Properties of the real line and weak forms of the axiom of choice

December 2, 2002

Abstract

We investigate the interrelations between weak forms of the axiom of choice restricted to non-empty subsets of reals.

1 Introduction and some preliminary results

This paper is a continuation of [5] and [8] and it is concerned with P. Howard's and J. E. Rubin's project “Consequences of the Axiom of Choice” [6]. Our aim is to clarify the interrelations between weak forms of the axiom of choice restricted to the real line. We use the notations established in [5], [6] and [8]. Furthermore, any statement “Form x” has been considered in [6] where all known implications between these forms are given in Table 1, see http://www.math.purdue.edu/~jer/Papers/conseq.html.

Below we list the statements we will be dealing with in the sequel.

Form 5, $\text{CC}_\omega(\mathbb{R})$ : For every family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty countable subsets of $\mathbb{R}$ there exists a set $c = \{c_i : i \in \omega\}$ such that for all $i \in \omega$, $c_i \in A_i$.

$\text{PCC}_\omega(\mathbb{R})$ : Every countable family of non-empty countable subsets of $\mathbb{R}$ has an infinite subfamily with a choice set. In [5] it is shown that $\text{PCC}_\omega(\mathbb{R})$ is equivalent to $\text{CC}_\omega(\mathbb{R})$.

Form 6, $\text{CUC}(\mathbb{R})$ : A countable union of countable subsets of $\mathbb{R}$ is countable.

Form 9 : Every infinite set has a countably infinite subset.

Form 13 : Every infinite subset of $\mathbb{R}$ has a countably infinite subset.
Form 26: The union of denumerably many sets each of power $2^{\aleph_0}$ has power $2^{\aleph_0}$.

Form 31, CUC: A countable union of countable sets is countable.

Form 34: $\aleph_1$ is a regular cardinal.

Form 35: The union of countably many meager subsets of $\mathbb{R}$ is meager.

Form 38: $\mathbb{R}$ can not be written as a countable union of countable sets.

Form 56: $H(2^{\aleph_0}) \neq \aleph_\omega$, where $H(2^{\aleph_0})$ is Hartog’s aleph, the least $\aleph$ not $\leq 2^{\aleph_0}$.

Form 79, AC($\mathbb{R}$): AC restricted to subsets of $\mathbb{R}$.

Form 85, $\omega_1$-CC($\mathbb{R}$): For every family $\mathcal{A} = \{A_i: i \in \omega\}$ of non-empty subsets of $\mathbb{R}$ there exists a family $\mathcal{B} = \{B_i: i \in \omega\}$ of non-empty countable sets such that for all $i \in \omega$, $B_i \subseteq A_i$.

Form 85 (C($\omega$, $\aleph_0$)): Every family of denumerable sets has a choice function.

Form 94, CC($\mathbb{R}$): AC($\mathbb{R}$) restricted to countable families.

Form 151, UT(WO,$\aleph_0$,WO): A well-ordered union of countable sets is well-orderable.

Form 169: There is an uncountable subset of $\mathbb{R}$ without a perfect subset (= closed without isolated points).

Form 170: $\aleph_1 \leq 2^{\aleph_0}$.

Form 203: Every partition of $\wp(\omega)$ into non-empty subsets has a choice function.

Form 212: AC($\mathbb{R}$) restricted to families of size $|\mathbb{R}|$.

Form 368: The set of all denumerable subsets of $\mathbb{R}$ has power $2^{\aleph_0}$.

Form 369. If $\mathbb{R}$ is partitioned into two sets, at least one of them has cardinality $2^{\aleph_0}$.

The following table is a submatrix of Table 1 in [6]. A “0” at position $(n,m)$ means that it is unknown whether statement $n$ implies statement $m$. A “1” means that $n$ implies $m$ and a “3” means that there is a Cohen model in which $n$ is true but $m$ is false.
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**Proposition 1**

(1) $5, 6, 38 \not\implies 74, 169$.  
(2) $13 \not\implies 34, 35, 38, 169$.  
(3) $34 \not\implies 74$ and $35 \not\implies 74, 169$.  
(4) $74 \not\implies 5, 6, 34, 35, 38, 94, 169, 211, 212$.  
(5) $74 \implies 13$.  
(6) $94 \not\implies 169, 212$.  
(7) $170 \implies 169$.  
(8) $170 \not\implies 74, 368$.  
(9) $169 \not\implies 5, 6, 13, 34, 35, 38, 74, 94, 211, 212, 369$.  
(10) $203 \implies 169, 212$.  
(11) $211 \not\implies 169, 212$.  
(12) $212 \not\implies 79, 203$ and $368$.  
(13) $368 \not\implies 13, 74, 79, 94, 203, 211, 212, 369$.  
(14) $368 \implies 169$.  
(15) $\neg 169 \implies 369$. Hence, $\neg 369 \implies 169$ and $169 \not\implies 369$.  

**Proof.** (1) In Cohen’s original model, model $M_1$ in [6], 6 and consequently
5 and 38, is true. On the other hand, in [4] it is shown that 74 implies 13 and since 13 is false in \( M_1 \) for the set \( A \) of the added Cohen reals, the independency follows. Now, in Truss’s model, \( M_{12}(8) \), where \( 8_1 \) is a singular cardinal, 6 is true whereas every subset of reals has a perfect subset in this model. Thus, 169 fails in \( M_{12}(8) \).

(2) In the Feferman–Lévy model, \( M_9 \) in [6], \( \mathbb{R} \) is expressed as a countable union of countable sets. Hence, each one of 35 and 38 fails in that model. In [4] it is shown that 74 is true of \( M_9 \). Hence, 13 holds in that model. In Truss’s model, \( M_{12}(\aleph) \) in [6], see also [15], 13 is true because every uncountable subset of the reals has a perfect subset, hence a subset of size \( \aleph_1 \). Thus, 169 fails in that model and since \( \aleph_1 \) is singular in \( M_{12}(\aleph) \), 34 fails too.

(3) 34, 35 hold in \( M_1 \) whereas 74 fails since 13 fails. 35 holds in Solovay’s model, \( M_{5}(\aleph) \), whereas 169 fails, see [6].

(4) 74 holds in \( M_9 \). Since 5, 6, 34, 35, 38, 211 and 212 all fail in \( M_9 \), the independence result follows. On the other hand, 74 holds in \( M_{5}(\aleph) \) whereas 169 fails, see [6].

(5) This has been proved in [4] and [10].

(6) In \( M_{5}(\aleph) \) 94 is true whereas 169 fails, see [6]. The second assertion of (6) has been proved in [8], Proposition 8.

(7) Assume on the contrary that every uncountable subset of reals has a perfect subset. By 170, this implies that \( \aleph_1 \) has a subset of size \( \aleph_1 \). But then \( 2^{\aleph_0} = |\aleph_1| \), hence, \( \mathbb{R} \) is well-ordered. It is well-known that this implies 169. A contradiction.

(8) In [5] it is shown that 170 is true in the model \( M_1 \). By (1) we have that 74 fails in that model. The second assertion of (8) is proved in [8], Proposition 9.

(9) In [5] we showed that 5 and 6 fail in \( M_9 \) by establishing that 5 implies 38. In view of the forthcoming Theorem 13, 169 is true in \( M_9 \). Moreover, 34, 35 and 38 fail in \( M_9 \), see [6]. Thus, 169 does not imply 5, 6, 34, 35, 38. Now, from the proof of (8) we deduce that 169 holds in \( M_1 \) whereas 13, 74, 94, 211, 212 fail in that model. Finally, since 170 holds in \( M_1 \), by (7) we also have that 169 holds in that model. On the other hand, it is known that 369 fails in \( M_1 \).

(10) In [5] we showed that 203 implies 170 and by (8) we deduce that 203 implies 169. The implication 203 \( \Rightarrow \) 212 has been proved in [8].

(11) 211 holds in \( M_{5}(\aleph) \) whereas 169 fails, see [6]. The second assertion has been proved in [8], Proposition 8.
(12) In [14] it is shown that 212 does not imply either of 79, 203 and in [8] it is shown that 212 does not imply 368.

(13) In [8] we showed that 368 is true in $\mathcal{M}$1. Since the forms 13, 74, 79, 94, 203, 211, 212 and 369 all fail in $\mathcal{M}$1, the independence result follows.

(14) Tarski [13] has shown that 368 implies 170. The conclusion now follows from (8).

(15) Assume the contrary and let $\mathbb{R} = A \cup B$ with $|A| \leq |\mathbb{R}|$ and $|B| < |\mathbb{R}|$. Since the union of two countable sets is countable, it follows that either $A$ of $B$ is uncountable. Thus, in view of $\neg 169$ it follows that either $A$ has the power of $\mathbb{R}$ or $B$ has the power of $\mathbb{R}$. $\square$

If $X, K$ are sets then $X^K$ denotes the set of all functions from $K$ into $X$ and $[X]^K$ denotes the set of all subsets of $X$ of cardinality $|K|$. $X^{< K}$ denotes the set of all functions from subsets of $K$ of cardinality $< |K|$ into $X$ and $[X]^{< K}$ denotes the set of all subsets of $X$ of cardinality $< |K|$.

Since $|[\mathbb{R}]^{\leq \omega}| = |\mathbb{R}|$ in ZF (Using the fact that each finite subset of $\mathbb{R}$ is well-ordered under the usual ordering of the reals, it is very easy to construct for each $n \in \omega$ a bijection $g_n : [\mathbb{R}]^n \to \mathbb{R}^n$. For $X \in [\mathbb{R}]^n$, define $g_n(X)(i) = \min(X \setminus \{g_n(X)(j) : j < i\})$. Then $|[\mathbb{R}]^{\leq \omega}| = |\mathbb{R}^{< \omega}|$ under the bijection $f$ defined by $f(X) = g_n(X)$ for $X \in [\mathbb{R}]^n$, $n \in \omega$. It is evident that $|\mathbb{R}^{< \omega}| \leq |\mathbb{R}^{\omega}|$. The function $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, n, n, \ldots)$ is 1:1. Since $\mathbb{R}^{\omega}$ with the Tychonoff product topology is a separable metrizable space, it follows that $|\mathbb{R}^{\omega}| = |\mathbb{R}|$ and consequently $|[\mathbb{R}]^{\leq \omega}| = |\mathbb{R}|$.)

it seems reasonable to expect that the statement:

(\*) "for every infinite subset $A$ of $\mathbb{R}$, $|[A]^{\leq \omega}| = |A|$"

is a theorem of ZF also. We show in Theorem 5 that this is not the case.

It can be readily verified in ZFC (=ZF+AC) that:

**Proposition 2** (ZFC) For every uncountable subset $X \subset \mathbb{R}$, $|[X]^{\leq \omega}| = |X|$ iff $CH$ (= the continuum hypothesis).

Since CH implies AC($\mathbb{R}$), one does not expect that Proposition 2 holds in the absence of AC. We show in Corollary 6 that this is not the case and Proposition 2 is a theorem of ZF. This means that some form of choice is incorporated in the statement: For every uncountable set $X$, $|[X]^{\leq \omega}| = |X|$. In particular, we show in Corollary 6 that the latter statement implies 31.
If $k$ is an ordinal number and $X$ is any infinite set then $kX$ stands for the assertion: “$|k \times X| = |X|$”. Likewise, $^kX$ stands for the proposition: “$|X|^k = |X|$”.

Clearly, $kX$ implies $X$ has a subset of size $k$. (If $f : k \times X \to X$ is a 1 : 1 function then $\{f((i, a)) : i \in k\}$, is the required subset of $X$ of size $k$.) In particular, for $k = \omega$, $\omega X$ implies $X$ has a countably infinite subset. Thus, the statement: “for every $X$, $\omega X$” implies 9. We show in Theorem 7 that the weaker statement “for every $X$, $2X$” still is strong enough to imply 9.

Clearly $2\setminus \mathbb{R}$ is true in ZF and one may ask whether the statement:

$$(**) \text{ “for every infinite subset } A \text{ of } \mathbb{R}, 2^A$$

is a theorem of ZF. We show in Theorem 7 that (**) implies 13 and since 13 fails in $\mathcal{M}1$, (**) is unprovable in ZF.

2 Main results

Theorem 3 $85$ implies $UT(WO, \aleph_0, WO)$. In particular, $85+34$ implies $CUC$.

Proof. Let $\mathcal{A} = \{A_i : i \in \aleph_0\}$, $\aleph_0$ a well-ordered cardinal, be a family of countable sets. Clearly, $\mathcal{C} = \bigcup \{\wp(A_i) : i \in \aleph_0\}$ is a family of countable sets.

Fix by 85, $f$ a choice function of the family $\mathcal{C}$. For every $i \in \aleph_0$ we construct via a transfinite induction on $\aleph_1$ a well-ordering of $A_i$ of type $v_i, v_i \in \aleph_1$.

For $j = 0$ put $a_{i0} = f(A_i)$, for $j = v + 1$ let $a_{ij} = f(A_i \setminus \{a_{iu} : u \leq v\})$ if $A_i \setminus \{a_{iu} : u \leq v\} \neq \emptyset$ and for $j$ a limit ordinal we let $a_{ij} = f(A_i \setminus \{a_{iu} : u < j\})$ if $A_i \setminus \{a_{iu} : u < j\} \neq \emptyset$.

Since $A_i$ is countable it follows that for some $j \in \aleph_1, A_i \setminus \{a_{iu} : u \in j\} = \emptyset$.

Let $$v_i = \min \{j : A_i \setminus \{a_{iu} : u \in j\} = \emptyset\}.$$ It follows that $\cup \mathcal{A} = \bigcup \{\{a_{ij} : j \in v_i\} : i \in \aleph_0\}$, being a well-ordered union of well-ordered sets, is well-ordered as required.

To see that $C(\infty, \aleph_0)+34$ implies $CUC$, fix a family of countable sets $\mathcal{A} = \{A_i : i \in \omega\}$. By $C(\infty, \aleph_0)$ $\cup \mathcal{A}$ is well-ordered. If $\cup \mathcal{A}$ is uncountable, then $\aleph_1 \supset \cup \mathcal{A}$ and $\aleph_1$ can be expressed as a countable union of countable sets. Thus, $\aleph_1$ is singular and 34 fails. This is a contradiction finishing the proof of the theorem. $\square$
Working as in the last theorem one can readily verify the following corollary:

**Corollary 4** $C(\omega \cdot \aleph, \aleph)$ ( = If $\aleph$ is a well-ordered cardinal then every family of sets of size $\leq \aleph$, has a choice function) implies $UT(WO, \aleph, WO)$ ( = If $\aleph$ is a well-ordered cardinal then the well-ordered union of sets of size $\leq \aleph$ is well-ordinalable).

**Theorem 5** (i) “For every uncountable set $X$, $|[X]^{\leq \omega}| = |X|$” implies “for every uncountable set $A$, every uncountable subset $X$ of $A$ has an uncountable well-ordered subset” $+$ CH. In particular, AC does not imply “for every uncountable set $X$, $|[X]^{\leq \omega}| = |X|$”.

(ii) “For every infinite set $X$, $|[X]^{< \omega}| = |X|$” implies 9. In particular, “for every infinite set $X \subset \mathbb{R}$, $|[X]^{< \omega}| = |X|$” implies 13.

**Proof.** (i) It suffices to show: $|[X]^{\leq \omega}| = |X|$ implies $X$ has an uncountable well-ordered subset. Fix such a set $X$ and let $f : [X]^{\leq \omega} \to X$ be a 1 : 1 and onto function. Let $x = f(\{x_1, x_2\})$, $x_1, x_2 \in X, x_1 \neq x_2$. We shall construct recursively a set $A = \{a_i : i \in \aleph_1\} \subset X$ as follows:

For $i = 0$ put $a_0 = x$.

For $i = v + 1$ a non-limit ordinal of $\aleph_1$ put $a_i = f(\{a_v\})$. Since $f$ is 1 : 1 it follows that $a_i \notin \{a_j : j \leq v\}$.

For $i$ a limit ordinal of $\aleph_1$ put $a_i = f(\{a_j : j < i\})$. As in the non-limit case $a_i \notin \{a_j : j < i\}$ terminating the induction.

The second assertion, i.e., the fact that CH holds, follows from $|\mathbb{R}| = |\mathbb{P}(\omega)| = |[\omega]^{\leq \omega}| \leq |[\aleph_1]^{\leq \omega}| = |\aleph_1|.$

(ii) This can be proved as in (i). \qed

**Corollary 6** (i) The statement: “for every infinite subset $A$ of $\mathbb{R}$, $|[A]^{< \omega}| = |A|$” is improvable in ZF.

(ii) “For every uncountable set $X$, $|[X]^{\leq \omega}| = |X|$” implies 31.

(iii) “For every infinite set $X$, $|[X]^{< \omega}| = |X|$” implies 9. In particular, “for every infinite set $X \subset \mathbb{R}$, $|[X]^{< \omega}| = |X|$” implies 13.

(iv) If $|[X]^{< |X|}| = |X|$ then $X$ is well-orderable. In particular, $|[\mathbb{R}]^{< |\mathbb{R}|}| = |\mathbb{R}|$ implies AC($\mathbb{R}$).

(v) “For every uncountable subset $X \subset \mathbb{R}$, $|[X]^{\leq \omega}| = |X|$” iff CH.

(vi) 368 implies 170.
**Proof.** (i) It is known, see [6], that 13 fails in the basic Cohen model, \( \mathcal{M}1 \) in [6], for the set \( A \) of the countably many added Cohen reals. Thus, our statement fails in \( \mathcal{M}1 \).

(ii) Our hypothesis implies CH which in turn implies 34, i.e., \( \aleph_1 \) is regular (CH \( \Rightarrow \) AC(\( \mathbb{R} \)) \( \Rightarrow \) 34). Now let \( A = \{ A_i : i \in \omega \} \) be a family of countable sets. If \( \bigcup A \) is uncountable, then in view of the proof of Theorem 5 (i), \( \bigcup A \) has a subset of size \( \aleph_1 \). Hence, \( \aleph_1 \) is singular, a contradiction. This completes the proof of (ii).

(iii) In the proof of Theorem 5 (i) we have only used \([X]^{<3} = |X|\).

(iv) Mimic the proof of Theorem 5 (i).

(v) (vi) These, in view of Theorem 5, are straightforward.

\[ \square \]

**Theorem 7** (i) “For every infinite set \( X, 2X \) implies \( 9 \). In particular, “for every infinite subset \( A \) of \( \mathbb{R}, 2A \)” implies 13.

(ii) For every infinite set \( X, 2X \) implies \( 9 \).

(iii) AC(\( \mathbb{R} \)) implies “for every infinite set \( X \), if \( |X^{<\omega}| = |X| \) then \( X \) has a well-ordinal subset of size \(|\mathbb{R}|\)”. In particular, in permutation models “\( |X^{<\omega}| = |X| \)” implies “\( X \) has a well-ordinal subset of size \(|\mathbb{R}|\)”.

**Proof.** The proof of this theorem is modeled on the proof of Theorem 5.

(i) Fix an infinite set \( X \) satisfying \( 2X \) and let \( f : X \times X \to X \) be a 1 : 1 and onto function. Let \( x = f((x_1, x_2)), x_1, x_2 \in X, x_1 \neq x_2 \). We shall construct inductively a set \( A = \{ a_i : i \in \aleph_0 \} \subset X \) as follows:

For \( i = 0 \) put \( a_0 = x \).

For \( i = v + 1 \) put \( a_i = f((a_v, a_v)) \). Since \( f \) is 1 : 1 it follows that \( a_i \notin \{ a_j : j \leq v \} \) terminating the induction and the proof of the (i).

(ii) Fix an infinite set \( X \) satisfying \( 2X \) and let \( f : 2 \times X \to X \) be a 1 : 1 and onto function. Via an easy induction we construct a set \( \{ x_i : i \in \omega \} \) as follows:

For \( i = 0 \) let \( x_0 = f((0, a)), a \in X \) is fixed and \( f((0, a)) \neq a \). (If \( f((0, a)) = a \) take \( x_0 = f((1, a)) \). As \( f \) is 1 : 1 \( x_0 \neq f((0, a)) \)).

For \( i = v + 1 \) let \( x_i = f((0, x_v)) \). Since \( f \) is 1 : 1 it is straightforward to verify that \( x_i \neq x_j \) for all \( i, j \in \omega, i \neq j \).

(iii) Fix an infinite set \( X \) satisfying \( |X^{<\omega}| = |X| \) and let \( f : X^{<\omega} \to X \) be a 1 : 1 and onto function. Let \( x = f((x_1, x_2)), x_1, x_2 \in X, x_1 \neq x_2 \). We shall construct inductively a set \( A = \{ a_i : i \in \aleph = |\mathbb{R}| \} \subset X \) as follows:

Let \( \{ h_j : j \in \aleph \} \) be an enumeration of all 1 : 1 functions from \( \omega \) into \( \omega \).

For \( i = 0 \) put \( a_0 = x \).
For \(i = v + 1\) a non-limit ordinal of \(\mathbb{N}\) put \(a_i = f((a_v, a_v))\).

For \(i\) a limit ordinal of \(\mathbb{N}\) put \(a_i = f((a_{h_i(0)}, a_{h_i(1)}, a_{h_i(2)}, \ldots))\). Since \(f\) is 1 : 1 it follows that for every \(i \in \mathbb{N}\), \(a_i \notin \{a_j : j \leq i\}\) finishing the proof of the theorem.

\[\square\]

Let (D) stand for the proposition: “every uncountable subset of \(\mathbb{R}\) has an uncountable well-orderable subset”. Since (D) clearly implies 13 and 368 holds in \(\mathcal{M}1\), see [8], but 13 fails in \(\mathcal{M}1\) we cannot expect that 368 implies (D). We show next that the conjunction of 368 with 212 implies (D).

**Theorem 8** (i) 212+368 implies “every uncountable subset of \(\mathbb{R}\) has an uncountable well-orderable subset”.

(ii) \(AC_\omega(2^{\aleph_0}, \mathbb{R}) + 368\) implies “a well-ordered union of countable subsets of \(\mathbb{R}\) can be well-ordered”.

**Proof.** (i) Fix \(A\) an uncountable subset of \(\mathbb{R}\). Since \([A]^{\leq \omega} \subseteq [\mathbb{R}]^{\leq \omega}\) we have by 212 and 368 that \([A]^{\leq \omega} = |\mathbb{R}|\). (By 212 \(A\) has a countably infinite subset and in ZF \([\omega]^{\omega} = |\mathbb{R}|\). Let, by 212, \(f\) be a choice function of \(A = \{A_x : x \in \mathbb{R}\}\), the family of all cocountable subsets of \(A\). On the basis of \(f\), and following ideas from the proof of Theorem 2, define recursively an uncountable subset \(\{a_i : i \in \aleph_1\}\) of \(A\).

(ii) This can be proved as in Theorem 2. \(\square\)

**Corollary 9** (i) 203 implies “every uncountable subset of \(\mathbb{R}\) has an uncountable well-orderable subset”.

(ii) 203 implies “a well-ordered union of countable subsets of \(\mathbb{R}\) can be well-ordered”.

**Proof.** 203 implies 212+368+\(AC_\omega(2^{\aleph_0}, \mathbb{R})\), see [5] and [8]. \(\square\)

**Theorem 10** (i) \(85 + AC(\omega_1, \mathbb{R})\) implies \([X]^{\leq \omega} \leq |X^\omega \times \omega_1|\).

(ii) \(\omega_1 + AC(\omega_1, \mathbb{R})\) implies 368.

**Proof.** (i) Let \(f\) be a choice function of \([X]^{\leq \omega}\). For every \(Y \in [X]^{\leq \omega}\) define recursively a 1:1 and onto function \(h_Y : \omega_1 \to Y\) as follows:

For \(i = 0\) let \(h_Y(0) = f(Y)\).

For \(i = v + 1\) let \(h_Y(i) = f(Y \setminus \{h_Y(j) : j \leq v\})\).

For \(i = v\) a limit ordinal of \(\omega_1\) let \(h_Y(i) = f(Y \setminus \{h_Y(j) : j \in v\})\).
Since $Y$ is countable it follows that for some $i \in \omega_1$, $\{h_Y(j) : j \in i\} = Y$. Let $i_Y$ be the smallest such ordinal of $\omega_1$. Clearly, $h_Y : i_Y \to Y$ is 1:1 and onto. It follows that the function $F : [X]^{\leq \omega} \to X^{< \omega_1}$, given by $F(Y) = h_Y$ is 1:1 meaning that $|X|^{\leq \omega} \leq |X^{< \omega_1}|$. In order to complete the proof it suffices to show that $|X^{< \omega_1}| = |X^\omega \times \omega_1|$. Clearly $X^{< \omega_1} = \bigcup \{X^v : v \in \omega_1\}$.

In [5] we showed that $AC(\omega_1, \mathbb{R})$, i.e., the axiom of choice for well-ordered families of non-empty subsets of $\mathbb{R}$, implies 170. The proof readily adapts to show that $AC(\omega_1, \mathbb{R})$ implies 170. Let $\mathcal{A} = \{A_v : v \in \omega_1\}$, where $A_v = \{f \in v^\omega : f \text{ is } 1:1\}$. In view of 170 we may consider $\omega_1 \subset \mathbb{R}$ and consequently we may think of $v^\omega$ as being a subset of $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ when considered as a Tychonoff product of $\omega$ copies of $\mathbb{R}$ taken with the usual topology is, in ZF, a separable metric space, we see that $|\mathbb{R}^\omega| = |\mathbb{R}|$. Thus, we can view $\mathcal{A}$ as a family of subsets of $\mathbb{R}$. Let, by $AC(\omega_1, \mathbb{R})$, $h$ be a choice function of $\mathcal{A}$. Let $F : \bigcup \{X^v : v \in \omega_1\} \to X^\omega \times \omega_1$ be the function given by:

$$F(f) = (f \circ h(v), v), v \in \omega_1, f \in X^v.$$  

It can be readily verified that $F$ is 1:1. Hence, $|\bigcup \{X^v : v \in \omega_1\}| \leq |X^\omega \times \omega_1|$ and consequently $|\mathbb{R}|^{\leq \omega} \leq |X^\omega \times \omega_1|$ as required.

(ii) Form part (i) of this theorem we have: $|\mathbb{R}|^{\leq \omega} \leq |\mathbb{R}^\omega \times \omega_1|$. Since, $|\mathbb{R}^\omega| = |\mathbb{R}|$, $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ and, by 170, $\omega_1 \subset \mathbb{R}$, we see that $|\mathbb{R}^\omega \times \omega_1| = |\mathbb{R}|$ and the desired result follows. \hfill \Box

In ZF$^0$ (= ZF minus foundation) 38+368 does not imply 26. Indeed, in the permutation model $\mathcal{N}55$ in [6], see also [9], 38+368 is true but the countable family $\mathcal{A}$ of circles has no choice. Furthermore, each circle has size $\mathbb{R}$. Thus, $\bigcup \mathcal{A}$ cannot have size $\mathbb{R}$ for otherwise $\mathcal{A}$ would have to have a choice set. We show next that 26 implies 38.

**Theorem 11** (i) 368 implies “the power set of every set $Y \subset \mathbb{R}$ which is expressible as a disjoint union of countable sets $Y = \bigcup \{A_i : i \in k\}$ has size $\leq |\mathbb{R}|^{\leq \omega}$”. In particular, 368 implies that “$\mathbb{R}$ cannot be expressed as a countable union of countable sets”.

(ii) 26 implies that “$\mathbb{R}$ cannot be expressed as a countable union of countable sets”.

**Proof.** (i) Fix $\mathcal{A} = \{A_n : n \in k\}$ a disjoint family of countable subsets of $\mathbb{R}$. Let, by 368, $f : [\mathbb{R}]^\omega \to \mathbb{R}^k$ be a 1:1 function. It can be readily verified that the function $H : \varphi(Y) \to \mathbb{R}^k$, $Y = \bigcup \mathcal{A}$ given by:

$$H(X)(i) = f(X \cap A_i)$$
is a 1:1 function. Thus, $|\omega(\gamma)| \leq |\mathbb{R}^x|$ as required.

The second assertion follows from the observation: $|\mathbb{R}^x| = |\mathbb{R}|$.

(ii) Assume the contrary and fix a disjoint family of countable sets $\mathcal{A} = \{A_i : i \in \omega\}$ covering $\mathbb{R}$. Clearly, for every $n \in \omega$, $|[A_n]^\omega| = |\mathbb{R}|$. Hence, by 26, we have $|\bigcup \{[A_n]^\omega : n \in \omega\}| = |\mathbb{R}|$. Thus, we may view each $[A_n]^\omega$ as a copy of $\mathbb{R}$, hence, $|\prod_{n \in \omega}[A_n]^\omega| = |\mathbb{R}|$. Continue now as in (i) to show that $|\mathbb{R}^x| \leq |\prod_{n \in \omega}[A_n]^\omega|$. By (i) this contradicts our assumption and completes the proof of the theorem. $\square$

In [6] the validity of the following implications

$170 \iff 56$, $56 \implies 170$, $38 \iff 56$, $56 \iff 38$ is indicated as unknown. We point out next that none of the above implications is true in ZF.

**Proposition 12** (i) $170 \not\iff 56$ and $56 \not\iff 170$. In particular, $\neg 56 \implies 170$.
(ii) $38 \not\iff 56$ and $56 \not\iff 38$.

**Proof.** Derrick and Drake [3] have constructed a forcing model, $\mathcal{M}10$ in [6], such that $(\forall n \in \omega)(\aleph_n \leq 2^{\aleph_0} \land \aleph_\omega \leq 2^{\aleph_0})$. Thus, in $\mathcal{M}10$, 170 is true and $H(2^{\aleph_0}) = \aleph_1$, meaning that 56 fails in this model. Furthermore, Derrick and Drake use a c.c. partially ordered set of forcing conditions to construct the model $\mathcal{M}10$. Therefore, cardinals are preserved and $\aleph_1$ is a regular cardinal in $\mathcal{M}10$. Since 170 is true in this model, it follows that $\mathbb{R}$ can not be expressed as a countable union of countable sets. Thus, form 38 is true of $\mathcal{M}10$.

On the other hand in the Feferman/Levy model, $\mathcal{M}9$ in [6], $\mathbb{R}$ is written as a countable union of countable sets, thus form 38 fails in $\mathcal{M}9$, and form 170 fails too, since the only well-orderable subsets of $\mathbb{R}$ are the countable sets, see Cohen [2], p. 146]. Therefore, in $\mathcal{M}9$, $H(2^{\aleph_0}) = \aleph_1 \neq \aleph_\omega$ and consequently 56 is true in that model. $\square$

**Theorem 13** (i) The negation of 169 implies $CC_\omega(\mathbb{R})$.
(ii) The negation of 169 implies $CUC(\mathbb{R})$.
(iii) $\omega$-$CC(\mathbb{R})$+169 does not imply $26+368$.
(iv) $\omega$-$CC(\mathbb{R})$+$\neg 169$ implies $CC(\mathbb{R})$.

**Proof.** Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of countable subsets of the real line $\mathbb{R}$.

(i) Without loss of generality we may assume that for every $i \in \omega$ $A_i \subset (i, i+1)$. It suffices to show that $\mathcal{A}$ has a partial choice function. If $\mathcal{A}$ has no
partial choice, then \( A = \cup A \) is uncountable. Thus, \( A \) has a perfect subset \( K \). Then \( |K| = |\mathbb{R}| \) and consequently there are infinitely many members of \( A \) meeting \( K \) non-trivially. Since for every \( i \in \omega \), \( K \cap A_i \subset A_i \) and the family of all closed subsets of \( \mathbb{R} \) has a choice function, see [9] and [12], it follows that \( A \) has a partial choice which is a contradiction.

(ii) If \( \cup A \) is uncountable then \( \neg \text{169} \) implies that \( \cup A \) has the power of the continuum. Thus, by (i) and the fact that 5 implies 38, see [5], we have that \( \cup A \) is countable.

(iii) In view of Theorem 11, 368 fails in \( \mathcal{M}9 \). On the other hand, it is known (see [4]) that \( \omega \)-CC(\( \mathbb{R} \)) holds in \( \mathcal{M}9 \). Moreover, 169 holds in \( \mathcal{M}9 \), as otherwise, by (i) we would have that 5 and consequently 38 would also hold in that model which is a contradiction. Finally, 26 fails in \( \mathcal{M}9 \) since by Theorem 11, 26 implies 38. \( \square \)

3 Summary

The proofs of the positive and independence results of this paper formulate the table of the introduction as follows:
References


[10] K. Keremedis and E. Tachtsis, Sequential compactness for subsets of $\mathbb{R}$ is countably productive in ZF, submitted manuscript.


