On Weierstrass compact pseudometric spaces and a weak form of the axiom of choice

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Abstract

We show that the countable multiple choice axiom CMC is equivalent to the assertion: Weierstrass compact pseudometric spaces are compact.

The countable axiom of choice CAC (Form 8 in [3]) is the assertion: For every set \( \mathcal{A} = \{ A_i : i \in \omega \} \) of non empty disjoint sets there exists a set \( C \) consisting of one and only one element from each element of \( \mathcal{A} \). The countable multiple axiom of choice, CMC, is the proposition:

Every set \( \mathcal{A} = \{ A_i : i \in \omega \} \) of non empty disjoint sets has a countable multiple choice. i.e. a family \( \mathcal{F} = \{ F_i : i \in \omega \} \) of finite non empty sets such that for every \( i \in \omega \), \( F_i \subseteq A_i \).

Let \((X, T)\) be a topological space. \( X \) is said to be countably compact iff every countable open cover \( \mathcal{U} \) of \( X \) has a finite subcover \( \mathcal{V} \). \( X \) is said to be Weierstrass compact iff every infinite set \( Y \subseteq X \) has a limit point \( a \). i.e., every open set \( O \) including \( a \) meets \( Y \) in an infinite set. WCC stands for the statement:

\[ \text{WCC} = \text{Weierstrass-compact pseudometric spaces are compact.} \]
In [1] it is asked what is the set theoretical status of WCC. The aim of this note is to show that WCC is equivalent to CMC.

**Lemma 1** CMC iff WCCC (Weierstrass-compact pseudometric spaces are countably compact).

**Proof.** (→) Assume on the contrary and let \((X, d)\) be a Weierstrass-compact but not a countably compact pseudometric space. Fix \(U = \{O_n : n \in \omega\}\) an open cover of \(X\) without a finite subcover. Clearly,

\[ Q = \{Q_n = \cup\{O_m : m \leq n\} : n \in \omega\} \]

is an ascending open cover of \(X\) such that no \(Q_n\) covers \(X\). Use CMC to pick a sequence of non empty finite sets \(\mathcal{F} = \{F_n \subset X \setminus Q_n : n \in \omega\}\). Let \(x \in O_n\) be a limit point of \(\cup \mathcal{F}\). Then \(O_n\) must contain infinitely many points of \(\cup \mathcal{F}\) contradicting the choice of \(F_n\)'s. Hence \(U\) has a finite subcover and \(X\) is countably compact as required.

(←). Assume on the contrary that WCCC holds but CMC fails.

**Claim 1.** ([2] and [4]). CMC iff for every countable family \(\mathcal{A}\) of disjoint non empty sets there exists an infinite set \(C \subset \cup \mathcal{A}\) such that for every \(A \in \mathcal{A}\), \(0 \leq |C \cap A| < \omega\).

**Proof of Claim 1.** (→). This is straightforward.

(←). Fix \(\mathcal{A} = \{A_n : n \in \omega\}\) a family of disjoint non empty sets. Put

\[ B = \{B_n = \prod_{m \leq n} A_m : n \in \omega\} \]

and let \(C = \{c_i : i \in \omega\}\) satisfy the conclusion of CMC for the infinite subfamily \(\{B_{n_i} : i \in \omega\}\) of \(B\). Based on \(C\) and taking projections we can easily construct inductively a set \(\mathcal{F} = \{F_n : n \in \omega\}\) satisfying CMC for \(\mathcal{A}\). □ (Claim 1)

Fix, by Claim 1, a countable family \(\mathcal{A} = \{A_i : i \in \omega\}\) of disjoint non empty sets having no infinite subfamily with a countable multiple choice set. Make \(X_i = A_i \cup \{i\}, i \notin A_i\) into a pseudometric space by requiring:

\[
    d_i(x, y) = \begin{cases} 
        1 & \text{if } ((x = i) \land (y \neq x)) \lor ((y = i) \land (y \neq x)) \\
        0 & \text{otherwise}
    \end{cases}.
\]

2
Let $X$ be the product of the family $\{(X_i, T_i) : i \in \omega\}$ endowed with the Tychonoff topology $T$, where $T_i$ is the topology induced by the pseudometric $d_i$. It can be readily verified that $T$ is the topology induced by the pseudometric $d$ given by $d(x, y) = \sum_{n \in \omega} \frac{1}{2^n} d_n(x(n), y(n))$.

**Claim 2.** $X$ is Weierstrass-compact.

**Proof of Claim 2.** Let $G$ be an infinite subset of $X$. As $\mathcal{A}$ has no partial multiple choice set, it follows that for every $g \in G$ there exists $n_g \in \omega$ $g(i) = i$ for all $i \geq n_g$. Put $G_n = \{g \in G : g(i) = i \text{ for all } i \geq n\}$. Clearly, for all but finitely many $n$’s $G_n$ is infinite (otherwise $\mathcal{A}$ would have a partial multiple choice set). Fix $n \in \omega$ such that $G_n$ is infinite. Since $X_1 \times X_2 \times, ..., X_{n-1}$ is a compact space homeomorphic to the subspace

$$X^n = \{x \in X : x(i) = i \text{ for all } i \geq n\}$$

and $G_n \subseteq X^n$, it follows by the compactness of $X^n$, that $G_n$ has a cluster point $x \in X^n$. It is easy to see that $x$ is a cluster point of $G_n$ in $X$. (Claim 2)

By Claim 2, $X$ is Weierstrass compact. Hence $X$ is countably compact. Thus $K = \{\pi_i^{-1}[A_i] : i \in \omega\}$ being a countable family of closed sets with the finite intersection property, satisfies $\cap K \neq \emptyset$ meaning that $\mathcal{A}$ has a choice function which is a contradiction. Hence CMC holds finishing the proof of Lemma 1.

**Lemma 2** CMC iff every compact pseudometric space $(X, d)$ has a dense subspace $Y$ which is written as a countable union of finite sets.

**Proof.** $(\rightarrow)$. Fix $(X, d)$ a compact pseudometric space. Using the compactness of $X$ one can show, as usual, that for every $n > 0$,

$$X_n = \{Y \in [X]^{<\omega} : d(x, Y) < 1/n \text{ for all } x \in X\} \neq \emptyset.$$ 

Put $Z = \{X_n : n \in \omega\}$ and let $F = \{F_n : n \in \omega\}$ be a countable multiple choice for $Z$. i.e., $\emptyset \neq F_n \in [X_n]^{<\omega}$. Clearly $Y = \cup\{P_n = \cup F_n : n \in \omega\}$ is a dense subset of $X$ which is expressed as a countable union of finite sets.

$(\leftarrow)$. Fix $\{X_n : n \in \omega\}$ a family of disjoint non empty sets. Put $X = (\cup\{X_n : n \in \omega\}) \cup \{\infty\}, \infty \notin \cup\{X_n : n \in \omega\}$ and let $d : X \times X \to \mathbb{R}$ be given by $d(x, y) = d(y, x)$ and

$$d(x, y) = \begin{cases} 
0 & \text{if } x, y \in X_n \text{ for some } n \in \omega \\
1/(n + 1) & \text{if } x = \infty \text{ and } y \in X_n \text{ for some } n \in \omega \\
\max\{1/n, 1/m\} & \text{if } x \in X_m, y \in X_n, m \neq n 
\end{cases}.$$
It can be readily verified that \((X, d)\) is a compact pseudometric space (every open set containing \(\infty\) includes all but finitely many \(X_n\)'s). Let \(Y = \cup \{Y_m : m \in \omega\}\) be the dense set which is guaranteed by the hypothesis. For every \(n \in \omega\) let \(F_n = Y_{m_n} \cap X_n\) where \(m_n\) is the first \(v \in \omega\) such that \(Y_v \cap X_n \neq \emptyset\). Clearly \(\{F_n : n \in \omega\}\) is a multiple choice set for the family \(\{X_n : n \in \omega\}\) finishing the proof of Lemma 2.

By the same proof of Lemma 2 we can show Corollary 3.

**Corollary 3** (i) \(CMC\) iff every compact pseudometric space \((X, d)\) has a dense subspace \(Y\) which is written as a well ordered union of finite sets.
(ii) \(CAC\) iff every compact pseudometric space \((X, d)\) is separable.

**Theorem 4** \(CMC\) iff \(WCC\).

**Proof.** (\(\rightarrow\)). Since \(WCC\) clearly implies \(WCCC\) which in turn is equivalent to \(CMC\), we see that \(WCC\) implies \(CMC\).

(\(\leftarrow\)). It suffices to show that \(WCCC\) implies \(WCC\). Fix \((X, d)\) a Weierstrass compact countably compact pseudometric space.

**Claim 1.** If \(X\) has a dense subset which can be written as a countable union of finite sets then \(X\) is compact.

**Proof of Claim 1.** Let \(Y = \cup \{Y_n \in [X]^{<\omega} : n \in \omega\}\) be the dense subset of \(X\). For every \(n \in \omega\) let \(B_n = \{D(y, 1/m) : y \in Y_n, m > 0\}\). Clearly each \(B_n\) is countable and \(B = \cup \{B_n : n \in \omega\}\) is a base for \(X\). In order to show that \(X\) is compact it suffices to show that every open cover \(U = \{U_i : i \in I\} \subseteq B\) has a finite subcover. For every \(n \in \omega\) let \(Q_n = \cup G_n, G_n = U \cap B_n\). Clearly each \(G_n\) is countable and \(Q = \{Q_n : n \in \omega\}\) is a countable open cover of \(X\). It follows that \(Q\) has a finite subcover say, \(Q_1, Q_2, ..., Q_v\). Since \(G = G_1 \cup G_2 \cup ..., \cup G_v\) is again a countable cover of \(X\) it follows that it has a finite subcover \(V\). Clearly \(V\) is a finite subcover of \(U\) and \(X\) is compact as required.

**Claim 2.** \(X\) is totally bounded.

**Proof of Claim 2.** Assume the contrary. Then there exists \(r > 0\) such that for every \(n > 0\), \(A_n = \{Y \in [X]^n : (\forall x, y \in Y)(x \neq y \rightarrow d(x, y) \geq r)\} \neq \emptyset\). Put \(A = \{A_n : n \in \omega\}\) and let \(F = \{F_n : n \in \omega\}\) be a multiple choice set for
Claim 3. If $X$ is totally bounded then $X$ is compact.

Proof of Claim 3. For every $n \in \omega$ put $P_n = \{Y \in [X]^{<\omega} : d(x, Y) < 1/n$ for all $x \in X\}$. As $X$ is totally bounded it follows that $P_n \neq \emptyset$. Set $P = \{P_n : n \in \omega\}$ and let $F = \{F_n : n \in \omega\}$ be a countable multiple choice for $P$. Clearly $Y = \bigcup (\cup F)$ is a dense subset of $X$ that is expressed as a countable union of finite sets. Thus, an application of Claim 1 shows that $X$ is compact finishing the proof of Claim 3 and Theorem 4.

References


