DE GRUYTER Forum Math. 2015; aop

Research Article

Vassilis Metaftsis* and Stratos Prassidis

On the K-theory of certain extensions of free groups

Abstract: Since $\mathrm{Hol}(F_n)$ embeds into $\mathrm{Aut}(F_{n+1})$, one can construct inductively the subgroups $\mathcal{H}_{(n)}$ of $\mathrm{Aut}(F_{n+1})$ by setting $\mathcal{H}_{(1)} = \mathrm{Hol}(F_2)$ and $\mathcal{H}_{(n)} = F_{n+1} \rtimes \mathcal{H}_{(n-1)}$. We show that the FJCw holds for $\mathcal{H}_{(n)}$. Moreover, we calculate the lower K-theory for the groups $\mathcal{H}_{(n)}$.

Keywords: Isomorphism conjecture, K-Theory, L-Theory, free group automorphisms, holomorph

MSC 2010: 19D35, 20F65

DOI: 10.1515/forum-2014-0214
Received December 11, 2014; revised July 4, 2015

Communicated by: Andrew Ranicki

1 Introduction

The Fibered Farrell–Jones Conjecture (FJC) is the main conjecture in geometric topology. It is used for the calculation of the obstruction groups that appear in geometric rigidity and in classification problems. In this paper, we are interested in the K- and L-theory FJC and its variation, the Fibered Farrell–Jones Conjecture with finite wreath products (FJCw). If we do not specify the prefix, by FJC we will mean either of the two versions of the conjecture. The FJCw has been proved for an extensive list of classes of groups. One notable case which remains open is the group $\mathrm{Aut}(F_n)$, the automorphism group of the free group on n letters. In [14], the K-FJC is proved for $n = 2$. Actually, in [14], the K-FJC is proved for $\mathrm{Hol}(F_2)$, the holomorph of F_2. We notice that the extension to K-FJCw is a direct computation. In this paper, we extend the result to certain subgroups of $\mathrm{Aut}(F_n)$ that are constructed from $\mathrm{Hol}(F_2)$. More precisely, there is a monomorphism $\mathrm{Hol}(F_n) \to \mathrm{Aut}(F_{n+1})$. We construct a sequence of groups with

$$\mathcal{H}_{(0)} = F_2, \quad \mathcal{H}_{(1)} = \mathrm{Hol}(F_2), \quad \mathcal{H}_{(n)} = F_{n+1} \rtimes \mathcal{H}_{(n-1)}, \; n \geq 2.$$

Notice that $\mathcal{H}_{(n)} < \mathrm{Hol}(F_{n+1})$.

The main result of the paper is the following theorem.

Theorem (Main Theorem). *The FJCw holds for the groups $\mathcal{H}_{(n)}$.*

As an application of the Main Theorem, we calculate the lower K-theory groups of $\mathcal{H}_{(n)}$ as follows.
(i) $K_i(\mathbb{Z}\mathcal{H}_{(n)}) = 0$, $i \leq -1$.
(ii) $\tilde{K}_0(\mathbb{Z}\mathcal{H}_{(n)}) \cong NK_0(\mathbb{Z}D_4) \oplus NK_0(\mathbb{Z}D_4)$.
(iii) $Wh(\mathcal{H}_{(n)}) \cong NK_1(\mathbb{Z}D_4) \oplus NK_1(\mathbb{Z}D_4)$.

The main point of the general FJCw is that the K- and L-groups of a group (and its finite wreath products) can be computed from the K- and L-theory of their virtually cyclic subgroups.

The FJCw satisfies the following properties (see [9, 18]).
(i) If the FJCw holds for a group G, then it holds for all the subgroups of G.

***Corresponding author: Vassilis Metaftsis:** Department of Mathematics, University of the Aegean, 83200 Karlovassi, Samos, Greece, e-mail: vmet@aegean.gr
Stratos Prassidis: Department of Mathematics, University of the Aegean, 83200 Karlovassi, Samos, Greece, e-mail: prasside@aegean.gr

(ii) If the FJCw holds for a group G contained in Γ as a subgroup of finite index, then the FJCw holds for Γ.
(iii) Let
$$1 \to H \xrightarrow{f} G \xrightarrow{g} K \to 1$$
be an exact sequence of groups. We assume that
(a) the FJCw holds for H and K;
(b) the FJCw holds for $g^{-1}(C)$, where C is any infinite cyclic subgroup of K.
Then, the FJCw holds for G.

For the proof of the Main Theorem, we use induction and properties (ii) and (iii) of the FJCw. We show that the groups that are the inverse images of infinite cyclic groups are either hyperbolic groups or CAT(0)-groups for which the FJCw holds.

In [14], it was shown that the finite subgroups of $\mathrm{Hol}(F_2)$ are isomorphic to one of the groups
$$\mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z}, \quad D_2, \quad D_4.$$

For the calculations of the lower K-groups we need to look at two types of subgroups: finite groups and groups that admit an epimorphism to $\mathbb{Z}$ with finite kernel. The third type of virtually cyclic groups (those that admit an epimorphism to the infinite dihedral group with finite kernel) is not needed in the calculation (see [7]).

The only subgroups of the second type are isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. This implies that the finite subgroups of $\mathcal{H}_{(n)}$ are isomorphic to one from the above list. Also, we show that the subgroups of the second type are isomorphic to products of finite groups times $\mathbb{Z}$. In other words, semi-direct products do not appear.

Moreover, we notice that the part of K-theory of $\mathcal{H}_{(n)}$ which is detected from the finite groups vanishes. The result follows from the calculation of the cokernel of the map from the K-theory detected from the finite subgroups to the total K-theory. For this, we use [1].

2 Preliminaries and notation

For a group G let $\mathrm{Aut}(G)$ be the group of automorphisms of G. The holomorph of G is the semi-direct product $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$ defined by the natural action of $\mathrm{Aut}(G)$ on G. Thus, we have the universal split extension
$$1 \to G \to \mathrm{Hol}(G) \to \mathrm{Aut}(G) \to 1$$
determined by G. In general, there is an embedding $E : \mathrm{Hol}(G) \to \mathrm{Aut}(G * \mathbb{Z})$ given by the following: for $g \in G$ by
$$E(g)(x) = \begin{cases} x, & x \in G, \\ gxg^{-1}, & x \in \mathbb{Z}, \end{cases}$$
and for $\alpha \in \mathrm{Aut}\, G$ by
$$E(\alpha)(x) = \begin{cases} \alpha(x), & x \in G, \\ x, & x \in \mathbb{Z}. \end{cases}$$
Thus, we can define the split group extension $(G * \mathbb{Z}) \rtimes (\mathrm{Hol}(G)) < \mathrm{Hol}(G * \mathbb{Z}) < \mathrm{Aut}((G * \mathbb{Z}) * \mathbb{Z})$.

Let F_{n-1} be the free group in $n-1$ generators. Inductively, we define $\mathcal{H}_{(i)}(G)$ to be
$$\mathcal{H}_{(0)}(G) = G, \quad \mathcal{H}_{(1)}(G) = \mathrm{Hol}(G), \quad \mathcal{H}_{(n)}(G) = (G * F_{n-1}) \rtimes (\mathcal{H}_{(n-1)}(G)), \ n \geq 2,$$
where $\mathcal{H}_{(n-1)}(G)$ is considered as a subgroup of $\mathrm{Aut}(G * F_{n-1})$ by the embedding given by repeatedly applying E. We write $\mathcal{H}_{(n)} = \mathcal{H}_{(n)}(F_2)$ for the group corresponding to F_2. Then, there is a split exact sequence
$$1 \to F_{n+1} \to \mathcal{H}_{(n)} \to E(\mathcal{H}_{(n-1)}) \to 1.$$

We are interested in the Fibered Farrell–Jones Conjecture (FJC) for the groups $\mathcal{H}_{(n)}$. We will review the general constructions. Let G be a group and let $\mathcal{C}$ be a class of subgroups. Then, $E_{\mathcal{C}}G$ denotes the classifying space of the class $\mathcal{C}$. We are interested in the following classes of subgroups of G.

- 1, the class of the trivial subgroup.
- $\mathcal{F}$, the class of finite subgroups.
- $\mathcal{FBC}$, the class of finite-by-cyclic subgroups.
- $\mathcal{VC}$, the class of virtually cyclic subgroups.
- $\mathcal{A}ll$, the class of all subgroups.

It is obvious that $1 \subset \mathcal{F} \subset \mathcal{FBC} \subset \mathcal{VC} \subset \mathcal{A}ll$. Instead of the classical theoretic FJC, we will consider the Isomorphism Conjecture with coefficients in an additive category $\mathcal{A}$ (with involution in the L-theory case). It is known that this implies also the Fibered Isomorphism Conjecture (see [4]).

(i) The K-FJC states that the assembly map

$$H_n^G(E_{\mathcal{VC}}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(E_{\mathcal{A}ll}G; \mathbf{K}_{\mathcal{A}}) = H_n^G(pt; \mathbf{K}_{\mathcal{A}})$$

is an isomorphism.

(ii) The L-FJC states that the assembly map

$$H_n^G(E_{\mathcal{VC}}G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(E_{\mathcal{A}ll}G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = H_n^G(pt; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})$$

is an isomorphism.

If a group satisfies this conjecture, we say that the group satisfies the FJC. We say that a group G satisfies the FJCw if the wreath product $G \wr H$ satisfies the FJC for each finite group H and with coefficients.

We need the following basic facts.

Remark 2.1. (i) Word hyperbolic groups satisfy the FJCw (see [2, 3]).
(ii) CAT(0)-groups satisfy the FJCw (see [18]).
(iii) Strongly poly-free groups or, more generally, weak strongly poly-surface groups satisfy the K-FJCw (see [16]).

Now, we recall some results that are relevant to the K-FJCw. In [1], it was shown that for a ring R the relative map

$$H_n^G(E_{\mathcal{F}}G; \mathbf{K}R^{-\infty}) \to H_n^G(E_{\mathcal{VC}}G; \mathbf{K}R^{-\infty})$$

is a split injection. Also, in [7], it was shown that the natural map

$$H_n^G(E_{\mathcal{FBC}}G; \mathbf{K}R^{-\infty}) \to H_n^G(E_{\mathcal{VC}}G; \mathbf{K}R^{-\infty})$$

is an isomorphism. Taking the corresponding cokernels, we have that

$$H_n^G(E_{\mathcal{F}}G \to E_{\mathcal{VC}}G; \mathbf{K}R^{-\infty}) \cong H_n^G(E_{\mathcal{F}}G \to E_{\mathcal{FBC}}G; \mathbf{K}R^{-\infty}).$$

Let the group G satisfy the condition $\mathcal{M}_{\mathcal{F}\subset\mathcal{FBC}}$ of [11], which states that every infinite group in $\mathcal{FBC}$ is contained in a unique maximal group in $\mathcal{FBC}$. Then,

$$\bigoplus_{V\in\mathcal{M}} H_n^{N_G(V)}(E_{\mathcal{F}}N_G(V) \to E_1 W_G(V); \mathbf{K}R^{-\infty}) \xrightarrow{\cong} H_n^G(E_{\mathcal{F}}G \to E_{\mathcal{FBC}}G; \mathbf{K}R^{-\infty}).$$

Here, $\mathcal{M}$ is a set of representatives of the conjugacy classes of the maximal infinite groups in $\mathcal{FBC}$ and $W_G(V) = N_G(V)/V$ is the Weyl group of V (see [11, Corollary 6.1]). Remark 6.2 in [11] implies that there is a spectral sequence

$$E_{p,q}^2 = H_p^{W_G(V)}(E_1 W_G(V); H_q^V(E_{\mathcal{F}}N_G(V) \to \{pt\}); \mathbf{K}R^{-\infty}) \Rightarrow H_{p+q}^{N_G(V)}(E_{\mathcal{F}}N_G(V) \to E_1 W_G(V); \mathbf{K}R^{-\infty}). \quad (2.1)$$

This is obtained by choosing $X = E_1 W_G(V)$ and noticing that $E_{\mathcal{F}}N_G(V) \times E_1 W_G(V)$ is a space of type $E_{\mathcal{F}}N_G(V)$ with the diagonal action. Also, [11, Example 6.3] implies that if $V = F \rtimes \mathbb{Z}$, then $H_q^V(E_{\mathcal{F}}N_G(V) \to \{pt\})$ is the non-connective version of Farrell's twisted Nil-term. Thus, the spectral sequence becomes

$$E_{p,q}^2 = H_p^{W_G(V)}(E_1 W_G(V); \mathbf{Nil}_R) \Rightarrow H_{p+q}^{N_G(V)}(E_{\mathcal{F}}N_G(V) \to E_1 W_G(V); \mathbf{K}R^{-\infty}). \quad (2.2)$$

Remark 2.2. In [14], the following were shown.

(i) The finite subgroups of $\mathrm{Aut}(F_2)$ and $\mathrm{Hol}(F_2)$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, D_2 and D_4. The maximal finite subgroups are $\mathbb{Z}/3\mathbb{Z}$ and D_4. From the construction, the same is true for $\mathcal{H}_{(n)}$ for all $n \geq 1$.

(ii) Although this is not explicitly shown in [14], up to isomorphism, there are various infinite $\mathcal{FBC}$ subgroups of $\mathrm{Hol}(F_2)$ which are isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. In fact, the construction of $\mathcal{H}_{(n)}$ shows that the infinite $\mathcal{FBC}$ subgroups are $F \times \mathbb{Z}$, where $F < \mathcal{H}_{(n)}$ is finite. The subgroup $\mathbb{Z}$ is a subgroup in the factors that are complementary to $\mathrm{Hol}(F_2)$. That is because each element that belongs to $F_i < \mathcal{H}_{(n)}$, $3 \leq i \leq n$, commutes with the subgroups of $\mathrm{Hol}(F_2)$.
Thus, the maximal infinite $\mathcal{FBC}$ subgroups are of the types $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}$ and $D_4 \times \mathbb{Z}$.

We will show that certain mapping tori of the free groups that are contained in $\mathcal{H}_{(n)}$ are CAT(0)-groups. Notice that, by the work of Brady [5], all mapping tori $F_2 \rtimes \mathbb{Z}$ contained in $F_2 \rtimes \mathrm{Aut}(F_2)$ are CAT(0). We show that this is true for all mapping tori $F_{n+1} \rtimes \mathbb{Z}$ contained in $F_{n+1} \rtimes E(\mathcal{H}_{(n-1)})$.

Proposition 2.3. *Let $G_n = F_{n+1} \rtimes \mathbb{Z} < F_{n+1} \rtimes E(\mathcal{H}_{(n-1)})$. Then, G_n is* CAT(0).

Proof. Set $G_n = F_{n+1} \rtimes \mathbb{Z} < \mathcal{H}_{(n)}$. By definition, the $\mathbb{Z}$-action on F_{n+1} is from an element of $\mathcal{H}_{(n-1)}$. This means that in the first two generators, it is an automorphism of the free group they generate, and on the other generators, it is conjugation by words on the previous generators. We will use induction. For $n = 1$, $G_1 = F_2 \rtimes \mathbb{Z}$, which is CAT(0) (see [5]).

For general n, let $F_{n+1} = \langle x_1, x_2, \dots, x_{n+1}\rangle$. Notice that $\mathcal{H}_{(n-1)} = F_n \rtimes \mathcal{H}_{(n-2)}$. Then, every $g \in \mathcal{H}_{(n-1)}$ can be written as $g = g_1 g_2$ with $g_1 \in F_n$ and $g_2 \in \mathcal{H}_{(n-2)}$. Then, the embedding $\mathcal{H}_{(n-1)}$ in $\mathrm{Aut}(F_{n+1})$ sends g to $\tilde{g}$ with

$$\tilde{g}(x_i) = g_2(x_i),\ i = 1, 2, \dots, n, \quad \tilde{g}(x_{n+1}) = g_1 x_{n+1} g_1^{-1}.$$

Then,

$$G_n = F_{n+1} \rtimes_{\tilde{g}} \mathbb{Z} = \langle t, x_1, x_2, \dots, x_{n+1} : t x_i t^{-1} = g_2(x_i),\ i = 1, 2 \dots, n,\ t x_{n+1} t^{-1} = g_1 x_{n+1} g_1^{-1}\rangle$$

with g_1 a word in x_i, $i = 1, 2, \dots, n$. Setting $\alpha = g_1^{-1} t$ and solving for t, we get

$$G_n = \langle \alpha, x_1, x_2, \dots, x_{n+1} : \alpha x_i \alpha^{-1} = g_1^{-1} g_2(x_i) g_1,\ i = 1, 2, \dots, n\rangle *_{\mathbb{Z}} \langle \alpha, x_{n+1} : [\alpha, x_{n+1}] = 1\rangle,$$

where $\mathbb{Z} = \langle\alpha\rangle$. Then, $G_n = H *_{\mathbb{Z}} \mathbb{Z}^2$. To characterize the group H, we set $\beta = g_1\alpha$ to get

$$H = \langle \beta, x_1, x_2, \dots, x_n : \beta x_i \beta^{-1} = g_2(x_i),\ i = 1, 2, \dots, n\rangle.$$

If $n = 2$, then $H = F_2 \rtimes \mathbb{Z} = G_1$. If $n > 2$, then $g_2 \in \mathcal{H}_{(n-2)}$. Thus, $H = G_{n-1}$. So, $G_n = G_{n-1} *_{\mathbb{Z}} \mathbb{Z}^2$, which is CAT(0) by induction and [6, Part II, Proposition 11.19]. □

The following result is in [18].

Corollary 2.4. *The groups G_n in Proposition 2.3 satisfy the FJCw.*

In [14], it was shown that the only infinite, virtually cyclic subgroup of type (I) in $\mathrm{Hol}(F_2)$ is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Corollary 2.5. *The group $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$ satisfies the FJCw.*

Proof. In [19], it was shown that CAT(0)-groups satisfy the FJCw. This means that finite extensions of CAT(0)-groups satisfy the FJCw. The result follows. □

Remark 2.6. In the Appendix, we will show that certain groups that appear in Corollary 2.5 are CAT(0).

We are also able to prove the following result.

Proposition 2.7. *Let D be a finite subgroup of* $\mathrm{Aut}(F_2)$ *and let $E(D)$ be its image in $\mathcal{H}_{(n)}$. Then, there are infinite cyclic-by-finite subgroups of the form $\mathbb{Z} \times E(D)$ in* $\mathrm{Hol}(F_n) \setminus \mathrm{Aut}(F_n)$, $n \geq 3$. *Moreover, every $G = F_m \rtimes (\mathbb{Z} \times E(D))$ in $\mathcal{H}_{(m-2)}$ is* CAT(0) *for all $m > n$.*

Proof. Assume that $F_n = \langle x_1, \dots, x_n\rangle$. By the definition of E, for all $\phi \in E(D)$ we have that $\phi(x_i) = x_i$ for every $i > 2$. Hence, in $\mathrm{Hol}(F_n)$, $\langle x_i\rangle$ with $i > 2$ commutes with ϕ for all $\phi \in E(D)$ and so $\langle x_i, E(D)\rangle$ is isomorphic to $\mathbb{Z} \times E(D)$ for every $i > 2$.

Let us now regard $\mathbb{Z} \times E(D)$ as a subgroup of $\mathrm{Hol}(F_n)$, $\mathbb{Z} = \langle x_n\rangle$, $n \geq 3$, and embed it in $\mathrm{Aut}(F_{n+1})$. Then, the action of $\mathbb{Z} = \langle \xi_{x_n}\rangle$ on x_{n+1} is conjugation by x_n and is trivial on every other generator of F_{n+1}. Then, G has a presentation of the form

$$\begin{aligned}G = \langle x_1, \dots, x_{n+1}, \xi_{x_n}, E(D) : [\xi_{x_n}, E(D)] = 1, [\xi_{x_n}, x_j] = 1,\ j < n+1,\\ \xi_{x_n} x_{n+1}\xi_{x_n}^{-1} = x_n x_{n+1} x_n^{-1}, [x_i, E(D)] = 1,\ i > 2\rangle.\end{aligned}$$

Set $z = x_n^{-1}\xi_{x_n}$ and get rid of x_n to get the presentation

$$\begin{aligned}G = \langle x_1, \dots, x_{n-1}, z, x_{n+1}, E(D) : [\xi_{x_n}, E(D)] = 1, [z, x_j] = 1,\ j < n+1, j \neq n,\\ [\xi_{x_n}, z] = 1, [x_i, E(D)] = 1,\ i > 2, i \neq n, [z, E(D)] = 1\rangle.\end{aligned}$$

Now, decompose G as an amalgamated free product $G = G_1 *_{\mathbb{Z}\times E(D)} G_2$ with

$$\begin{aligned}G_1 &= \langle x_1, x_2, \xi_{x_n}, E(D)\rangle \cong F_2 \rtimes (\mathbb{Z} \times E(D)),\\ G_2 &= \langle x_3, \dots, x_{n-1}, x_{n+1}, z, \xi_{x_n}, E(D)\rangle \cong \langle x_3, \dots, x_{n_1}, x_{n+1}, z, \xi_{x_n}\rangle \times E(D).\end{aligned}$$

Now, notice that the subgroup generated by $\langle x_3, \dots, x_{n_1}, x_{n+1}, z, \xi_{x_n}\rangle$ has a presentation

$$\langle x_3, \dots, x_{n_1}, x_{n+1}, z, \xi_{x_n} : [\xi_{x_n}, z] = 1, [\xi_{x_n}, x_j] = 1 \text{ for all } j < n, [z, x_{n+1}] = 1\rangle,$$

which makes it a right-angled Artin group and so is CAT(0) by [8]. Thus, G is CAT(0) by [6, Part II, Proposition 11.19]. □

Remark 2.8. In the last proposition, we showed that the group G is CAT(0) and thus it satisfies the FJCw. But we can use Proposition 2.3 to show directly that G satisfies the FJCw. This is done as in Corollary 2.5.

Remark 2.9. Note the following two properties of the groups described in Lemma 2.7.
(i) Every such subgroup is contained in a maximal cyclic-by-finite subgroup. This is an immediate consequence of the fact that $\mathrm{Aut}(F_2)$ decomposes as an amalgamated free product with maximal elements of finite order.
(ii) The normalizer of every maximal such subgroup coincides with the normalizer of its finite subgroup in $\mathrm{Aut}(F_2)$.

Now, let us introduce some notation from [14]. The group $\mathrm{Aut}(F_2)$ admits a presentation of the form

$$\begin{aligned}\langle p, x, y, \tau_a, \tau_b : x^4 = p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3\tau_b^{-1}\tau_a, p^{-1}\tau_a p = x^{-1}\tau_a x = y^{-1}\tau_a y = \tau_b,\\ p^{-1}\tau_b p = \tau_a, x^{-1}\tau_b x = \tau_a^{-1}, y^{-1}\tau_b y = \tau_a^{-1}\tau_b\rangle,\end{aligned}$$

where τ_a, τ_b are the inner automorphisms of F_2 corresponding to a, b, respectively. Moreover, any element of $\mathrm{Aut}(F_2)$ can be written uniquely in the form $p^r u(x, y)x^{2s} w(\tau_a, \tau_b)$, where $r, s \in \{0, 1\}$, $w(\tau_a, \tau_b)$ is a reduced word in $\mathrm{Inn}(F_2)$ and $u(x, y)$ is a reduced word, where x, y, y^{-1} are the only powers of x, y appearing (see [12, 13]).

Moreover, a presentation for $\mathrm{GL}_2(\mathbb{Z})$ is given by

$$\mathrm{GL}_2(\mathbb{Z}) = \langle P, X, Y : X^4 = P^2 = (PX)^2 = (PY)^2 = 1, X^2 = Y^3\rangle$$

and $\mathrm{Aut}(F_2)$ maps homomorphically onto $\mathrm{GL}_2(\mathbb{Z})$ by $p \mapsto P$, $x \mapsto X$, $y \mapsto Y$, $\tau_a, \tau_b \mapsto 1$.

Lemma 2.10. *Let D_4 be the subgroup of* $\mathrm{Aut}(F_2)$ *generated by $\langle p, x\rangle$. Then, the normalizer $N_{\mathrm{Aut}(F_2)}(D_4)$ of D_4 in* $\mathrm{Aut}(F_2)$ *is D_4 itself.*

Proof. Let $p^r u(x, y)x^{2s} w(\tau_a, \tau_b)$ be an element of $\mathrm{Aut}(F_2)$ that belongs to the normalizer of D_4. Then, it necessarily conjugates elements of order 4 to elements of order 4. But the only elements of order 4 in $\mathrm{Aut}(F_2)$ are conjugates of $x^{\pm 1}$ (see [13]). Hence, we have the relation

$$p^r ux^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} p^{-r} = x^{\pm 1} \tag{2.3}$$

or, equivalently,

$$ux^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} = p^r x^{\pm 1} x^{-r},$$

i.e.,

$$ux^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} = x^{\pm 1}.$$

Now, project this relation to $\mathrm{GL}_2(\mathbb{Z})$. It reduces to $U(X, Y)X^{2s} X X^{-2s} U^{-1} = X^{\pm 1}$ or $UXU^{-1} = X^{\pm 1}$. This last relation implies that $U = X$ and, therefore, $u = x$, since the projection maps y to $D_6 \setminus D_2$ and x to $D_4 \setminus D_2$ and, therefore, U and X freely generate a free group.

Thus, (2.3) reduces to $w(\tau_a, \tau_b)xw^{-1}(\tau_a, \tau_b) = x^{\pm 1}$, which implies that $w = 1$. Hence, the only words of $\mathrm{Aut}(F_2)$ that normalize x are of the form $p^r x^s$, hence $N_{\mathrm{Aut}(F_2)}(D_4) = D_4$. □

Corollary 2.11. *The normalizer* $N_{\mathcal{H}_{(n)}}(D_4 \times \mathbb{Z}) = D_4 \times \mathbb{Z}$.

Proof. This follows from Lemma 2.10 and Remark 2.9. □

3 The lower K-theory for $\mathcal{H}_{(n)}$

The algebraic calculations of the previous section allow us to prove the theoretic Isomorphism Conjecture for the groups $\mathcal{H}_{(n)}$ using induction.

Theorem 3.1. *The groups* $\mathcal{H}_{(n)}$ *satisfy the FJCw.*

Proof. We will use induction on n. For $n = 0$, $\mathcal{H}_{(0)} = F_2$, for which the FJCw holds (see [2, 3]). For $n \geq 1$ we have an exact sequence

$$1 \to F_{n+1} \to \mathcal{H}_{(n)} \xrightarrow{p} \mathcal{H}_{(n-1)} \to 1.$$

We assume that the FJCw holds for $\mathcal{H}_{(n-1)}$. Given that the FCJw holds for F_{n+1}, it suffices to show that the FJCw holds for $p^{-1}(C)$, where C is an infinite cyclic subgroup of $\mathcal{H}_{(n-1)}$. But in that case, Proposition 2.3 implies that $p^{-1}(C)$ is CAT(0). Thus, it satisfies the FJCw. □

Corollary 3.2. *The FJCw holds for* $\mathrm{Aut}(F_2)$ *and* $\mathrm{Hol}(F_2)$.

Proof. Since $\mathrm{Hol}(F_2) = \mathcal{H}_{(1)}$, the FJCw holds for $\mathrm{Hol}(F_2)$. Also, $\mathrm{Aut}(F_2) < \mathrm{Hol}(F_2)$ and thus the FJCw holds for $\mathrm{Aut}(F_2)$. □

Using Theorem 3.1, we calculate the lower K-theory of $\mathcal{H}_{(n)}$.

Theorem 3.3. *The groups* $K_q(\mathbb{Z}\mathcal{H}_{(n)}) = 0$ *for* $q \leq -1$. *For* $q = 0, 1$ *the reduced K-groups are*

$$\tilde{K}_q(\mathbb{Z}\mathcal{H}_{(n)}) = \begin{cases} 0, & n = 0, 1 \\ NK_q(\mathbb{Z}D_4) \oplus NK_q(\mathbb{Z}D_4), & n \geq 2 \end{cases}$$

Proof. For $n = 0$, $\mathcal{H}_{(0)} = F_2$, which is a hyperbolic group, and the result is well known. For $n = 1$, $\mathcal{H}_{(1)} = \mathrm{Hol}(F_2)$, and the result was proved in [14]. So we may assume that $n \geq 2$. Since the groups $\mathcal{H}_{(n)}$ satisfy the K-FJCw, we have that

$$\begin{aligned} K_q(\mathbb{Z}\mathcal{H}_{(n)}) &\cong H_q^{\mathcal{H}_{(n)}}(E_{\mathcal{FBC}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) \\ &\cong H_q^{\mathcal{H}_{(n)}}(E_{\mathcal{F}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{M}} H_q^{N_{\mathcal{H}_{(n)}} V}(E_{\mathcal{F}} N_{\mathcal{H}_{(n)}} V \to E_1 W_{\mathcal{H}_{(n)}} V; \mathbf{K}\mathbb{Z}^{-\infty}), \end{aligned}$$

where $\mathcal{M}$ is a set of representatives of the conjugacy classes of maximal infinite groups in $\mathcal{FBC}$.

The calculations of [14] show that $H_n^{\mathcal{H}(n)}(E_{\mathcal{F}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) = 0$ for $n \leq 1$. For each of the summands, there is a spectral sequence

$$E_{p,q}^2 = H_p^{W_{\mathcal{H}(n)}V}(E_1 W_{\mathcal{H}_{(n)}} V; H_q^V(E_{\mathcal{F}}V \to pt; \mathbf{K}\mathbb{Z}^{-\infty})) \Rightarrow H_{p+q}^{N_{\mathcal{H}(n)}V}(E_{\mathcal{F}}N_{\mathcal{H}_{(n)}}V \to E_1 W_{\mathcal{H}_{(n)}}V; \mathbf{K}\mathbb{Z}^{-\infty}).$$

In Remark 2.2, it was shown that the maximal infinite groups in $\mathcal{H}_{(n)}$ are of the types $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$, $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$, $D_4 \times \mathbb{Z}$, and there is only one conjugacy class for each of the groups $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$ and $D_4 \times \mathbb{Z}$.

(i) If V is $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$, then $H_q^V(E_{\mathcal{F}}V \to pt; \mathbf{K}_{\mathbb{Z}}) = 0$, $q \leq 1$ because the Nil-groups of the two cyclic groups vanish. Thus, for these groups (cf. the spectral sequence (2.2)),

$$H_n^{N_{\mathcal{H}(n)}V}(E_{\mathcal{F}}N_{\mathcal{H}_{(n)}}V \to E_1 W_{\mathcal{H}_{(n)}}V; \mathbf{K}\mathbb{Z}^{-\infty}) = 0, \quad i \leq 1.$$

(ii) For $V = D_4 \times \mathbb{Z}$, we have that $N_{\mathcal{H}_{(n)}}V = V$ (cf. Corollary 2.11) and the spectral sequence (cf. the spectral sequence (2.1)) reduces for $q \leq 1$ to the isomorphism

$$H_q^{N_{\mathcal{H}(n)}V}(E_{\mathcal{F}}N_{\mathcal{H}_{(n)}}V \to E_1 W_{\mathcal{H}_{(n)}}V; \mathbf{K}\mathbb{Z}^{-\infty}) \cong H_q^V(E_{\mathcal{F}}V \to pt; \mathbf{K}\mathbb{Z}^{-\infty}) \cong NK_q(\mathbb{Z}D_4) \oplus NK_q(\mathbb{Z}D_4).$$

It is known that $NK_q(\mathbb{Z}D_4) = 0$ for $q \leq -1$ and that it is infinitely generated for $q = 0, 1$ (see [20]). Combining the above information, we have the following for $n \geq 2$.

(i) $K_i(\mathbb{Z}\mathcal{H}_{(n)}) = 0$, $i \leq -1$.

(ii) $\tilde{K}_0(\mathbb{Z}\mathcal{H}_{(n)}) \cong NK_0(\mathbb{Z}D_4) \oplus NK_0(\mathbb{Z}D_4)$.

(iii) $Wh(\mathcal{H}_{(n)}) \cong NK_1(\mathbb{Z}D_4) \oplus NK_1(\mathbb{Z}D_4)$. □

Remark 3.4. In [20], it was shown that $NK_0(\mathbb{Z}D_4)$ is isomorphic to the direct sum of an infinite free $\mathbb{Z}_2$-module with a countably infinite free $\mathbb{Z}_4$-module. Also, $NK_1(\mathbb{Z}D_4)$ is a countably infinite torsion group of exponent 2 or 4.

4 Concluding remarks

In general, if the group G is linear (i.e., it admits a faithful finite-dimensional real or complex representation), then the K-FJCw can be proved for G. The problem is that $\mathrm{Aut}(F_n)$ is not linear for $n \geq 3$. The group that was used to show that $\mathrm{Aut}(F_n)$ is not linear is the Formanek–Procesi group FP (see [10]). This group is given by a split extension

$$1 \to F_3 \xrightarrow{f} FP \xrightarrow{p} F_2 \to 1$$

and has the presentation

$$FP = \langle \alpha_1, \alpha_2, \alpha_3, \phi_1, \phi_2 : \phi_i\alpha_j\phi_i^{-1} = \alpha_j,\ \phi_i\alpha_3\phi_i^{-1} = \alpha_3\alpha_i,\ i, j = 1, 2\rangle.$$

In [10], it was shown that FP is not linear and $FP < \mathrm{Aut}(F_3)$. On the other hand, it is obvious that the group G is not word hyperbolic (since it contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$) and it is not known if it is CAT(0). The point here is that FP has “enough” CAT(0)-subgroups so that it satisfies the FJCw.

Proposition 4.1. *The group FP satisfies the FJCw.*

Proof. Let V be an infinite cyclic subgroup of F_2 that is generated by a word on ϕ_1 and ϕ_2 and their inverses. Then, the action of the generator on F_3 fixes the first two generators and sends α_3 to the element $\alpha_3 c$, where c is a word in α_1, α_2 and their inverses. Then, $F_3 \rtimes V$ is a CAT(0)-group from [17, Theorem 4.4]. Thus, it satisfies the FJCw. Therefore, FP satisfies the FJCw. □

A Appendix

We will show the result stated in Remark 2.6, that certain groups of type $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$ are CAT(0).

Our investigation, similar to the one in [14, Proposition 3.2], shows that subgroups of $\mathrm{Aut}(F_2)$ isomorphic to $F_2 \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ occur when the action of $\mathbb{Z}/2\mathbb{Z}$ on F_2 is of the form

$$t(x_1) = x_1^{-1}, \quad t(x_2) = x_2 \text{ or } t(x_1) = x_1, \quad t(x_2) = x_2^{-1}.$$

We show that in all the above cases these subgroups are CAT(0).

Lemma A.1. *Let $\mathbb{Z}/2\mathbb{Z} \cong \langle t_1 \rangle < \mathrm{Aut}(F_2)$ be such that $t_1(x_1) = x_1^{-1}$ and $t_1(x_2) = x_2$, as above. Let $\mathbb{Z} \cong \langle t_2 \rangle < \mathrm{Hol}(F_2)$ so that $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} < \mathrm{Hol}(F_2)$. Then, we have the following cases.*
(i) *If $t_2 \notin \mathrm{Aut}(F_2)$, then $t_2 = x_2^k \in F_2$, $k \in \mathbb{Z}$, $k \neq 0$.*
(ii) *If $t_2 \in \mathrm{Aut}(F_2)$, then $t_2(x_1) = x_2^k x_1^{\pm 1} x_2^{-k}$, $t_2(x_2) = x_2^{\pm 1}$, $k \neq 0$ (four cases).*

Proof. First, we assume that $t_2 \notin \mathrm{Aut}(F_2)$. Because t_1 and t_2 commute, t_1 acts trivially on t_2. Let $w(x_1, x_2)$ be the word in F_2 representing t_2. Then, $w(x_1^{-1}, x_2) = w(x_1, x_2)$. This means that x_1 does not appear in $w(x_1, x_2)$. Thus, $w(x_1, x_2) = x_2^k$.

Now, let $t_2 \in \mathrm{Aut}(F_2)$. Then, $t_2(x_1) = w_1(x_1, x_2)$ and $t_2(x_2) = w_2(x_1, x_2)$. Since $t_1 t_2 = t_2 t_1$, we have that

$$w_1(x_1^{-1}, x_2) = w_1(x_1, x_2)^{-1}, \quad w_2(x_1^{-1}, x_2) = w_2(x_1, x_2).$$

As before, the second relation implies that the word $w_2 = x_2^k$, $k \in \mathbb{Z}$. Looking at the first relation, we get that $w_1 = c x_1^{\ell} c^{-1}$, $\ell \in \mathbb{Z}$. But w_1 and w_2 must be a generating set for F_2. This means that $k, \ell \in \{\pm 1\}$. Also,

$$\begin{aligned}(t_1 \circ t_2)(x_1) &= t_1(c(x_1, x_2) x_1^{\pm 1} c(x_1, x_2)^{-1}) = c(x_1^{-1}, x_2) x_1^{\mp 1} c(x_1^{-1}, x_2)^{-1},\\ (t_2 \circ t_1)(x_1) &= t_2(x_1^{-1}) = c(x_1, x_2) x_1^{\mp 1} c(x_1, x_2)^{-1}.\end{aligned}$$

Since $t_1 \circ t_2 = t_2 \circ t_1$, we have $c(x_1, x_2) = c(x_1^{-1}, x_2)$ and thus $c = x_2^k$, $k \in \mathbb{Z}$. □

Lemma A.2. *Let $G = F_2 \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, where the generator t_1 of $\mathbb{Z}/2\mathbb{Z}$ acts as*

$$t_1(x_1) = x_1^{-1}, \quad t_1(x_2) = x_2,$$

and the generator t_2 of $\mathbb{Z}$ acts as

$$t_2(x_1) = x_2^k x_1^{\pm 1} x_2^{-k}, \quad t_2(x_2) = x_2^{\pm 1}.$$

Then, G is a CAT(0)-group.

Proof. The group G has the presentation

$$\langle t_1, t_2, x_1, x_2 : t_1 x_1 t_1^{-1} = x_1^{-1},\ t_1 x_2 t_1^{-1} = x_2,\ t_1^2 = [t_1, t_2] = 1,\ t_2 x_1 t_2^{-1} = x_2^k x_1^{\pm 1} x_2^{-k},\ t_2 x_2 t_2^{-1} = x_2^{\pm 1} \rangle.$$

We set $\xi = x_2^{-k} t_2$. Then, the presentation becomes

$$\langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2,\ t_1^2 = [t_1, \xi] = 1,\ \xi x_1 \xi^{-1} = x_1^{\pm 1},\ \xi x_2 \xi^{-1} = x_2^{\pm 1} \rangle.$$

Now, we consider four cases.

Case 1. In this case,

$$G = \langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1},\ t_1 x_2 t_1^{-1} = x_2,\ t_1^2 = [t_1, \xi] = 1,\ \xi x_1 \xi^{-1} = x_1,\ \xi x_2 \xi^{-1} = x_2 \rangle.$$

Now, let $L_1 = \langle x_2, t_1, \xi : t_1^2 = [t_1, x_2] = [t_1, \xi] = [x_2, \xi] = 1 \rangle < G$, which is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$ and, thus, it is CAT(0). Also, let $L_2 = \langle x_1, t_1, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, \xi x_1 \xi^{-1} = x_1, t_1^2 = [t_1, \xi] = 1 \rangle$. Then,

$$L_2 = \langle \xi \rangle \times \langle x_1, t_1 : t_1 x_1 t_1^{-1} = x_1^{-1} \rangle \cong \mathbb{Z} \times D_\infty.$$

The infinite dihedral group is a CAT(0)-group because it is a Coxeter group (see [15]). Thus, L_2 is a CAT(0)-group, as a direct product of CAT(0)-groups. Also, $L = \langle t_1, \xi : t_1^2 = [t_1, \xi] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. But then, $G = L_1 *_L L_2$ is CAT(0) (see [6, Part II, Corollary 11.19]).

Case 2. We assume that

$$G = \langle t_1, x_1, x_2, \xi : t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ t_1^2 = [t_1, \xi] = 1,\ \xi x_1\xi^{-1} = x_1^{-1},\ \xi x_2\xi^{-1} = x_2\rangle.$$

By setting $\xi_2 = t_1\xi$ and rewriting the presentation, we are back in Case 1.

Case 3. We assume that

$$G = \langle t_1, x_1, x_2, \xi : t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ t_1^2 = [t_1, \xi] = 1,\ \xi x_1\xi^{-1} = x_1,\ \xi x_2\xi^{-1} = x_2^{-1}\rangle.$$

We repeat the same method as before. Let

$$L_1 = \langle x_2, t_1, \xi : t_1^2 = [t_1, x_2] = [t_1, \xi] = 1,\ \xi x_2\xi^{-1} = x_2^{-1}\rangle \cong \langle t_1 : t_1^2 = 1\rangle \times \langle x_2, \xi : \xi x_2\xi^{-1} = x_2^{-1}\rangle.$$

Therefore, $L_1 \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z} \rtimes \mathbb{Z})$. Now, the second group can be written as an HNN-extension $\mathbb{Z}*_r$, where r is the non-trivial automorphism of $\mathbb{Z}$. Then, $\mathbb{Z} \rtimes \mathbb{Z}$ is a CAT(0)-group (see [6, Part II, Corollary 11.22]) and, thus, L_1 is CAT(0). Now, L and L_2 are as in Case 1 and, thus, $G = L_1 *_L L_2$ is CAT(0).

Case 4. We assume that

$$G = \langle t_1, x_1, x_2, \xi : t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ t_1^2 = [t_1, \xi] = 1,\ \xi x_1\xi^{-1} = x_1^{-1},\ \xi x_2\xi^{-1} = x_2^{-1}\rangle.$$

Again, set $\xi_2 = t_1\xi$ and rewrite the presentation to arrive at Case 3. □

Proposition A.3. *The group $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$ is a CAT(0)-group.*

Proof. Let t_1 be the generator of $\mathbb{Z}/2\mathbb{Z}$ and let t_2 be the generator of $\mathbb{Z}$. We will consider two cases.

Case 1. Let $t_2 \in \mathrm{Hol}(F_2)$ and $t_2 \notin \mathrm{Aut}(F_2)$. From Case 1 of Lemma A.1, t_2 is an element x_2^k, $k \in \mathbb{Z}$, $k \neq 0$. Then, $G = F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ has the presentation

$$\begin{aligned}\langle x_1, x_2, \dots, x_n, t_1, t_2 : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1}, t_1x_it_1^{-1} = x_i,\ i = 2, \dots, n,\\ t_2x_1t_2^{-1} = x_1,\ t_2x_2t_2^{-1} = x_2,\ t_2x_it_2^{-1} = x_2^kx_ix_2^{-k},\ i = 3, \dots, n,\ [t_1, t_2] = 1\rangle.\end{aligned}$$

We change the generators by setting $\xi = x_2^{-k}t_2$. First, notice that $[t_1, \xi] = 1$ because t_1 commutes with t_2 and x_2. Then, the presentation becomes

$$\begin{aligned}\langle x_1, x_2, \dots, x_n, t_1, \xi : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_it_1^{-1} = x_i,\ i = 2, \dots, x_n,\\ \xi x_1\xi^{-1} = x_2^{-k}x_1x_2^k,\ \xi x_i\xi^{-1} = x_i,\ i = 2, \dots, x_n,\ [t_1, \xi] = 1\rangle.\end{aligned}$$

Set $K_1 = \langle t_1, \xi, x_3, \dots, x_n : t_1^2 = [t_1, \xi] = [t_1, x_i] = [\xi, x_i]1,\ i = 3, \dots, n\rangle < F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Then, K_1 is isomorphic to $\mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$, which is a CAT(0)-group. Let $K = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle t_1, t_2\rangle$, which is a virtually infinite cyclic group. Also, set $K_2 < G$ with presentation

$$\langle x_1, x_2, t_1, \xi : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ \xi x_1\xi^{-1} = x_2^{-k}x_1x_2^k,\ \xi x_2\xi^{-1} = x_2,\ [t_1, \xi] = 1\rangle.$$

Notice that $G = K_1 *_K K_2$. In order to show that G is CAT(0), it suffices to show that K_2 is a CAT(0) group.

To that end, we change generators in K_2 by setting $\zeta = x_2^k\xi$. Then, the presentation of K_2 becomes

$$\langle x_1, x_2, t_1, \zeta : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ \zeta x_1\zeta^{-1} = x_1,\ \zeta x_2\zeta^{-1} = x_2,\ [t_1, \zeta] = 1\rangle.$$

For Case 1 of Lemma A.2, K_2 is CAT(0) and we are done.

Case 2. We assume that $t_2 \in \mathrm{Aut}(F_2)$. Using Case 2 of Lemma A.2, the group G has the presentation

$$\begin{aligned}\langle x_1, x_2, \dots, x_n, t_1, t_2 : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_it_1^{-1} = x_i,\ i = 2, \dots, x_n,\\ t_2x_1t_2^{-1} = x_2^kx_1^{\pm1}x_2^{-k},\ t_2x_2t_2^{-1} = x_2^{\pm1},\ t_2x_it_2^{-1} = x_i,\ i = 3, \dots, n,\ [t_1, t_2] = 1\rangle.\end{aligned}$$

Set

$$K_1 = \langle t_1, t_2, x_3, \dots, x_n : t_1^2 = [t_1, t_2] = [t_1, x_i] = [t_2, x_i] = 1,\ i = 3, \dots, n \rangle \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$$

and $K = \langle t_1, t_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which are two subgroups of G. Finally, set K_2 to be

$$\langle t_1, t_2, x_1, x_2 : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ t_2x_1t_2^{-1} = x_2^k x_1^{\pm 1} x_2^{-k},\ t_2x_2t_2^{-1} = x_2^{\pm 1},\ [t_1, t_2] = 1 \rangle.$$

It is obvious that $G = K_1 *_K K_2$. To show that G is CAT(0), it suffices to show that K_2 is a CAT(0) group.

Set $\zeta = x_2^{-k} t_2$ and the presentation becomes

$$\langle t_1, x_1, x_2, \zeta : t_1^2 = 1,\ t_1x_1t_1^{-1} = x_1^{-1},\ t_1x_2t_1^{-1} = x_2,\ \zeta x_1\zeta^{-1} = x_1^{\pm 1},\ \zeta x_2\zeta^{-1} = x_2^{\pm 1},\ [t_1, \zeta] = 1 \rangle,$$

which is CAT(0) from Lemma A.2. □

The reader should notice that there are more possibilities for the $\mathbb{Z}/2\mathbb{Z}$ action on the $F_2 \rtimes \mathbb{Z}$ subgroups of $\mathrm{Aut}(F_2)$.

Acknowledgment: We would like to thank the anonymous referee for helping us improve our exposition.

Funding: Research supported by the European Union (European Social Fund – ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) – Research Funding Program: THALIS.

References

[1] A. C. Bartels, On the domain of the assembly map in algebraic K-theory, *Algebr. Geom. Topol.* **3** (2003), 1037–1050.
[2] A. Bartels and W. Lück, The Borel conjecture for hyperbolic and CAT(0)-groups, *Ann. of Math. (2)* **175** (2012), no. 2, 631–689.
[3] A. Bartels, W. Lück and H. Reich, The K-theoretic Farrell–Jones conjecture for hyperbolic groups, *Invent. Math.* **172** (2008), no. 1, 29–70.
[4] A. Bartels and H. Reich, Coefficients for the Farrell–Jones conjecture, *Adv. Math.* **209** (2007), no. 1, 337–362.
[5] T. Brady, Complexes of nonpositive curvature for extensions of F_2 by $\mathbb{Z}$, *Topology Appl.* **63** (1995), no. 3, 267–275.
[6] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. 319, Springer, Berlin, 1999.
[7] J. F. Davis, F. Quinn and H. Reich, Algebraic K-theory over the infinite dihedral group: A controlled topology approach, *J. Topol.* **4** (2001), no. 3, 505–528.
[8] M. W. Davis and T. Januszkiewicz, Right-angled Artin groups are commensurable with right-angled Coxeter groups, *J. Pure Appl. Algebra* **153** (2000), no. 3, 229–235.
[9] F. T. Farrell and X. Wu, Isomorphism conjecture for Baumslag–Solitar groups, *Proc. Amer. Math. Soc.* **143** (2015), no. 8, 3401–3406.
[10] E. Formanek and C. Procesi, The automorphism group of a free group is not linear, *J. Algebra* **149** (1992), no. 2, 494–499.
[11] W. Lück and M. Weiermann, On the classifying space of the family of virtually cyclic subgroups, *Pure Appl. Math. Q.* **8** (2012), no. 2, 497–555.
[12] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*, Interscience, New York, 1966.
[13] S. Meskin, Periodic automorphisms of the two-generator free group, in: *Proceedings of the Second International Conference on the Theory of Groups* (Canberra 1973), Lecture Notes in Math. 372, Springer, Berlin (1974), 494–498.
[14] V. Metaftsis and S. Prassidis, On the vanishing of the lower K-theory of the holomorph of a free group on two generators, *Forum Math.* **24** (2012), no. 4, 861–874.
[15] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D thesis, The Ohio State University, 1988.
[16] S. K. Roushon, Algebraic K-theory of groups wreath product with finite groups, *Topology Appl.* **154** (2007), no. 9, 1921–1930.
[17] P. Samuelson, On CAT(0) structures for free-by-cyclic groups, *Topology Appl.* **153** (2006), no. 15, 2823–2833.
[18] C. Wegner, The K-theoretic Farrell–Jones conjecture for CAT(0)-groups, *Proc. Amer. Math. Soc.* **140** (2012), no. 3, 779–793.
[19] C. Wegner, The Farrell–Jones conjecture for virtually solvable groups, preprint (2013), http://arxiv.org/abs/1308.2432.
[20] C. Weibel, NK_0 and NK_1 of the groups C_4 and D_4, *Comment. Math. Helv.* **84** (2009), no. 2, 339–349.