Nonconstructive Properties of Well-Ordered $T_2$ Topological Spaces

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Abstract We show that none of the following statements is provable in Zermelo-Fraenkel set theory ($ZF$) answering the corresponding open questions from Brunner in “The axiom of choice in topology”:

(i) For every $T_2$ topological space $(X, T)$ if $X$ is well-ordered, then $X$ has a well-ordered base,

(ii) For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then each open cover of $X$ has a well-ordered open refinement,

(iii) For every $T_2$ topological space $(X, T)$, if $X$ has a well-ordered dense subset, then there exists a function $f : X \times W \to T$ such that $W$ is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

1 Introduction Let $(X, T)$ be a $T_2$ topological space and let $\mathcal{B}$ be a base for $X$. Clearly,

$$|T| \leq |2^X|$$

(1)

and

$$|X| \leq |2^{|\mathcal{B}|}|.$$  

(2)

(The map $f : X \to \mathcal{P}(\mathcal{B}) (= \text{the powerset of } \mathcal{B})$, $f(x) = \{B \in \mathcal{B} : x \in B\}$ is obviously 1 : 1). We then have the following proposition.

Proposition 1.1 In Fraenkel-Mostowski permutation models, a $T_2$ topological space $(X, T)$ is well-ordered if and only if $X$ has a well-ordered base.

Proof: From (1) and the fact that in every permutation model Form 91 in Howard and Rubin [4], PW : The powerset of a well-ordered set can be well-ordered holds, we have that if $X$ is well-ordered, then $T$ is well-ordered. Similarly from (2) it follows that if $X$ has a well-ordered base, then $X$ is well-ordered.

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In Cohen models however Proposition 1.1 may fail. Indeed, in the basic Cohen model, model $\mathcal{M}1$ of [4], the real line $\mathbb{R}$ with the standard topology has a countable base, but $\mathbb{R}$ is not well-ordered. There remains the question:

*If $(X, T)$ is a well-ordered $T_2$ topological space, then does $X$ have a well-ordered base?*

Motivated by this question, Brunner [1] defined the following statements:

(A1) Form 148 in [4]: For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then $X$ has a well-ordered base.

(A2) For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then there exists a function $f: X \times W \to T$ such that $W$ is a well-ordered set and $f(\{x\} \times W)$ is a neighborhood base at $x$ for each $x \in X$.

(A3) For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then each open cover of $X$ has a well-ordered open refinement.

(A4) For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then $X$ satisfies $(\ast)$: if $O \subseteq T$ covers $X$, there is a mapping $f: X \to T$ such that $x \in f(x)$ and $f[X]$ refines $O$.

(A5) For every $T_2$ topological space $(X, T)$, if $X$ is well-ordered, then $(\ast)$ is a hereditary property of $X$.

(A6) For every $T_2$ topological space $(X, T)$, if $X$ has a well-ordered dense subset, then there exists a function $f: X \times W \to T$ such that $W$ is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

Clearly, each of the above statements is a theorem of ZFC (ZF with the axiom of choice AC, Form 1 in [4]). Brunner [1] asks whether these statements are provable in ZF minus the axiom of regularity ($ZF^0$) and Howard and Rubin [4] ask whether 148 implies AC. The aim of this paper is to show that none of (Ai), $i = 1, 2, 3, 4, 5, 6$, is a theorem of ZF and that 148 does not imply AC in $ZF^0$. In particular, we show that

1. (A1), (A2), and (A6) are equivalent to AC in ZF,
2. (A3), (A4), and (A5) imply Form 13 in [4]: *Every Dedekind finite subset (i.e., it has no countably infinite subset) of $\mathbb{R}$ is finite.*

Before setting out with proofs let us make a straightforward remark on the interrelation between the statements (A1) up to (A5):

(i) (A1) $\iff$ (A2).

(ii) (A1) $\implies$ (A3).

(iii) (A3) $\iff$ (A4) $\iff$ (A5).

For any undefined topological notion the reader is referred to Willard [9].
2 Results We begin by observing the following.

Theorem 2.1 \((A1)\) does not imply AC in \(ZF^0\).

Proof: Let \(\mathcal{N}\) be the basic Fraenkel model (model \(\mathcal{N}_1\) in [4]). By Proposition 1.1 we have that \((A1)\) holds in \(\mathcal{N}\). On the other hand, AC fails in \(\mathcal{N}\) (see [4]) meaning that \((A1)\) does not imply AC in \(ZF^0\) as required.\(\square\)

However in \(ZF\), \((A1)\) is equivalent to AC as Theorem 2.3 clarifies. In particular, we show that both \((A1)\) and \((A6)\) are equivalent to the set-theoretic principle PW (see the introduction) which in \(ZF\) is known to be equivalent to AC (see Felgner and Jech [3]). We recall first the notion of an independent family of sets.

Definition 2.2 Let \(\theta \geq \omega\) be an ordinal number. A family \(\mathcal{A} \subseteq \mathcal{P}(\theta)\) is said to be independent if and only if for any finite collection \(A_1, \ldots, A_m, B_1, \ldots, B_n\) of distinct elements of \(\mathcal{A}\), \(|A_1 \cap \cdots \cap A_m \cap (E \setminus B_1) \cap \cdots \cap (E \setminus B_n)| = |\theta|\).

Theorem 2.3 In \(ZF\) the following statements are equivalent:

\(\begin{align*}
(i) & \quad PW, \\
(ii) & \quad (A1), \\
(iii) & \quad (A6).
\end{align*}\)

Proof: \((i) \rightarrow (ii)\) This is straightforward.

\((ii) \rightarrow (i)\) Fix an ordinal number \(\kappa \geq \omega\) and let \(\mathcal{A} = \{a_i : i \in 2^\kappa\} \subseteq \mathcal{P}(\kappa)\) be an independent family (see Kunen [6], Exercise (A6), p. 288). The existence of such a family can be proved in \(ZF^0\). We show that \(2^\kappa\) is well-ordered.

For each \(i \in 2^\kappa\), let \(G_i = \{x \in \mathcal{P}(\kappa) : |x \triangle a_i| < \omega\} \) where \(\triangle\) denotes the operation of symmetric difference. Since for all \(i, j \in 2^\kappa, i \neq j, a_i \triangle a_j\) is infinite, we have that \(G_i \cap G_j = \emptyset\). Put \(G = \bigcup \{G_i : i \in 2^\kappa\}\). For each \(x \in [\kappa]^{<\omega}\) (= \(\{x \in \mathcal{P}(\kappa) : |x| < \omega\}\)), \(i \in 2^\kappa\) and \(g \in G_i\), put

\[B(x, i, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \emptyset\}. \tag{3}\]

Claim 2.4 The family \(\{B(x, i, g) : x \in [\kappa]^{<\omega}, i \in 2^\kappa, g \in G_i\}\) is a cover of \([\kappa]^{<\omega}\).

Proof of Claim 2.4: Fix \(x \in [\kappa]^{<\omega}\) and let \(i \in 2^\kappa\). Then \(a_i \setminus x \in G_i\) and \(x \in B(x, i, a_i \setminus x)\) finishing the proof of the Claim 2.4. \(\square\)

Let \(\mathcal{B} = \{B(x, g) : x \in [\kappa]^{<\omega}, g = \bigcup Q, Q \in [G]^{<\omega}\}\) where \(B(x, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \emptyset\}\).

Claim 2.5 \(\mathcal{B}\) is a base for a \(T_2\) topology \(T_\mathcal{B}\) on \([\kappa]^{<\omega}\).

Proof of Claim 2.5: By Claim 2.4 we have that \(\mathcal{B}\) is a cover of \([\kappa]^{<\omega}\). On the other hand, if \(x \in B(x_1, g_1) \cap B(x_2, g_2)\), then since \(x \cap g_1 = x \cap g_2 = \emptyset\), we have that \(B(x, g_1 \cup g_2) \in \mathcal{B}\) and clearly, \(x \in B(x, g_1 \cup g_2) \subseteq B(x_1, g_1) \cap B(x_2, g_2)\). Therefore, \(\mathcal{B}\) is a base. We show now that \(\mathcal{B}\) generates a \(T_2\) topology on \([\kappa]^{<\omega}\). Fix \(x, y \in [\kappa]^{<\omega}\) with \(x \neq y\) and let \(g \in G\) be such that \((x \cup y) \cap g = \emptyset\) (take, for example, \(i \in 2^\kappa\) and put \(g = a_i \setminus (x \cup y)\)). Then \(V_x = B(x, g \cup (y \setminus x))\) and \(V_y = B(y, g \cup (x \setminus y))\) are disjoint neighborhoods of \(x\) and \(y\), respectively. Assume otherwise and let \(z \in V_x \cap V_y\). Then \(x \subseteq z, z \cap (g \cup (y \setminus x)) = \emptyset\), and \(y \subseteq z, z \cap (g \cup (x \setminus y)) = \emptyset\). Thus,
Since (\(\kappa\)^{<\omega}, T_\kappa) is a well-ordered T_2 space, let by (A1) \(\mathcal{W} = \{ W_j : j \in \mathbb{N} \}\) be a well-ordered base. Consider now the open cover \(U = \{ B(\emptyset, i, g) : i \in 2^\kappa, g \in G_i \}\) where \(B(x, i, g)\) is given by (3). Then \(\mathcal{V} = \{ V \in \mathcal{W} : V \subseteq U \text{ for some } U \in \mathcal{U} \}\) is clearly a well-ordered open refinement of \(\mathcal{U}\). For every \(V \in \mathcal{V}\), let \(H_V = \{ i \in 2^\kappa : \exists g \in G_i, V \subseteq B(\emptyset, i, g) \}\). (6)

**Claim 2.6** For each \(V \in \mathcal{V}\), \(H_V\) is finite.

*Proof of Claim 2.6:* Assume the contrary and let \(V_0 \in \mathcal{V}\) be such that \(H_{V_0}\) is infinite. As each \(G_i\) can be well-ordered uniformly (\([a_i \triangle x : x \in [\kappa]^{<\omega}\) is a uniform well-ordering of \(G_i\)) we may define an infinite set \(\{ g_i \in G_i : i \in H_{V_0} \}\) such that \(V_0 \subseteq B(\emptyset, i, g_i)\) for all \(i \in H_{V_0}\). Fix \(B(x_0, g)\) a basic open set contained in \(V_0\). Then \(g = g_i_1 \cup g_i_2 \cup \cdots \cup g_i_\omega\) for some \(n \in \omega\) and \(g_i_j \in G_i, j \leq n\). Since \(B(x_0, g) \subseteq \bigcap (B(\emptyset, i, g_i) : i \in H_{V_0})\), we have that \((\bigcup \{ g_i : i \in H_{V_0} \}) \setminus g = \emptyset\) (otherwise fix \(i \in H_{V_0}\) and \(y \in g_i \setminus g\), then \(x_0 \cup \{ y \} \in B(x_0, g) \setminus B(\emptyset, i, g_i)\), a contradiction). Since \(|g_i \setminus a_i| < \omega\), it follows immediately that for all \(i \in H_{V_0}\), \(F_i = a_i \setminus (a_i \cup a_{i_1} \cup \cdots \cup a_{i_n})\) is finite. This contradicts the fact that \(\mathcal{A}\) is an independent family and completes the proof of Claim 2.6. \(\square\)

Since \(\mathcal{A}\) is an independent family, \(\mathcal{U}\) has no finite subcover. Furthermore, as \(\mathcal{W}\) is a base it is clear that \(2^\kappa = \bigcup (H_V : V \in \mathcal{V})\) and since \(\kappa\) is well-ordered, \(2^\kappa\) is linearly ordered (e.g., lexicographically). Thus, each \(H_V\) is well-ordered and consequently \(2^\kappa\) is well-ordered finishing the proof of (ii) \(\rightarrow\) (i).

(i) \(\rightarrow\) (iii) Since in ZF, AC \(\iff\) PW, this is straightforward.

(iii) \(\rightarrow\) (i) Fix an ordinal number \(\kappa\). Since \(|\kappa| < |2^\kappa|\) we may assume without loss of generality that \(\kappa \subseteq 2^\kappa\). Let \(\mathcal{W}' = \{ W_f : f \in 2^\kappa \setminus \kappa \}\) be an independent family of subsets of \(\kappa\). Define a topology \(T\) on \(X = 2^\kappa\) by requiring: All points in \(\kappa\) to be isolated whereas neighborhoods of \(f \in 2^\kappa \setminus \kappa\) are all sets of the form

\[V_f = \{ f \} \cup (W_f \setminus (\bigcup Q \cup A)), \; Q \in [\mathcal{W} \setminus \{ W_f \}]^{<\omega}, \; A \in [W_f]^{<\omega}.\]

\((X, T)\) is a T_2 space. Indeed, let \(x, y \in X, x \neq y\). We consider the following cases.

**Case 1:** \(x, y \in \kappa\). Then \(\{x\}, \{y\}\) are the required disjoint neighborhoods of \(x\) and \(y\), respectively.
Case 2: $x \in \kappa$, $y \in 2^\kappa \setminus \kappa$. Then $\{x\}, \{y\} \cup (W_y \setminus \{x\})$ are the required disjoint neighborhoods of $x$ and $y$, respectively.

Case 3: $x, y \in 2^\kappa \setminus \kappa$. Then $\{x\} \cup (W_x \setminus W_y), \{y\} \cup (W_y \setminus W_x)$ are the required disjoint neighborhoods of $x$ and $y$, respectively.

Thus, $(X, T)$ is a $T_2$ space having the well-ordered set $\kappa$ as a dense subset. Adjoin an extra point $\infty$ to $X$ and extend the topology $T$ by declaring neighborhoods of $\infty$ to be all supersets of $\{\infty\}$ missing finitely many sets $\{f\} \cup W_f, f \in 2^\kappa \setminus \kappa$. Thus, each neighborhood of $\infty$ misses only finitely many elements of $2^\kappa \setminus \kappa$. Clearly $Y = X \cup \{\infty\}$ with the extended topology $T^\infty$ is a $T_2$ space having $\kappa$ as a dense subset.

Let, by (A6), $\{Z_i : i \in \mathbb{N}\}$ be a well-ordered family of neighborhoods of $\{\infty\}$ such that $\{\infty\} = \bigcap \{Z_i : i \in \mathbb{N}\}$. Then $2^\kappa \setminus \kappa = \bigcup \{(2^\kappa \setminus \kappa) \setminus Z_i : i \in \mathbb{N}\}$ and by the above each set $(2^\kappa \setminus \kappa) \setminus Z_i$ is finite. As $2^\kappa$ is linearly ordered, $(2^\kappa \setminus \kappa) \setminus Z_i$ is well-ordered. Thus, $2^\kappa \setminus \kappa$ is well-ordered finishing the proof of (iii) $\rightarrow$ (i) and of the theorem. \qed

Remark 2.7 The statement “If $(X, T)$ is a $T_2$ space with a well-ordered dense subset, then each open cover of $X$ has a well-ordered open refinement” has also been considered in [1] where it is shown not to be a theorem of ZF; in the basic Cohen model, the Moore plane (see Steen and Seebach [8], Example 82) is a separable $T_2$ space having an open cover with no well-ordered open refinement. Via the latter proof, Brunner implicitly suggests that the above statement implies a well-known weak choice principle, namely, the axiom of choice for families of nonempty subsets of $\mathbb{R}$, AC($\mathbb{R}$), and Form [79 A] in [4]. However, following the proof of Theorem 2.3 we deduce that the above statement is equivalent to AC in ZF. Indeed, let $(X, T)$ be the $T_2$ space of Theorem 2.3 and let $O = \{\{f\} \cup W_f : f \in 2^\kappa \setminus \kappa\} \cup \{\{x\} : x \in \kappa\}$. Clearly, $O$ is an open cover of $X$. Let $V = \{V_i : i \in \mathbb{N}\}$ be a well-ordered open refinement of $O$. For each $f \in 2^\kappa \setminus \kappa$, let $i_f$ be the least $i \in \mathbb{N}$ such that $f \in V_i$. Then $V_{i_f} \subseteq \{g\} \cup W_g$ for some $g \in 2^\kappa \setminus \kappa$. Necessarily, $g = f$ and consequently the function $f \mapsto V_{i_f}$ is 1:1 meaning that $2^\kappa$ is well-ordered.

Brunner [1] (*) of a $T_2$ space (see the Introduction) as topologically the most interesting (among the other fifteen properties of $T_2$ spaces he considers in [1]) because as he points out it is both a weakening of metacompact and well-ordered local weight.

In Theorem 2.8 we show that the statement (A3) implies Form 13. Since (13) fails in the basic Cohen model for the set of the countably many added Cohen reals (see Cohen [2]) we have that (A3) is not provable in ZF. Thus, neither of the statements (A4) or (A5) is provable in ZF.

Theorem 2.8 (A3) implies the statement: For every ordinal number $\kappa \geq \omega$, every infinite subset of $2^\kappa$ has an infinite well orderable subset. In particular, for $\kappa = \omega$, (A3) implies Form 13.

Proof: Fix an ordinal number $\kappa \geq \omega$ and let $A$ be an infinite subset of $2^\kappa$. Assume that $A$ has no infinite well-orderable subset and let $\mathcal{A}_i, G_i, G$ and $\{\kappa\}^{<\omega}, T_\mathcal{B}$ be defined as in the proof of Theorem 2.3. By (A3) let $\mathcal{W} = \{W_j : j \in \mathbb{N}\}$ be a well-ordered open refinement of the open cover $\mathcal{U} = \{B(\emptyset, i, g) : i \in A, g \in G_j\}$ where $B(x, i, g)$ is given by (3) in the proof of Theorem 2.3. For each $j \in \mathbb{N}$, let

$$H_j = \{i \in A : \exists g \in G_j, W_j \subseteq B(\emptyset, i, g)\}.$$
As in Claim 2.6 of the proof of Theorem 2.3 it can be shown that $H_j$ is finite for all $j \in \aleph$ and since $\mathcal{U}$ has no finite subcover, it follows that there is an infinite set $M \subseteq \aleph$ such that $H_j \neq H_{j'}$ for all $j, j' \in M, j \neq j'$. Consequently, $\mathcal{H} = \bigcup \{H_j : j \in M\}$ is infinite and as $2^\kappa$ is linearly ordered it follows that $\mathcal{H}$ is an infinite well-ordered subset of $A$. This contradicts our assumption and completes the proof of the theorem. □

**Remark 2.9** In view of Theorem 2.8 one expects that (13) does not imply back (A3). Indeed, this is the case. In particular, Monro ([7], p. 37) constructs a symmetric extension $\mathcal{N}$ of a countable transitive model of $ZF + V = L$ such that $\mathcal{N}$ satisfies $AC(\mathbb{R})$ (hence, it satisfies 13), but there is a cardinal $\kappa \in \mathcal{N}$ and an infinite subset of $2^\kappa$ (the set $G^*$ of p. 37) having no countably infinite subsets in $\mathcal{N}$. Thus, the conclusion of Theorem 2.8 fails for the ordinal number $\kappa$ and consequently (A3) fails in $\mathcal{N}$ as well.

### 3 Summary

The following diagram summarizes the results of the paper.

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\text{AC} \equiv (A1) \equiv (A2) \equiv (A6) \\
\downarrow \\
(A3) \equiv (A4) \equiv (A5) \\
\downarrow \\
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### REFERENCES


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