# On Weierstrass compact pseudometric spaces and a weak form of the axiom of choice

Kyriakos Keremedis
Department of Mathematics, University of the Aegean, Karlovasi, 83200, GREECE.
FAX : 0273 - 33896
e-mail : kker@aegean.gr

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**Abstract**

We show that the countable multiple choice axiom CMC is equivalent to the assertion: Weierstrass compact pseudometric spaces are compact.

The *countable axiom of choice* CAC (Form 8 in [3]) is the assertion: *For every set* $\mathcal{A} = \{A_i : i \in \omega\}$ *of non empty disjoint sets there exists a set* $C$ *consisting of one and only one element from each element of* $\mathcal{A}$. The *countable multiple axiom of choice*, CMC, is the proposition:

> *Every set* $\mathcal{A} = \{A_i : i \in \omega\}$ *of non empty disjoint sets has a countable multiple choice. i.e. a family* $\mathcal{F} = \{F_i : i \in \omega\}$ *of finite non empty sets such that for every* $i \in \omega$, $F_i \subseteq A_i$.

Let $(X, T)$ be a topological space. $X$ is said to be *countably compact* iff every countable open cover $\mathcal{U}$ of $X$ has a finite subcover $\mathcal{V}$. $X$ is said to be *Weierstrass compact* iff every infinite set $Y \subset X$ has a limit point $a$. i.e., every open set $O$ including $a$ meets $Y$ in an infinite set. WCC stands for the statement:

WCC= *Weierstrass-compact pseudometric spaces are compact.*

In [1] it is asked what is the set theoretical status of WCC. The aim of this note is to show that WCC is equivalent to CMC.

**Lemma 1** *CMC iff WCCC (Weierstrass-compact pseudometric spaces are countably compact).*

**Proof**. ($\rightarrow$) Assume on the contrary and let $(X, d)$ be a Weierstrass-compact but not a countably compact pseudometric space. Fix $\mathcal{U} = \{O_n : n \in \omega\}$ an open cover of $X$ without a finite subcover. Clearly,

$$Q = \{Q_n = \cup\{O_m : m \leq n\} : n \in \omega\}$$

is an ascending open cover of $X$ such that no $Q_n$ covers $X$. Use CMC to pick a sequence of non empty finite sets $\mathcal{F} = \{F_n \subset X \backslash Q_n : n \in \omega\}$. Let $x \in O_n$ be a limit point of $\cup\mathcal{F}$. Then $O_n$ must contain infinitely many points of $\cup\mathcal{F}$ contradicting the choice of $F_n$'s. Hence $\mathcal{U}$ has a finite subcover and $X$ is countably compact as required.

($\leftarrow$). Assume on the contrary that WCCC holds but CMC fails.

**Claim 1**. ([2] and [4]). CMC iff *for every countable family $\mathcal{A}$ of disjoint non empty sets there exists an infinite set $C \subset \cup\mathcal{A}$ such that for every $A \in \mathcal{A}$,* $0 \leq |C \cap A| < \omega$.

**Proof of Claim 1**. ($\rightarrow$). This is straightforward.

($\leftarrow$). Fix $\mathcal{A} = \{A_n : n \in \omega\}$ a family of disjoint non empty sets. Put

$$\mathcal{B} = \{B_n = \prod_{m \leq n} A_m : n \in \omega\}$$

and let $C = \{c_{n_i} : i \in \omega\}$ satisfy the conclusion of CMC for the infinite subfamily $\{B_{n_i} : i \in \omega\}$ of $\mathcal{B}$. Based on $C$ and taking projections we can easily construct inductively a set $\mathcal{F} = \{F_n : n \in \omega\}$ satisfying CMC for $\mathcal{A}$. ▮ (Claim 1)

Fix, by Claim 1, a countable family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint non empty sets having no infinite subfamily with a countable multiple choice set. Make $X_i = A_i \cup \{i\}, i \notin A_i$ into a pseudometric space by requiring:

$$d_i(x, y) = \begin{cases} 1 \text{ if } [((x = i) \wedge (y \neq x)) \vee ((y = i) \wedge (y \neq x))] \\ 0 \text{ otherwise} \end{cases}.$$

Let $X$ be the product of the family $\{(X_i, T_i) : i \in \omega\}$ endowed with the Tychonoff topology $T$, where $T_i$ is the topology induced by the pseudometric $d_i$. It can be readily verified that $T$ is the topology induced by the pseudometric $d$ given by $d(x, y) = \sum_{n\in\omega} \frac{1}{2^n} d_n(x(n), y(n))$.

**Claim 2**. $X$ is Weierstrass-compact.

**Proof of Claim 2**. Let $G$ be an infinite subset of $X$. As $\mathcal{A}$ has no partial multiple choice set, it follows that for every $g \in G$ there exists $n_g \in \omega$ $g(i) = i$ for all $i \geq n_g$. Put $G_n = \{g \in G : g(i) = i \text{ for all } i \geq n\}$. Clearly, for all but finitely many $n$'s $G_n$ is infinite (otherwise $\mathcal{A}$ would have a partial multiple choice set). Fix $n \in \omega$ such that $G_n$ is infinite. Since $X_1 \times X_2 \times, ..., \times X_{n-1}$ is a compact space homeomorphic to the subspace

$$X^n = \{x \in X : x(i) = i \ \textit{for all} \ i \geq n\}$$

and $G_n \subseteq X^n$, it follows by the compactness of $X^n$, that $G_n$ has a cluster point $x \in X^n$. It is easy to see that $x$ is a cluster point of $G_n$ in $X$. ∎ (Claim 2)

By Claim 2, $X$ is Weierstrass compact. Hence $X$ is countably compact. Thus $K = \{\pi_i^{-1}[A_i] : i \in \omega\}$ being a countable family of closed sets with the finite intersection property, satisfies $\cap K \neq \emptyset$ meaning that $\mathcal{A}$ has a choice function which is a contradiction. Hence CMC holds finishing the proof of Lemma 1. ∎

**Lemma 2** *CMC iff every compact pseudometric space $(X, d)$ has a dense subspace $Y$ which is written as a countable union of finite sets.*

**Proof**. ($\rightarrow$). Fix $(X, d)$ a compact pseudometric space. Using the compactness of $X$ one can show, as usual, that for every $n > 0$,

$X_n = \{Y \in [X]^{<\omega} : d(x, Y) < 1/n \text{ for all } x \in X\} \neq \emptyset$. Put

$Z = \{X_n : n \in \omega\}$ and let $F = \{F_n : n \in \omega\}$ be a countable multiple choice for $Z$. i.e., $\emptyset \neq F_n \in [X_n]^{<\omega}$. Clearly $Y = \cup\{P_n = \cup F_n : n \in \omega\}$ is a dense subset of $X$ which is expressed as a countable union of finite sets.

($\leftarrow$). Fix $\{X_n : n \in \omega\}$ a family of disjoint non empty sets. Put $X = (\cup\{X_n : n \in \omega\}) \cup \{\infty\}, \infty \notin \cup\{X_n : n \in \omega\}$ and let $d : X \times X \rightarrow \mathbb{R}$ be given by $d(x, y) = d(y, x)$ and

$$d(x,y) = \begin{cases} 0 \text{ if } x, y \in X_n \text{ for some } n \in \omega \\ 1/(n+1) \text{ if } x = \infty \text{ and } y \in X_n \text{ for some } n \in \omega \\ \max\{1/n, 1/m\} \text{ if } x \in X_m, y \in X_n,\ m \neq n \end{cases}.$$

It can be readily verified that $(X, d)$ is a compact pseudometric space (every open set containing $\infty$ includes all but finitely many $X_n$'s). Let $Y = \cup\{Y_m : m \in \omega\}$ be the dense set which is guaranteed by the hypothesis. For every $n \in \omega$ let $F_n = Y_{m_n} \cap X_n$ where $m_n$ is the first $v \in \omega$ such that $Y_v \cap X_n \neq \emptyset$. Clearly $\{F_n : n \in \omega\}$ is a multiple choice set for the family $\{X_n : n \in \omega\}$ finishing the proof of Lemma 2. ∎

By the same proof of Lemma 2 we can show Corollary 3.

**Corollary 3** *(i) CMC iff every compact pseudometric space $(X, d)$ has a dense subspace $Y$ which is written as a well ordered union of finite sets.*
*(ii) CAC iff every compact pseudometric space $(X, d)$ is separable.*

**Theorem 4** *CMC iff WCC.*

**Proof**. ($\leftarrow$). Since WCC clearly implies WCCC which in turn is equivalent to CMC, we see that WCC implies CMC.

($\rightarrow$). It suffices to show that WCCC implies WCC. Fix $(X, d)$ a Weierstrass compact countably compact pseudometric space.

**Claim 1**. If $X$ has a dense subset which can be written as a countable union of finite sets then $X$ is compact.

**Proof of Claim 1.** Let $Y = \cup\{Y_n \in [X]^{<\omega} : n \in \omega\}$ be the dense subset of $X$. For every $n \in \omega$ let $B_n = \{D(y, 1/m) : y \in Y_n, m > 0\}$. Clearly each $B_n$ is countable and $\mathcal{B} = \cup\{B_n : n \in \omega\}$ is a base for $X$. In order to show that $X$ is compact it suffices to show that every open cover $U = \{U_i : i \in I\} \subseteq \mathcal{B}$ has a finite subcover. For every $n \in \omega$ let $Q_n = \cup G_n, G_n = U \cap B_n$. Clearly each $G_n$ is countable and $Q = \{Q_n : n \in \omega\}$ is a countable open cover of $X$. It follows that $Q$ has a finite subcover say, $Q_1, Q_2, ..., Q_v$. Since $G = G_1 \cup G_2 \cup, ..., \cup G_v$ is again a countable cover of $X$ it follows that it has a finite subcover $\mathcal{V}$. Clearly $\mathcal{V}$ is a finite subcover of $U$ and $X$ is compact as required.

**Claim 2**. $X$ is totally bounded.

**Proof of Claim 2**. Assume the contrary. Then there exists $r > 0$ such that for every $n > 0$, $A_n = \{Y \in [X]^n : (\forall x, y \in Y)(x \neq y \rightarrow d(x, y) \geq r)\} \neq \emptyset$. Put $A = \{A_n : n \in \omega\}$ and let $F = \{F_n : n \in \omega\}$ be a multiple choice set for

$A$. Put $Z = \overline{\cup\{W_n = \cup F_n : n \in \omega\}}$. Arguing as in Claim 1 one can easily see that $Z$ is a compact space. Thus the open cover $U$ by balls of radius $r/2$ has a finite subcover. Assume that $D(x_1, r/2), D(x_2, r/2), ..., D(x_v, r/2)$ covers $Z$. It is easy to see that there exists $j \leq v$ and $x, y \in D(x_j, r/2)$ such that $d(x, y) \geq r$. This contradiction establishes Claim 2.

**Claim 3**. If $X$ is totally bounded then $X$ is compact.

**Proof of Claim 3**. For every $n \in \omega$ put $P_n = \{Y \in [X]^{<\omega} : d(x, Y) < 1/n$ for all $x \in X\}$. As $X$ is totally bounded it follows that $P_n \neq \emptyset$. Set $P = \{P_n : n \in \omega\}$ and let $F = \{F_n : n \in \omega\}$ be a countable multiple choice for $P$. Clearly $Y = \cup(\cup F)$ is a dense subset of $X$ that is expressed as a countable union of finite sets. Thus, an application of Claim 1 shows that $X$ is compact finishing the proof of Claim 3 and Theorem 4. ∎

# References


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