Properties of the real line and weak forms of the axiom of choice

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Abstract

We investigate the interrelations between weak forms of the axiom of choice restricted to non-empty subsets of reals.

1 Introduction and some preliminary results

This paper is a continuation of [5] and [8] and it is concerned with P. Howard's and J. E. Rubin's project "Consequences of the Axiom of Choice" [6]. Our aim is to clarify the interrelations between weak forms of the axiom of choice restricted to the real line. We use the notations established in [5], [6] and [8]. Furthermore, any statement "Form x" has been considered in [6] where all known implications between these forms are given in Table 1, see http://www.math.purdue.edu/~jer/Papers/conseq.html.

Below we list the statements we will be dealing with in the sequel.

Form 5, $\mathbf{CC}_\omega(\mathbb{R})$: For every family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty countable subsets of $\mathbb{R}$ there exists a set $c = \{c_i : i \in \omega\}$ such that for all $i \in \omega$, $c_i \in A_i$.

$\mathbf{PCC}_\omega(\mathbb{R})$: Every countable family of non-empty countable subsets of $\mathbb{R}$ has an infinite subfamily with a choice set. In [5] it is shown that $\mathrm{PCC}_\omega(\mathbb{R})$ is equivalent to $\mathrm{CC}_\omega(\mathbb{R})$.

Form 6, $\mathbf{CUC}(\mathbb{R})$: A countable union of countable subsets of $\mathbb{R}$ is countable.

Form 9 : Every infinite set has a countably infinite subset.

Form 13 : Every infinite subset of $\mathbb{R}$ has a countably infinite subset.

Form 26 : The union of denumerably many sets each of power $2^{\aleph_0}$ has power $2^{\aleph_0}$.

Form 31, CUC : A countable union of countable sets is countable.

Form 34 : $\aleph_1$ is a regular cardinal.

Form 35: The union of countably many meager subsets of $\mathbb{R}$ is meager.

Form 38 : $\mathbb{R}$ can not be written as a countable union of countable sets.

Form 56 : $H(2^{\aleph_0}) \neq \aleph_\omega$, where $H(2^{\aleph_0})$ is Hartog's aleph, the least $\aleph$ not $\leq 2^{\aleph_0}$.

Form 79, AC($\mathbb{R}$) : AC restricted to subsets of $\mathbb{R}$.

AC$_\omega$($\mathbb{R}$) : AC($\mathbb{R}$) restricted to countable subsets of $\mathbb{R}$.

AC$_\omega$($2^{\aleph_0}, \mathbb{R}$) : AC$_\omega$($\mathbb{R}$) restricted to $|\mathbb{R}|$ sized families.

AC($\omega_1, \mathbb{R}$) : AC($\mathbb{R}$) restricted to ω_1 sized families.

Form 85 ($C(\infty, \aleph_0)$) : Every family of denumerable sets has a choice function.

Form 94, CC($\mathbb{R}$) : AC($\mathbb{R}$) restricted to countable families.

ω-CC($\mathbb{R}$) : For every family $A = \{A_i : i \in \omega\}$ of non-empty subsets of $\mathbb{R}$ there exists a family $B = \{B_i : i \in \omega\}$ of non-empty countable sets such that for all $i \in \omega$, $B_i \subset A_i$.

Form 151, UT(WO,$\aleph_0$,WO) : A well-ordered union of countable sets is well-orderable.

Form 169 : There is an uncountable subset of $\mathbb{R}$ without a perfect subset (= closed without isolated points).

Form 170 : $\aleph_1 \leq 2^{\aleph_0}$.

Form 203 : Every partition of $\wp(\omega)$ into non-empty subsets has a choice function.

Form 212 : AC($\mathbb{R}$) restricted to families of size $|\mathbb{R}|$.

Form 368 : The set of all denumerable subsets of $\mathbb{R}$ has power $2^{\aleph_0}$.

Form 369. If $\mathbb{R}$ is partitioned into two sets, at least one of them has cardinality $2^{\aleph_0}$.

The following table is a submatrix of Table 1 in [6]. A "0" at position (n,m) means that it is unknown whether statement n implies statement m. A "1" means that n implies m and a "3" means that there is a Cohen model in which n is true but m is false.

	5	**6**	**13**	**34**	**35**	**38**	**74**	**79**	**94**	**169**	**170**	**203**	**211**	**212**	**368**	**369**
5	1	0	3	3	0	1	0	3	3	0	3	3	3	3	3	3
6	1	1	3	3	0	1	0	3	3	0	3	3	3	3	3	3
13	3	3	1	0	0	0	0	3	3	0	3	3	3	3	3	0
34	3	3	3	1	0	1	0	3	3	3	3	3	3	3	3	3
35	0	0	3	0	1	1	0	3	3	0	3	3	3	3	3	3
38	3	3	3	3	0	1	0	3	3	0	3	3	3	3	3	3
74	0	0	0	0	0	0	1	3	0	0	3	3	0	0	3	0
79	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
94	1	1	1	1	1	1	1	3	1	0	3	3	0	0	3	0
169	0	0	0	0	0	0	0	3	0	1	3	3	0	0	3	0
170	3	3	3	3	0	0	0	3	3	0	1	3	3	3	0	3
203	1	1	1	1	1	1	1	0	1	0	1	1	1	0	1	1
211	1	1	1	1	1	1	1	3	1	0	3	3	1	0	3	0
212	1	1	1	1	1	1	1	0	1	0	0	0	1	1	0	1
368	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
369	0	0	0	3	0	0	0	3	3	3	3	3	3	3	3	1

Proposition 1 *(1) 5, 6, 38* $\not\Longrightarrow$ *74, 169.*
(2) 13 $\not\Longrightarrow$ *34, 35, 38, 169.*
(3) 34 $\not\Longrightarrow$ *74 and 35* $\not\Longrightarrow$ *74, 169.*
(4) 74 $\not\Longrightarrow$ *5, 6, 34, 35, 38, 94, 169, 211, 212.*
(5) 74 $\Longrightarrow$ *13.*
(6) 94 $\not\Longrightarrow$ *169, 212.*
(7) 170 $\Longrightarrow$ *169.*
(8) 170 $\not\Longrightarrow$ *74, 368.*
(9) 169 $\not\Longrightarrow$ *5, 6, 13, 34, 35, 38, 74, 94, 211, 212, 369.*
(10) 203 $\Longrightarrow$ *169, 212.*
(11) 211 $\not\Longrightarrow$ *169, 212.*
(12) 212 $\not\Longrightarrow$ *79, 203 and 368.*
(13) 368 $\not\Longrightarrow$ *13, 74, 79, 94, 203, 211, 212, 369.*
(14) 368 $\Longrightarrow$ *169.*
(15) $\neg$*169* $\Longrightarrow$ *369. Hence,* $\neg$*369* $\Longrightarrow$ *169 and 169* $\not\Longrightarrow$ *369.*

Proof. (1) In Cohen's original model, model $\mathcal{M}1$ in [6], 6 and consequently

5 and 38, is true. On the other hand, in [4] it is shown that 74 implies 13 and since 13 is false in $\mathcal{M}1$ for the set A of the added Cohen reals, the independency follows. Now, in Truss's model, $\mathcal{M}12(\aleph)$, where $\aleph$ is a singular cardinal, 6 is true whereas every subset of reals has a perfect subset in this model. Thus, 169 fails in $\mathcal{M}12(\aleph)$.

(2) In the Feferman-Lévy model, $\mathcal{M}9$ in [6], $\mathbb{R}$ is expressed as a countable union of countable sets. Hence, each one of 35 and 38 fails in that model. In [4] it is shown that 74 is true of $\mathcal{M}9$. Hence, 13 holds in that model. In Truss's model, $\mathcal{M}12(\aleph)$ in [6], see also [15], 13 is true because every uncountable subset of the reals has a perfect subset, hence a subset of size $|\mathbb{R}|$. Thus, 169 fails in that model and since $\aleph_1$ is singular in $\mathcal{M}12(\aleph)$, 34 fails too.

(3) 34, 35 hold in $\mathcal{M}1$ whereas 74 fails since 13 fails. 35 holds in Solovay's model, $\mathcal{M}5(\aleph)$, whereas 169 fails, see [6].

(4) 74 holds in $\mathcal{M}9$. Since 5, 6, 34, 35, 38, 211 and 212 all fail in $\mathcal{M}9$, the independence result follows. On the other hand, 74 holds in $\mathcal{M}5(\aleph)$ whereas 169 fails, see[6].

(5) This has been proved in [4] and [10].

(6) In $\mathcal{M}5(\aleph)$ 94 is true whereas 169 fails, see [6]. The second assertion of (6) has been proved in [8], Proposition 8.

(7) Assume on the contrary that every uncountable subset of reals has a perfect subset. By 170, this implies that $\aleph_1$ has a subset of size $|\mathbb{R}|$. But then $|2^{\aleph_0}| = |\aleph_1|$, hence, $\mathbb{R}$ is well-ordered. It is well-known that this implies 169. A contradiction.

(8) In [5] it is shown that 170 is true in the model $\mathcal{M}1$. By (1) we have that 74 fails in that model. The second assertion of (8) is proved in [8], Proposition 9.

(9) In [5] we showed that 5 and 6 fail in $\mathcal{M}9$ by establishing that 5 implies 38. In view of the forthcoming Theorem 13, 169 is true in $\mathcal{M}9$. Moreover, 34, 35 and 38 fail in $\mathcal{M}9$, see [6]. Thus, 169 does not imply 5, 6, 34, 35, 38. Now, from the proof of (8) we deduce that 169 holds in $\mathcal{M}1$ whereas 13, 74, 94, 211, 212 fail in that model. Finally, since 170 holds in $\mathcal{M}1$, by (7) we also have that 169 holds in that model. On the other hand, it is known that 369 fails in $\mathcal{M}1$.

(10) In [5] we showed that 203 implies 170 and by (8) we deduce that 203 implies 169. The implication $203 \Longrightarrow 212$ has been proved in [8].

(11) 211 holds in $\mathcal{M}5(\aleph)$ whereas 169 fails, see [6]. The second assertion has been proved in [8], Proposition 8.

(12) In [14] it is shown that 212 does not imply either of 79, 203 and in [8] it is shown that 212 does not imply 368.

(13) In [8] we showed that 368 is true in $\mathcal{M}1$. Since the forms 13, 74, 79, 94, 203, 211, 212 and 369 all fail in $\mathcal{M}1$, the independence result follows.

(14) Tarski [13] has shown that 368 implies 170. The conclusion now follows from (8).

(15) Assume the contrary and let $\mathbb{R} = A \cup B$ with $|A| < |\mathbb{R}|$ and $|B| < |\mathbb{R}|$. Since the union of two countable sets is countable, it follows that either A of B is uncountable. Thus, in view of $\neg 169$ it follows that either A has the power of $\mathbb{R}$ or B has the power of $\mathbb{R}$. $\square$

If X, K are sets then X^K denotes the set of all functions from K into X and $[X]^K$ denotes the set of all subsets of X of cardinality $|K|$. $X^{<K}$ denotes the set of all functions from subsets of K of cardinality $< |K|$ into X and $[X]^{<K}$ denotes the set of all subsets of X of cardinality $< |K|$.

Since $|[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$ in ZF (Using the fact that each finite subset of $\mathbb{R}$ is well-ordered under the usual ordering of the reals, it is very easy to construct for each $n \in \omega$ a bijection $g_n : [\mathbb{R}]^n \to \mathbb{R}^n$. For $X \in [\mathbb{R}]^n$, define $g_n(X)(i) = \min(X \backslash \{g_n(X)(j) : j < i\})$. Then $|[\mathbb{R}]^{<\omega}| = |\mathbb{R}^{<\omega}|$ under the bijection f defined by $f(X) = g_n(X)$ for $X \in [\mathbb{R}]^n$, $n \in \omega$. It is evident that $|\mathbb{R}^{<\omega}| \leq |\mathbb{R}^\omega|$; The function $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, n, n, \ldots)$ is 1:1. Since $\mathbb{R}^\omega$ with the Tychonoff product topology is a separable metrizable space, it follows that $|\mathbb{R}^\omega| = |\mathbb{R}|$ and consequently $|[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$.)
it seems reasonable to expect that the statement:

(*) "*for every infinite subset A of $\mathbb{R}$, $|[A]^{<\omega}| = |A|$*"

is a theorem of ZF also. We show in Theorem 5 that this is not the case.

It can be readily verified in ZFC (=ZF+AC) that:

Proposition 2 *(ZFC) For every uncountable subset $X \subset \mathbb{R}$, $|[X]^{\leq\omega}| = |X|$ iff CH (= the continuum hypothesis).*

Since CH implies AC($\mathbb{R}$), one does not expect that Proposition 2 holds in the absence of AC. We show in Corollary 6 that this is not the case and Proposition 2 is a theorem of ZF. This means that some form of choice is incorporated in the statement: *For every uncountable set X, $|[X]^{\leq\omega}| = |X|$.* In particular, we show in Corollary 6 that the latter statement implies 31.

If k is an ordinal number and X is any infinite set then kX stands for the assertion: "$|k \times X| = |X|$". Likewise, ${}^kX$ stands for the proposition: "$|X^k| = |X|$".

Clearly, kX implies X has a subset of size k. (If $f : k \times X \to X$ is a $1:1$ function then $\{f((i,a)) : i \in k\}$, is the required subset of X of size k.) In particular, for $k = \omega$, ωX implies X has a countably infinite subset. Thus, the statement: "for every X, ωX" implies 9. We show in Theorem 7 that the weaker statement "for every X, $2X$" still is strong enough to imply 9.

Clearly ${}^2\mathbb{R}$ is true in ZF and one may ask whether the statement:

(**) "*for every infinite subset A of $\mathbb{R}$, ${}^2A$*"

is a theorem of ZF. We show in Theorem 7 that (**) implies 13 and since 13 fails in $\mathcal{M}1$, (**) is unprovable in ZF.

2 Main results

Theorem 3 *85 implies $UT(WO, \aleph_0, WO)$. In particular, 85+34 implies CUC.*

Proof. Let $\mathcal{A} = \{A_i : i \in \aleph\}$, $\aleph$ a well-ordered cardinal, be a family of countable sets. Clearly, $\mathcal{C} = \cup\{\wp(A_i) : i \in \aleph\}$ is a family of countable sets. Fix by 85, f a choice function of the family $\mathcal{C}$. For every $i \in \aleph$ we construct via a transfinite induction on $\aleph_1$ a well-ordering of A_i of type $v_i, v_i \in \aleph_1$.

For $j = 0$ put $a_{i0} = f(A_i)$, for $j = v + 1$ let $a_{ij} = f(A_i \backslash \{a_{iu} : u \leq v\})$ if $A_i \backslash \{a_{iu} : u \leq v\} \neq \emptyset$ and for j a limit ordinal we let $a_{ij} = f(A_i \backslash \{a_{iu} : u < j\})$ if $A_i \backslash \{a_{iu} : u < j\} \neq \emptyset$.

Since A_i is countable it follows that for some $j \in \aleph_1$, $A_i \backslash \{a_{iu} : u \in j\} = \emptyset$. Let

$$v_i = \min\{j : A_i \backslash \{a_{iu} : u \in j\} = \emptyset\}.$$

It follows that $\cup\mathcal{A} = \cup\{\{a_{ij} : j \in v_i\} : i \in \aleph\}$, being a well-ordered union of well-ordered sets, is well-ordered as required.

To see that C(∞,$\aleph_0$)+34 implies CUC, fix a family of countable sets $\mathcal{A} = \{A_i : i \in \omega\}$. By C($\infty$,$\aleph_0$) $\cup\mathcal{A}$ is well-ordered. If $\cup\mathcal{A}$ is uncountable, then $\aleph_1 \subset \cup\mathcal{A}$ and $\aleph_1$ can be expressed as a countable union of countable sets. Thus, $\aleph_1$ is singular and 34 fails. This is a contradiction finishing the proof of the theorem. $\square$

Working as in the last theorem one can readily verify the following corollary:

Corollary 4 *$C(\infty, \aleph)$ (= If $\aleph$ is a well-ordered cardinal then every family of sets of size $\leq \aleph$, has a choice function) implies $UT(WO, \aleph, WO)$ (= If $\aleph$ is a well-ordered cardinal then the well-ordered union of sets of size $\leq \aleph$ is well-orderable).*

Theorem 5 *(i) "For every uncountable set X, $|[X]^{\leq\omega}| = |X|$" implies "for every uncountable set A, every uncountable subset X of A has an uncountable well-ordered subset" + CH. In particular, AC does not imply "for every uncountable set X, $|[X]^{\leq\omega}| = |X|$".*
(ii) "For every infinite set X, $|[X]^{<\omega}| = |X|$" implies 9. In particular, "for every infinite set $X \subset \mathbb{R}$, $|[X]^{<\omega}| = |X|$" implies 13.

Proof. (i) It suffices to show: $|[X]^{\leq\omega}| = |X|$ implies X has an uncountable well-ordered subset. Fix such a set X and let $f : [X]^{\leq\omega} \to X$ be a $1:1$ and onto function. Let $x = f(\{x_1, x_2\}), x_1, x_2 \in X, x_1 \neq x_2$. We shall construct recursively a set $A = \{a_i : i \in \aleph_1\} \subset X$ as follows:

For $i = 0$ put $a_0 = x$.

For $i = v + 1$ a non-limit ordinal of $\aleph_1$ put $a_i = f(\{a_v\})$. Since f is $1:1$ it follows that $a_i \notin \{a_j : j \leq v\}$.

For i a limit ordinal of $\aleph_1$ put $a_i = f(\{a_j : j < i\})$. As in the non-limit case $a_i \notin \{a_j : j < i\}$ terminating the induction.

The second assertion, i.e., the fact that CH holds, follows from $|\mathbb{R}| = |\wp(\omega)| = |[\omega]^{\leq\omega}| \leq |[\aleph_1]^{\leq\omega}| = |\aleph_1|$.

(ii) This can be proved as in (i). $\square$

Corollary 6 *(i) The statement :* "for every infinite subset A of $\mathbb{R}$, $|[A]^{<\omega}| = |A|$" *is unprovable in ZF.*
(ii) "For every uncountable set X, $|[X]^{\leq\omega}| = |X|$" implies 31.
(iii) "For every infinite set X, $|[X]^{<3}| = |X|$" implies 9. In particular, "for every infinite set $X \subset \mathbb{R}$, $|[X]^{<3}| = |X|$" implies 13.
(iv) If $|[X]^{<|X|}| = |X|$ then X is well-orderable. In particular, $|[\mathbb{R}]^{<|\mathbb{R}|}| = |\mathbb{R}|$ implies $AC(\mathbb{R})$.
(v) "For every uncountable subset $X \subset \mathbb{R}$, $|[X]^{\leq\omega}| = |X|$" iff CH.
(vi) 368 implies 170.

Proof. (i) It is known, see [6], that 13 fails in the basic Cohen model, $\mathcal{M}1$ in [6], for the set A of the countably many added Cohen reals. Thus, our statement fails in $\mathcal{M}1$.

(ii) Our hypothesis implies CH which in turn implies 34, i.e., $\aleph_1$ is regular (CH $\Rightarrow$ AC($\mathbb{R}$) $\Rightarrow$ 34). Now let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family of countable sets. If $\cup\mathcal{A}$ is uncountable, then in view of the proof of Theorem 5 (i), $\cup\mathcal{A}$ has a subset of size $\aleph_1$. Hence, $\aleph_1$ is singular, a contradiction. This completes the proof of (ii).

(iii) In the proof of Theorem 5 (i) we have only used $|[X]^{<3}| = |X|$.

(iv) Mimic the proof of Theorem 5 (i).

(v), (vi) These, in view of Theorem 5, are straightforward. $\square$

Theorem 7 *(i) "For every infinite set X, ${}^2X$" implies 9. In particular,* "for every infinite subset A of $\mathbb{R}$, ${}^2A$" *implies 13.*
(ii) For every infinite set $X, 2X$ implies 9.
(iii) AC($\mathbb{R}$) implies "for every infinite set X, if $|X^{\leq\omega}| = |X|$ then X has a well-orderable subset of size $|\mathbb{R}|$". In particular, in permutation models "$|X^{\leq\omega}| = |X|$" implies "X has a well-orderable subset of size $|\mathbb{R}|$".

Proof. The proof of this theorem is modeled on the proof of Theorem 5.

(i) Fix an infinite set X satisfying ${}^2X$ and let $f : X \times X \to X$ be a $1:1$ and onto function. Let $x = f((x_1, x_2)), x_1, x_2 \in X, x_1 \neq x_2$. We shall construct inductively a set $A = \{a_i : i \in \aleph_0\} \subset X$ as follows:

For $i = 0$ put $a_0 = x$.

For $i = v + 1$ put $a_i = f((a_v, a_v))$. Since f is $1:1$ it follows that $a_i \notin \{a_j : j \leq v\}$ terminating the induction and the proof of the (i).

(ii) Fix an infinite set X satisfying $2X$ and let $f : 2 \times X \to X$ be a $1:1$ and onto function. Via an easy induction we construct a set $\{x_i : i \in \omega\}$ as follows:

For $i = 0$ let $x_0 = f((0, a))$, where $a \in X$ is fixed and $f((0, a)) \neq a$. (If $f((0, a)) = a$ take $x_0 = f((1, a))$. As f is $1:1$ $x_0 \neq f((0, a))$).

For $i = v + 1$ let $x_i = f((0, x_v))$. Since f is $1:1$ it is straightforward to verify that $x_i \neq x_j$ for all $i, j \in \omega, i \neq j$.

(iii) Fix an infinite set X satisfying $|X^{\leq\omega}| = |X|$ and let $f : X^{\leq\omega} \to X$ be a $1:1$ and onto function. Let $x = f((x_1, x_2)), x_1, x_2 \in X, x_1 \neq x_2$. We shall construct inductively a set $A = \{a_i : i \in \aleph = |\mathbb{R}|\} \subset X$ as follows:

Let $\{h_j : j \in \aleph\}$ be an enumeration of all $1:1$ functions from ω into ω.

For $i = 0$ put $a_0 = x$.

For $i = v + 1$ a non-limit ordinal of $\aleph$ put $a_i = f((a_v, a_v))$.

For i a limit ordinal of $\aleph$ put $a_i = f((a_{h_i(0)}, a_{h_i(1)}, a_{h_i(2)}, \ldots))$. Since f is $1:1$ it follows that for every $i \in \aleph$, $a_i \notin \{a_j : j \in i\}$ finishing the proof of the theorem. $\square$

Let (D) stand for the proposition: "*every uncountable subset of* $\mathbb{R}$ *has an uncountable well-orderable subset* ". Since (D) clearly implies 13 and 368 holds in $\mathcal{M}1$, see [8], but 13 fails in $\mathcal{M}1$ we cannot expect that 368 implies (D). We show next that the conjunction of 368 with 212 implies (D).

Theorem 8 *(i) 212+368 implies "every uncountable subset of* $\mathbb{R}$ *has an uncountable well-orderable subset".*
(ii) $AC_\omega(2^{\aleph_0}, \mathbb{R})$*+368 implies "a well-ordered union of countable subsets of* $\mathbb{R}$ *can be well-ordered".*

Proof. (i) Fix A an uncountable subset of $\mathbb{R}$. Since $[A]^{\leq\omega} \subset [\mathbb{R}]^{\leq\omega}$ we have by 212 and 368 that $|[A]^{\leq\omega}| = |\mathbb{R}|$. (By 212 A has a countably infinite subset and in ZF $|[\omega]^\omega| = |\mathbb{R}|$). Let, by 212, f be a choice function of $\mathcal{A} = \{A_x : x \in \mathbb{R}\}$, the family of all cocountable subsets of A. On the basis of f, and following ideas from the proof of Theorem 2, define recursively an uncountable subset $\{a_i : i \in \aleph_1\}$ of A.

(ii) This can be proved as in Theorem 2. $\square$

Corollary 9 *(i) 203 implies "every uncountable subset of* $\mathbb{R}$ *has an uncountable well-orderable subset".*
(ii) 203 implies "a well-ordered union of countable subsets of $\mathbb{R}$ *can be well-ordered".*

Proof. 203 implies 212+368+$\mathrm{AC}_\omega(2^{\aleph_0}, \mathbb{R})$, see [5] and [8]. $\square$

Theorem 10 *(i) 85+*$AC(\omega_1, \mathbb{R})$ *implies* $|[X]^{\leq\omega}| \leq |X^\omega \times \omega_1|$.
(ii) $AC_\omega(\mathbb{R})$*+*$AC(\omega_1, \mathbb{R})$ *implies 368.*

Proof. (i) Let f be a choice function of $[X]^{\leq\omega}$. For every $Y \in [X]^{\leq\omega}$ define recursively a 1:1 and onto function $h_Y : \omega_1 \to Y$ as follows:

For $i = 0$ let $h_Y(0) = f(Y)$.

For $i = v + 1$ let $h_Y(i) = f(Y \backslash \{h_Y(j) : j \leq v\})$.

For $i = v$ a limit ordinal of ω_1 let $h_Y(i) = f(Y \backslash \{h_Y(j) : j \in v\})$.

Since Y is countable it follows that for some $i \in \omega_1$, $\{h_Y(j) : j \in i\} = Y$. Let i_Y be the smallest such ordinal of ω_1. Clearly, $h_Y : i_Y \to Y$ is 1:1 and onto. It follows that the function $F : [X]^{\leq\omega} \to X^{<\omega_1}$, given by $F(Y) = h_Y$ is 1:1 meaning that $|[X]^{\leq\omega}| \leq |X^{<\omega_1}|$. In order to complete the proof it suffices to show that $|X^{<\omega_1}| = |X^\omega \times \omega_1|$. Clearly $X^{<\omega_1} = \cup\{X^v : v \in \omega_1\}$.

In [5] we showed that AC(WO,$\mathbb{R}$), i.e., the axiom of choice for well-ordered families of non-empty subsets of $\mathbb{R}$, implies 170. The proof readily adapts to show that AC($\omega_1, \mathbb{R}$) implies 170. Let $\mathcal{A} = \{A_v : v \in \omega_1\}$, where $A_v = \{f \in v^\omega : f \text{ is } 1:1\}$. In view of 170 we may consider $\omega_1 \subset \mathbb{R}$ and consequently we may think of v^ω as being a subset of $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ when considered as a Tychonoff product of ω copies of $\mathbb{R}$ taken with the usual topology is, in ZF, a separable metric space, we see that $|\mathbb{R}^\omega| = |\mathbb{R}|$. Thus, we can view $\mathcal{A}$ as a family of subsets of $\mathbb{R}$. Let, by AC($\omega_1, \mathbb{R}$), h be a choice function of $\mathcal{A}$. Let $F : \cup\{X^v : v \in \omega_1\} \to X^\omega \times \omega_1$ be the function given by:

$$F(f) = (f \circ h(v), v), v \in \omega_1, f \in X^v.$$

It can be readily verified that F is $1:1$. Hence, $|\cup\{X^v : v \in \omega_1\}| \leq |X^\omega \times \omega_1|$ and consequently $|[X]^{\leq\omega}| \leq |X^\omega \times \omega_1|$ as required.

(ii) Form part (i) of this theorem we have: $|[\mathbb{R}]^{\leq\omega}| \leq |\mathbb{R}^\omega \times \omega_1|$. Since, $|\mathbb{R}^\omega| = |\mathbb{R}|, |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ and, by 170, $\omega_1 \subset \mathbb{R}$, we see that $|\mathbb{R}^\omega \times \omega_1| = |\mathbb{R}|$ and the desired result follows. □

In ZF^0 (= ZF minus foundation) 38+368 does not imply 26. Indeed, in the permutation model $\mathcal{N}55$ in [6], see also [9], 38+368 is true but the countable family $\mathcal{A}$ of circles has no choice. Furthermore, each circle has size $\mathbb{R}$. Thus, $\cup\mathcal{A}$ cannot have size $\mathbb{R}$ for otherwise $\mathcal{A}$ would have to have a choice set. We show next that 26 implies 38.

Theorem 11 *(i) 368 implies "the power set of every set $Y \subset \mathbb{R}$ which is expressible as a disjoint union of countable sets $Y = \cup\{A_i \subset \mathbb{R} : i \in k\}$ has size $\leq |\mathbb{R}^k|$". In particular, 368 implies that "$\mathbb{R}$ cannot be expressed as a countable union of countable sets".*
(ii) 26 implies that "$\mathbb{R}$ cannot be expressed as a countable union of countable sets".

Proof. (i) Fix $\mathcal{A} = \{A_n : n \in k\}$ a disjoint family of countable subsets of $\mathbb{R}$. Let, by 368, $f : [\mathbb{R}]^\omega \to \mathbb{R}$ be a 1:1 function. It can be readily verified that the function $H : \wp(Y) \to \mathbb{R}^k$, $Y = \cup\mathcal{A}$ given by:

$$H(X)(i) = f(X \cap A_i)$$

is a $1:1$ function. Thus, $|\wp(Y)| \leq |\mathbb{R}^k|$ as required.

The second assertion follows from the observation: $|\mathbb{R}^\omega| = |\mathbb{R}|$.

(ii) Assume the contrary and fix a disjoint family of countable sets $\mathcal{A} = \{A_i : i \in \omega\}$ covering $\mathbb{R}$. Clearly, for every $n \in \omega, |[A_n]^\omega| = |\mathbb{R}|$. Hence, by 26, we have $|\cup\{[A_n]^\omega : n \in \omega\}| = |\mathbb{R}|$. Thus, we may view each $[A_n]^\omega$ as a copy of $\mathbb{R}$, hence, $|\prod_{n\in\omega}[A_n]^\omega| = |\mathbb{R}|$. Continue now as in (i) to show that $|[\mathbb{R}]^\omega| \leq |\prod_{n\in\omega}[A_n]^\omega|$. By (i) this contradicts our assumption and completes the proof of the theorem. $\square$

In [6] the validity of the following implications

$170 \Longrightarrow 56$, $56 \Longrightarrow 170$, $38 \Longrightarrow 56$, $56 \Longrightarrow 38$ is indicated as unknown. We point out next that none of the above implications is true in ZF.

Proposition 12 *(i) $170 \not\Longrightarrow 56$ and $56 \not\Longrightarrow 170$. In particular, $\neg 56 \Longrightarrow 170$.*
(ii) $38 \not\Longrightarrow 56$ and $56 \not\Longrightarrow 38$.

Proof. Derrick and Drake [3] have constructed a forcing model, $\mathcal{M}10$ in [6], such that $(\forall n \in \omega)(\aleph_n \leq 2^{\aleph_0} \ \wedge \ \aleph_\omega \not\leq 2^{\aleph_0})$. Thus, in $\mathcal{M}10$, 170 is true and $H(2^{\aleph_0}) = \aleph_\omega$ meaning that 56 fails in this model. Furthermore, Derrick and Drake use a ccc partially ordered set of forcing conditions to construct the model $\mathcal{M}10$. Therefore, cardinals are preserved and $\aleph_1$ is a regular cardinal in $\mathcal{M}10$. Since 170 is true in this model, it follows that $\mathbb{R}$ can not be expressed as a countable union of countable sets. Thus, form 38 is true of $\mathcal{M}10$.

On the other hand in the Feferman/Lévy model, $\mathcal{M}9$ in [6], $\mathbb{R}$ is written as a countable union of countable sets, thus form 38 fails in $\mathcal{M}9$, and form 170 fails too, since the only well-orderable subsets of $\mathbb{R}$ are the countable sets, see Cohen [[2], p. 146]. Therefore, in $\mathcal{M}9$, $H(2^{\aleph_0}) = \aleph_1 \neq \aleph_\omega$ and consequently 56 is true in that model. $\square$

Theorem 13 *(i) The negation of 169 implies $CC_\omega(\mathbb{R})$.*
(ii) The negation of 169 implies $CUC(\mathbb{R})$.
(iii) ω-$CC(\mathbb{R})$+169 does not imply 26+368.
(iv) ω-$CC(\mathbb{R})$+$\neg$169 implies $CC(\mathbb{R})$.

Proof. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of countable subsets of the real line $\mathbb{R}$.

(i) Without loss of generality we may assume that for every $i \in \omega$ $A_i \subset (i, i+1)$. It suffices to show that $\mathcal{A}$ has a partial choice function. If $\mathcal{A}$ has no

partial choice, then $A = \cup\mathcal{A}$ is uncountable. Thus, A has a perfect subset K. Then $|K| = |\mathbb{R}|$ and consequently there are infinitely many members of $\mathcal{A}$ meeting K non-trivially. Since for every $i \in \omega$, $\overline{K \cap A_i} \subset A_i$ and the family of all closed subsets of $\mathbb{R}$ has a choice function, see [9] and [12], it follows that $\mathcal{A}$ has a partial choice which is a contradiction.

(ii) If $\cup\mathcal{A}$ is uncountable then $\neg 169$ implies that $\cup\mathcal{A}$ has the power of the continuum. Thus, by (i) and the fact that 5 implies 38, see [5], we have that $\cup\mathcal{A}$ is countable.

(iii) In view of Theorem 11, 368 fails in $\mathcal{M}9$. On the other hand, it is known (see [4]) that ω-CC($\mathbb{R}$) holds in $\mathcal{M}9$. Moreover, 169 holds in $\mathcal{M}9$, as otherwise, by (i) we would have that 5 and consequently 38 would also hold in that model which is a contradiction. Finally, 26 fails in $\mathcal{M}9$ since by Theorem 11, 26 implies 38. $\square$

3 Summary

The proofs of the positive and independence results of this paper formulate the table of the introduction as follows:

	5	**6**	**13**	**34**	**35**	**38**	**74**	**79**	**94**	**169**	**170**	**203**	**211**	**212**	**368**	**369**
5	1	0	3	3	0	1	**3**	3	3	**3**	3	3	3	3	3	3
6	1	1	3	3	0	1	**3**	3	3	**3**	3	3	3	3	3	3
13	3	3	1	**3**	**3**	**3**	0	3	3	**3**	3	3	3	3	3	0
34	3	3	3	1	0	1	**3**	3	3	3	3	3	3	3	3	3
35	0	0	3	0	1	1	**3**	3	3	**3**	3	3	3	3	3	3
38	3	3	3	3	0	1	**3**	3	3	**3**	3	3	3	3	3	3
74	**3**	**3**	**1**	**3**	**3**	**3**	1	3	**3**	**3**	3	3	**3**	**3**	3	0
79	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
94	1	1	1	1	1	1	1	3	1	**3**	3	3	0	**3**	3	0
169	**3**	**3**	**3**	**3**	**3**	**3**	**3**	3	**3**	1	3	3	**3**	**3**	3	**3**
170	3	3	3	3	0	0	**3**	3	3	**1**	1	3	3	3	**3**	3
203	1	1	1	1	1	1	1	0	1	**1**	1	1	1	**1**	1	1
211	1	1	1	1	1	1	1	3	1	**3**	3	3	1	**3**	3	0
212	1	1	1	1	1	1	1	**3**	1	0	0	**3**	1	1	**3**	1
368	0	0	**3**	0	0	**1**	**3**	**3**	**3**	**1**	1	**3**	**3**	**3**	1	**3**
369	0	0	0	3	0	0	0	3	3	3	3	3	3	3	3	1

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