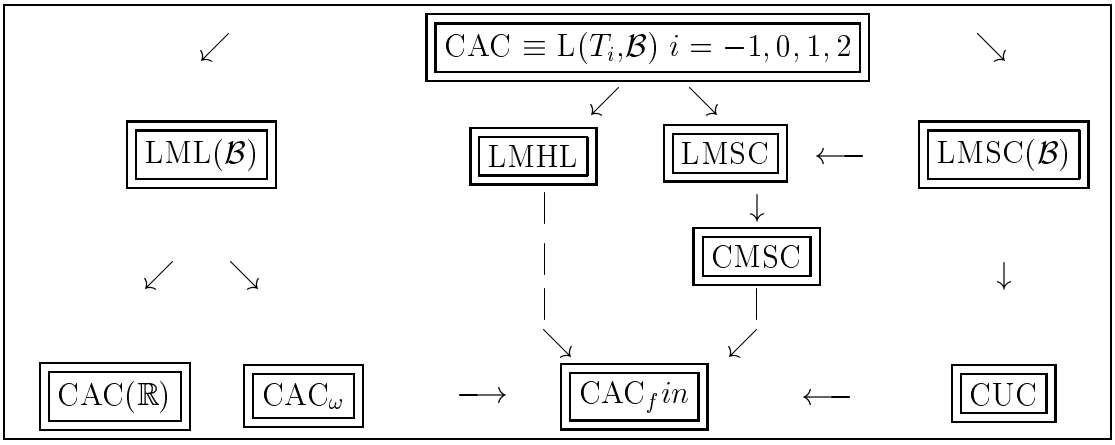
Lemma 3.4 *(i) CAC($\mathbb{R}$) does not imply neither LMSC($\mathcal{B}$) nor LML($\mathcal{B}$).*
(ii) CAC_{fin} does not imply neither LMSC($\mathcal{B}$) nor LML($\mathcal{B}$).
(iii) CUC (hence, CAC_ω) does not imply neither LML($\mathcal{B}$) nor LMS($\mathcal{B}$) (= Lindelöf metric spaces with respect to a base $\mathcal{B}$ are separable).
(iv) CAC_ω does not imply LM($\mathcal{B}, 2^\omega$).

Proof. (i) and (ii). CAC($\mathbb{R}$) and CAC_{fin} hold in the model $\mathcal{N}41$ in [5], whereas both CAC_ω and CUC fail. Also in the Cohen type model $\mathcal{M}12(\aleph)$ (see, [5]) CAC_{fin} holds whereas CUC fails. Due to Corollary 2.2, LMSC($\mathcal{B}$) fails in $\mathcal{M}12(\aleph)$ also.

(iii). It is evident that LML($\mathcal{B}$) implies that $\mathbb{N}$ is Lindelöf. Therefore, LML($\mathcal{B}$) implies CAC($\mathbb{R}$). Now in Cohen's original model, $\mathcal{M}1$ in [5], CUC holds whereas CAC($\mathbb{R}$) and consequently LML($\mathcal{B}$) fails. Further, in $\mathcal{M}1$ there exists a metric space (X, d) with a countable base $\mathcal{B}$, hence Lindelöf with respect to $\mathcal{B}$, which is not separable. Thus, LMS($\mathcal{B}$) fails in $\mathcal{M}1$.

(iv). In model $\mathcal{N}18$ in [5], CAC_ω holds but CUC fails. Thus, by Lemma 3.1, LM($\mathcal{B}, 2^\omega$) fails also in $\mathcal{N}18$. □

4 Summary



Questions. (i). Is CAC$_{fin}$ strictly weaker than LMSC or CMSC?
(ii). Does CUC imply LMSC or LMS?
(iii). Does LMSC or LMS imply CUC?
(iv). Does CMSC imply LMSC?
(v). Does LMSC imply LMS in ZF^0?

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