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Groups with infinite virtual cohomological dimension which act freely on $\mathbb{R}^m \times S^{n-1}$

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Abstract

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In this paper free and proper discontinuous actions on $\mathbb{R}^m \times S^{n-1}$ are constructed of groups with infinite virtual cohomological dimension.

Introduction

In this paper there is an extension of the results given in [4], where free actions of groups with finite virtual cohomological dimension on $\mathbb{R}^m \times S^{n-1}$ have been studied. The purpose of this paper is to find groups G, with infinite virtual cohomological dimension, which act freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$ for some m and n. As it is stated in [17, p. 518], any group which acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$ must satisfy periodicity in ordinary cohomology in high dimensions. The groups which appear in [14, 15] satisfy this condition and many of them have infinite virtual cohomological dimension. So, those groups are candidates for such actions. In [4], it was proven that a countable group G with $\mathrm{vcd}(G) < \infty$ acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$ if and only if it has periodic Farrell cohomology [4, Theorem A]. (The Farrell cohomology is an extension of the Tate cohomology to infinite groups with finite virtual cohomological dimension [5].) In this paper, we study countable groups G with $\mathrm{vcd}(G) = \infty$ and for which the Farrell cohomology is defined [7, 8]. It turns out that if such a group acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$, then it has periodic Farrell cohomology.

Notice that if a group G, with finite generalized cohomological dimension, for which Farrell cohomology is defined [7, 8], acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$, then it will have periodic cohomology after a finite number of steps [17]. Then, Proposition 3 asserts that G has periodic Farrell cohomology.

The main result of this paper (Theorem 10) is that there are groups with infinite cohomological dimension for which Farrell cohomology is defined and it is periodic which act freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$.

The method of proof of the above result is based on the methods developed in [4]. We use groups G which act on a finite-dimensional acyclic complex X with finite or cohomologically nice isotropy groups. This acyclic complex replaces the complex $\mathscr{E}G$ constructed in [12] when $\mathrm{vcd}(G) < \infty$. Then, as in [4], we construct a finite-dimensional free G-complex E and a Hurewicz fibration $\pi : E \to X$ with fibers the polarized complexes constructed in [16] for finite groups and in [4] for infinite groups with finite virtual cohomological dimension. The crucial point of the construction is that we choose groups G for which X can be chosen to be a tree and this has the advantage of avoiding the obstruction theory used in [4] for extending the fibration over higher skeleta. Once E is constructed, standard methods produce a free and properly discontinuous action of G on $\mathbb{R}^m \times S^{n-1}$.

Let G be a countable group of Ikenaga's class C_∞ (see [7] and Section 1 for the definition). Then G acts on a finite-dimensional acyclic simplicial complex and the isotropy groups of simplices have 'better' cohomological properties than G. This suggests an inductive process for constructing free and properly discontinuous actions of such groups on $\mathbb{R}^m \times S^{n-1}$. The following problem generalizes the question asked by Wall in [18] for groups with finite virtual cohomological dimension and answered in [4].

Problem. A countable group of class C_∞ acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$, for some m and n, if and only if it has periodic (extended) Farrell cohomology.

One more interesting question is about group actions on $\mathbb{R}^m \times S^{n-1}$ with compact quotient. In [9], there are given classes of groups which act freely, properly discontinuously, and with compact orbit space. In [6], there are constructed such actions of groups which contain dihedral subgroups, disproving a conjecture by F.T. Farrell.

For the inverse of this problem, there is a question asked by Wall in [17]:

Question. Let G be a group which acts freely, properly discontinuously, and with compact quotient on $\mathbb{R}^m \times S^{n-1}$ for some m and n. Does G have finite virtual cohomological dimension?

1. Farrell cohomology

In [7, 8], Ikenaga defines Farrell cohomology for groups with infinite virtual cohomological dimension. More precisely, for a group G, he defines the generalized cohomological dimension of G:

$$\mathbf{cd}(G) = \sup\{k \mid \operatorname{Ext}^k_G(M, F) \neq 0,\ M \text{ is } \mathbb{Z}\text{-free},\ F \text{ is } \mathbb{Z}G\text{-free}\}\ .$$

All the modules will be assumed to be left modules.

In [7], it is proven that:

(i) If G is finite $\mathbf{cd}(G) = 0$.

(ii) $\mathbf{cd}(G) \leq \mathrm{cd}(G)$ with equality if $\mathrm{cd}(G) < \infty$.

(iii) If $H < G$, then $\mathbf{cd}(H) \leq \mathbf{cd}(G)$ with equality if H has finite index in G. In particular, if $\mathrm{vcd}(G) < \infty$, then $\mathbf{cd}(G) = \mathrm{vcd}(G)$.

The Farrell cohomology for groups G, with $\mathrm{vcd}(G) < \infty$, is defined using complete resolutions [5, 2]. A complete resolution of a $\mathbb{Z}G$-module M is an acyclic chain complex, $\{F_k\}_{k\in\mathbb{Z}}$, consisting of projective $\mathbb{Z}G$-modules which agrees with a projective resolution in sufficiently high positive dimensions.

In [7], it is proven that if a group G has finite generalized cohomological dimension, and if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a complete resolution, then the Farrell cohomology of G is well defined. His definition extends the definition of the Farrell cohomology for groups with finite virtual cohomological dimension (see [2, Chapter X]). Most of the properties of the ordinary Farrell cohomology are also valid in this extended version. For example the Farrell cohomology functors satisfies Shapiro's Lemma and they are effaceable and coeffaceable.

Lemma 1. *Let G be a group with $\mathbf{cd}(G) < \infty$. Let F_* be a complete resolution which agrees with a projective resolution P_* in sufficiently high dimensions. Then there is a chain map $i : F_* \to P_*$ which is the identity in high dimensions and any two such maps are chain homotopic through a homotopy which is zero in high dimensions.*

Proof. In [7, p. 422], the chain map i is constructed. The proof that any two such maps are chain homotopic follows the method of the proof of Proposition 13 in [7]: Let $i,i' : F_* \to P_*$ be two chain maps which are the identity in high dimensions. We define a chain homotopy $s = 0$ between i and i' in high dimensions. We extend this homotopy on all dimensions: Assume that $s_{n-1} : F_{n-1} \to P_n$ has been defined for $n \geq k$. Let d,d' denote the differentials in F_*,P_*, respectively. First, define a map $s' : \ker d_{k-2} \to P_{k-1}$ as follows: Let $x \in \ker d_{k-2} = \operatorname{Im} d_{k-1}$, and there is $y \in F_{k-1}$so that $x = d_{k-1}(y)$. Set $s'(x) = i(y) - i'(y) - d'_k s_{k-1}(y)$. This is a well-defined map: Let $y' \in F_{k-1}$ with $x = d_{k-1}(y) = d_{k-1}(y')$, then $y - y' \in \ker d_{k-1} = \operatorname{Im} d_k$, and there is $z \in F_k$ so that $y - y' = d_k(z)$. Then:

$$\begin{aligned}
&i(y) - i'(y) - d'_k s_{k-1}(y) - (i(y') - i'(y') - d'_k s_{k-1}(y')) \\
&\quad = i(y - y') - i'(y - y') - d'_k s_{k-1}(y - y') \\
&\quad = i d_k(z) - i' d_k(z) - d'_k s_{k-1} d_k(z) \\
&\quad = d'_k i(z) - d'_k i'(z) - d'_k s_{k-1} d_k(z) \\
&\quad = d'_k(i(z) - i'(z) - s_{k-1} d_k(z))\ .
\end{aligned}$$

But by the inductive assumption, $i - i' = s_{k-1}d_k + d'_{k-1}s_k$. Therefore,

$$d'_k(i(z) - i'(z) - s_{k-1}d_k(z)) = -d'_k d'_{k-1}s_k(z) = 0$$

and the map s is well defined. Using the Corollary in [7, p. 425], we can extend the map s' to a map $s_{k-2} : F_{k-2} \to P_{k-1}$. This way we can extend s to a chain homotopy between i and i'. □

Remark. If $\mathbf{cd}G \le n$ and F_* is a complete resolution of $\mathbb{Z}$, using Proposition 14 of [7], we can choose the chain map above to be the identity in dimensions higher than $n - 1$.

Corollary 2. *Let G be a group with $\mathbf{cd}(G) \le n$ and assume that $\mathbb{Z}$ admits a complete resolution over $\mathbb{Z}G$. Then for each $\mathbb{Z}G$-module M there is a natural map $\iota : H^*(G, M) \to \hat{H}^*(G, M)$ which is epimorphism for $* = n$ and isomorphism for $* > n$.*

Proof. Immediate from the above remark. □

Let G be a group with $\mathbf{cd}(G) < \infty$, and assume that $\mathbb{Z}$ admits a complete resolution. In [7], the definition of cup products in the Farrell cohomology is given which is compatible with the cup products in the ordinary cohomology via the map ι.

In [7], a special class of groups with finite generalized cohomological dimension is defined. It is the class C_∞. They are groups which act on finite-dimensional acyclic simplicial complexes in such a way that the generalized cohomological dimensions of the isotropy groups of the cells are bounded. The action of G on the complexes will be assumed always simplicial, i.e. G acts on a simplicial complex X by permuting the simplices. Let C_0 be the class of finite groups and a group G belongs to the class C_n if there is an acyclic G-complex X, such that:

(i) The isotropy group, G_σ, of the cell σ is of class C_{n-1}.

(ii) $\sup\{\dim \sigma + \mathbf{cd}(G_\sigma) \mid \sigma \text{ is a cell of } X\} < \infty$.

An acyclic G-simplicial complex X which satisfies (i) and (ii) above is called admissible.

Define the class $C_\infty = \bigcup_{n \in \mathbb{N}} C_n$.

The groups in the class C_∞ have finite generalized cohomological dimension [7, Theorem 1] and the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a complete resolution [7, Theorem 2]. For further properties of the groups in the class C_∞, see [7, Chapter V].

Let G be a group in the class C_∞. Then there is an admissible G-complex X. We may assume that the action is order preserving and the isotropy group of a cell fixes the cell pointwise [7, p. 446]. Then there is a spectral sequence

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} \hat{H}^q(G_\sigma, M) \Rightarrow \hat{H}^{p+q}(G, M)(S),$$

where Σ_p is a set of representatives for the p cells of $X \bmod G$, and M is a $\mathbb{Z}G$-module. This spectral sequence generalizes the spectral sequence (4.1) of Chapter X in [2]. Let X_p be the set of p cells of X. If σ is a cell of X and $g \in G$, let $c(g^{-1})^* : \hat{H}^*(G_\sigma, M) \to \hat{H}^*(G_{g\sigma}, M)$ be the conjugation induced isomorphism. Write $c(g^{-1})^*(u) = gu$. We summarize the properties of this spectral sequence (the proofs are similar to the proofs given in the case $\mathrm{vcd}(G) < \infty$ in [2, Chapter X, Section 4]).

(i) E_1^{pq} can be identified with the subgroup of $\prod_{\sigma \in X_p} \hat{H}^q(G_\sigma, M)$ consisting of those families $(u_\sigma)_{\sigma \in X_p}$ such that $gu_\sigma = u_{g\sigma}$ for all $g \in G$, $\sigma \in X_p$. The differential d_1^{pq} is the restriction to this subgroup of the map

$$d : \prod_{\sigma \in X_p} \hat{H}^q(G_\sigma, M) \to \prod_{\sigma \in X_{p+1}} \hat{H}^q(G_\sigma, M)$$

defined by combining restriction and face maps as follows:

Let $\tau = (v_0, v_1, \ldots, v_{p+1}) \in X_{p+1}$ with $v_0 < v_1 < \cdots < v_{p+1}$, let $\tau_i = (v_0, \ldots, \hat{v}_i, \ldots, v_{p+1})$, $i = 0, 1, \ldots, p+1$, and let $\rho_i : \hat{H}^q(G_{\tau_i}, M) \to \hat{H}^q(G_\tau, M)$ be the restriction map. Then d is given:

$$d(u_\sigma)(\tau) = \sum_{i=0}^{p+1} (-1)^i \rho_i u_{\tau_i}\,.$$

(ii) In particular, $E_2^{0q} = \ker d_1^{0q}$ can be identified with the subgroup of $\prod_{v \in X_0} \hat{H}^q(G_v, M)$ consisting of those families $(u_v)_{v \in X_0}$ satisfying the two conditions

(1) $gu_v = u_{gv}$ for all $g \in G$, $v \in X_0$,

(2) if e is a 1-cell of X with vertices v_0, v_1, then u_{v_0} and u_{v_1} restrict to the same element of $\hat{H}^q(G_e, M)$.

Notice that the cup product on the cellular cochains of X with values in a commutative ring R with identity and a diagonal of a complete resolution F_* [7, pp. 438–440] induce a multiplicative structure on the spectral sequence above with coefficients in the ring R. We summarize the multiplicative properties of the spectral sequence (the proofs are the same as in [2, Chapter X, Proposition 4.5, Proposition 4.6]):

(iii) The differential d_r is a derivation with respect to the product on E_r, i.e.

$$d_r(uv) = d_r(u)v + (-1)^{\deg u} u d_r(v)$$

(iv) The product on $E_{r+1} = H(E_r)$ is obtained from the product on E_r by passage to homology.

(v) The product on E_1 is given by the composite of the cup product in the Farrell cohomology and the cup product in the cochains of X. More precisely, the product on E_1 is given by

$$\hat{H}^q(G, C^p(X)) \otimes \hat{H}^{q'}(G, C^{p'}(X)) \to \hat{H}^{q+q'}(G, C^p(X) \otimes C^{p'}(X)) \\ \to \hat{H}^{q+q'}(G, C^{p+p'}(X)) .$$

(vi) The product on E_r is associative for $r \geq 1$ and commutative for $r \geq 2$.

(vii) The product on E_∞ is compatible with the usual product on $\hat{H}^*(G, R)$ under the identification on E_∞ with $\mathrm{Gr}\,\hat{H}^*(G, R)$.

(viii) The kernel of the restriction map

$$\mathrm{res} : \hat{H}^q(G, R) \to \prod_{v \in \Sigma_0} \hat{H}^q(G_v, R) = E_1^{0q}$$

is nilpotent. Notice that res is just the edge homomorphism in the spectral sequence. This property is a weak version of Quillen's [10] theorem for groups G with $\mathrm{vcd}(G) < \infty$, which states that the restriction map

$$\mathrm{res} : \hat{H}^q(G, \mathbb{Z}) \to \varprojlim \hat{H}^q(H, \mathbb{Z}) ,$$

where H ranges over the finite subgroups of G (or even better over the elementary abelian subgroups of G), is an F-isomorphism, i.e. the kernel and cokernel of the map consist of nilpotent elements.

2. Groups with periodic Farrell cohomology

The formulation of the Farrell cohomology suggests that we can extend the definition of groups with periodic Farrell cohomology from the class of groups with finite virtual cohomological dimension to the class of groups with finite generalized cohomological dimension.

Definition. Let G be a group with $\mathbf{cd}(G) < \infty$, and assume that $\mathbb{Z}$ admits a complete resolution. Then G has *periodic Farrell cohomology* if there is a unit u, of degree $q \neq 0$, in the cohomology ring $\hat{H}^*(G, \mathbb{Z})$. In this case the map

$$\cup\, u : \hat{H}^i(G, M) \to \hat{H}^{i+1}(G, M)$$

is an isomorphism for each $i \in \mathbb{Z}$, and for each $\mathbb{Z}G$-module M.

Talelli defines cohomological periodicity for infinite groups [14, 15]. She defines a group G to have period q after k-steps if a resolution which is periodic after k-steps, i.e. there is an exact sequence

$$(*) \qquad 0 \to R_{k+q} \to P_{k+q-1} \to \cdots \to P_k \xrightarrow{\partial_k} P_{k-1} \cdots \to P_0 \to \mathbb{Z} \to 0 ,$$

where P_i, $0 \leq i \leq k+q-1$, are projective $\mathbb{Z}G$-modules and ∂_k can be factored as

$P_k \xrightarrow{\alpha} R_k \xrightarrow{\alpha'} P_{k-1}$, where α is an epimorphism, α' is a monomorphism and $R_k = R_{k+q}$. It follows that a group has periodic cohomology after k-steps if and only if the functors $H^n(G, -)$ and $H^{n+q}(G, -)$ are naturally isomorphic for all $n \geq k+1$.

In the following proposition we compare the two definitions:

Proposition 3. *Let G be a group with* $\mathbf{cd}(G) < \infty$. *Then the following are equivalent*:

(i) *The trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a complete resolution*, $\mathbf{cd}(G) \leq k$, *and G has periodic Farrell cohomology of period q.*

(ii) *G has period q after k-steps.*

Proof. Assume that (i) holds. Then there is an element $u \in \hat{H}^q(G, \mathbb{Z})$, $q \neq 0$, such that

$$\cup\, u : \hat{H}^i(G, M) \rightarrow \hat{H}^{i+q}(G, M)$$

is an isomorphism for each $i \in \mathbb{Z}$, and for each $\mathbb{Z}G$-module M. This isomorphism is natural with respect to change of modules [7, p. 441]. By Corollary 2, there is a natural isomorphism between the factors $H^n(G, -)$ and $H^{n+q}(G, -)$ for $n \geq k+1$. There are two possibilities:

(1) Assume that (i) holds and $\mathbf{cd}(G) = 0$. Then, by Proposition 1.9 in [14], G is a finite group with periodic Tate cohomology of period q.

(2) Assume that (i) holds and $\mathbf{cd}(G) > 0$. Then, by Proposition 1.8 in [14], G has periodic cohomology after k-steps.

Assume, now, that (ii) holds. Then the sequence (*) yields an exact sequence

$$(**) \qquad 0 \rightarrow R \rightarrow P_{k+q-1} \rightarrow \cdots \rightarrow P_k \rightarrow R \rightarrow 0\,.$$

By splicing together from both sides the sequence (**), we get a complete resolution of $\mathbb{Z}$. It is part of a projective resolution in dimensions higher than $k-1$. Then, by Proposition 15 in [7], $\mathbf{cd}(G) \leq k$. Therefore, the Farrell cohomology of G is defined. It remains to be proven that G has periodic Farrell cohomology. Notice that the exact sequence (*) yields an exact sequence:

$$(***) \qquad 0 \rightarrow R \rightarrow P_{k-1} \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0\,.$$

By breaking the above sequence in short exact sequences and using the fact that $\hat{H}^i(G, P) = 0$ for all $i \in \mathbb{Z}$ and all projective $\mathbb{Z}G$-modules P, we see that the iterated coboundary map

$$\delta : \hat{H}^i(G, \mathbb{Z}) \rightarrow \hat{H}^{i+k}(G, R)$$

is an isomorphism for all $i \in \mathbb{Z}$. The same argument applied to (**) shows that the iterated coboundary map

$$\delta' : \hat{H}^i(G, R) \rightarrow \hat{H}^{i+q}(G, R)\,,$$

is an isomorphism for all $i \in \mathbb{Z}$. Combining δ and δ', we get an isomorphism

$$\hat{H}^i(G,\mathbb{Z}) \xrightarrow{\delta} \hat{H}^{i+k}(G,R) \xrightarrow{\delta'} \hat{H}^{i+k+q}(G,R) \xrightarrow{\delta^{-1}} \hat{H}^{i+q}(G,\mathbb{Z}) .$$

Denote this isomorphism by Δ. Since Δ is a composition of coboundary maps and their inverses, Property $2'$ in [7, p. 442] states that Δ satisfies

$$\Delta(a \cup b) = \Delta(a) \cup b .$$

If we set $a = 1 \in \hat{H}^0(G,\mathbb{Z})$, then we get that

$$\Delta(b) = \Delta(1 \cup b) = \Delta(1) \cup b ,$$

for all $b \in \hat{H}^*(G,\mathbb{Z})$. So the map Δ is given as a cup product with the element $\Delta(1) = u \in \hat{H}^q(G,\mathbb{Z})$. In particular, the map

$$\Delta : \hat{H}^{-q}(G,\mathbb{Z}) \to \hat{H}^0(G,\mathbb{Z})$$

is an isomorphism. So there is an element $v \in \hat{H}^{-q}(G,\mathbb{Z})$ such that $\Delta(v) = 1 \Rightarrow u \cup v = 1$. Therefore, u is a unit, and G has periodic Farrell cohomology. □

Remarks. (i) If $G \in C_\infty$, then $\mathbf{cd}(G) \leq k$ and G has periodic Farrell cohomology if and only if G has periodic cohomology after k-steps.

(ii) If G is a group with $\mathbf{cd}(G) = 0$, $\mathbb{Z}$ admits a complete resolution, and G has periodic Farrell cohomology, then G is a finite group.

Let $C_n(1)$ be the subclass of C_n consisting of those groups for which the admissible G-complex can be chosen to be a tree. As before, $C_\infty(1) = \bigcup_{n \in \mathbb{N}} C_n(1)$. Let G belong to $C_\infty(1)$ and let X be a tree which is an admissible G-complex. We can assume that G acts on X without inversions. If $Y = X/G$, then G is isomorphic to the fundamental group of a graph of groups $(\mathbf{G}, Y)$ [13, Chapter I, Section 5]. Notice also that the set of vertices (edges) of X mod G is equal to the set of vertices (edges) of the graph Y. For σ a cell in Y, G_σ denotes the isotropy group of a lifting of the cell σ in X.

Lemma 4. *Let G be a group. Then the following are equivalent*:

(1) $G \in C_\infty(1)$.

(2) *G is the fundamental group of a graph of groups* $(\mathbf{G}, Y)$, *where* $\sup\{\mathbf{cd}(G_v) \mid v$ *a vertex of* $Y\} < \infty$.

Proof. Assume that $G \in C_\infty(1)$. Since

$$\mathbf{cd}(G) \le \max\{\sup\{\mathbf{cd}(G_v) \mid v \text{ a vertex of } Y\} + \sup\{\mathbf{cd}(G_e) + 1 \mid e \text{ an edge of } Y\}\} < \infty\,,$$

it follows that $\sup\{\mathbf{cd}(G_v) \mid v \text{ a vertex of } Y\} < \infty$.

Conversely, let G be the fundamental group of a graph of groups $(\mathbf{G}, Y)$ and $\sup\{\mathbf{cd}(G_v) \mid v \text{ a vertex of } Y\} < \infty$. Choose a maximal tree T and an orientation A of Y. Then G acts on a tree X which is the universal covering of $(\mathbf{G}, Y)$ relative to T and A [13, Chapter I, Section 5]. Since $\sup\{\mathbf{cd}(G_v) \mid v \text{ a vertex of } Y\} < \infty$,

$$\mathbf{cd}(G) \le \max\{\sup\{\mathbf{cd}(G_v) \mid v \text{ a vertex of } Y\} + \sup\{\mathbf{cd}(G_e) + 1 \mid e \text{ an edge of } Y\}\} < \infty$$

and therefore G belongs to $C_\infty(1)$. $\square$

Let $G \in C_\infty(1)$, X a tree which is an admissible G-complex, $Y = X/G$ and A an orientation of Y. For each $e \in A$, there are monomorphisms

$$i_{e,o} : G_e \to G_{o(e)} \quad \text{and} \quad i_{e,t} : G_e \to G_{t(e)}\,.$$

Assume that the groups G_v have periodic Farrell cohomology of period q for each $v \in \text{ver } Y$. A collection $\{g_v \mid v \in \text{ver } Y\}$ of units in $\hat{H}^q(G_v, \mathbb{Z})$ is called compatible if

$$i^*_{e,o}(g_{o(e)}) = i^*_{e,t}(g_{t(e)}) \quad \text{for each } e \in A\,.$$

In [3], Theorem 1 states that there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \bigoplus_{e \in A} \mathbb{Z}[G/G_e] \to \bigoplus_{v \in \text{ver } Y} \mathbb{Z}[G/G_v] \to \mathbb{Z} \to 0\,.$$

If M is $\mathbb{Z}G$-module we have an exact sequence

$$0 \to M \to \mathrm{Hom}_{\mathbb{Z}}\Big(\bigoplus_{v \in \text{ver } Y} \mathbb{Z}[G/G_v], M\Big) \to \mathrm{Hom}_{\mathbb{Z}}\Big(\bigoplus_{e \in A} \mathbb{Z}[G/G_e], M\Big) \to 0\,.$$

This induces a long exact sequence in Farrell cohomology:

$$\text{(SE)} \qquad \cdots \to \hat{H}^i(G, M) \to \hat{H}^i\Big(G, \prod_{v \in \text{ver } Y} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/G_v], M)\Big) \to \hat{H}^i\Big(G, \prod_{e \in A} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/G_e], M)\Big) \to \hat{H}^{i+1}(G, M) \to \cdots$$

Then, as in [1, Section 2.4], we obtain an exact sequence

$$\text{(E)} \qquad \cdots \to \hat{H}^i(G, M) \xrightarrow{\alpha} \prod_{v \in \text{ver } Y} \hat{H}^i(G_v, M)$$
$$\xrightarrow{\rho} \prod_{e \in A} \hat{H}^i(G_e, M) \xrightarrow{\partial} \hat{H}^{i+1}(G, M) \to \cdots$$

where α is given by the product of the restriction maps, ∂ is given as the composition of the coboundary map of the sequence (SE) and the product of isomorphisms given by Shapiro's Lemma, and

$$\rho\left(\prod_v f_v\right) = \prod_e \left(i^*_{e,o}(f_{o(e)}) - i^*_{e,t}(f_{t(e)})\right),$$

where $f_v \in \hat{H}^i(G_v, M)$. This is proven, for ordinary cohomology, also in [3]. Notice that a collection of units $\{g_v \mid v \in \text{ver } Y\}$ is compatible if and only if

$$\rho\left(\prod_v g_v\right) = 0 .$$

Proposition 5. *Let G be a group of class $C_\infty(1)$, X be a tree which is an admissible G-complex and $Y = X/G$ a graph with orientation A. Then the following are equivalent*:

(i) *G has periodic Farrell cohomology of period q.*

(ii) *for each $v \in \text{ver } Y$, the isotropy group G_v of v has period q and there is a compatible family $\{g_v \mid v \in \text{ver } Y\}$ of units in $\hat{H}^q(G_v, \mathbb{Z})$.*

Proof. We follow the proof of the corresponding result for ordinary cohomology given in [15, Theorem 2.1]. We first assume that G has periodic Farrell cohomology of period q. Then there is a unit $u \in \hat{H}^q(G, \mathbb{Z})$. Set $u_v = \text{res}^G_{G_v}(u) \in \hat{H}^q(G_v, \mathbb{Z})$ for each $v \in \text{ver } Y$. Then $\{u_v \mid v \in \text{ver } Y\}$ is a collection of units. But this collection belongs to the Im α and from the exact sequence (E) its image under ρ is zero and therefore it is a compatible family of units.

We assume now that (ii) holds. Let $\{u_v \mid v \in \text{ver } Y\}$ be a compatible collection of units. Then

$$\rho\left(\prod_v u_v\right) = 0$$

and, from the exact sequence (E) there is $u \in \hat{H}^q(G, \mathbb{Z})$ so that $u_v = \text{res}^G_{G_v}(u)$ for each $v \in \text{ver } Y$. It remains to be proven that u is a unit in $\hat{H}^q(G, \mathbb{Z})$. From the naturality properties of the exact sequence (E), we obtain a commutative diagram:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \hat{H}^i(G,M) \arrow[r, "\alpha"] \arrow[d, "\cup u"] & \prod_{v\in \mathrm{ver}\, Y} \hat{H}^i(G_v,M) \arrow[r, "\rho"] \arrow[d, "\prod_v(\cup u_v)"] & \prod_{e\in A} \hat{H}^i(G_e,M) \arrow[r, "\partial"] \arrow[d, "\prod_e(\cup u_v)"] & \hat{H}^{i+1}(G,M) \arrow[r] \arrow[d, "\cup u"] & \cdots \\
\cdots \arrow[r] & \hat{H}^{i+q}(G,M) \arrow[r, "C\alpha"] & \prod_{v\in \mathrm{ver}\, Y} \hat{H}^{i+q}(G_v,M) \arrow[r, "\rho"] & \prod_{e\in A} \hat{H}^{i+q}(G_e,M) \arrow[r, "\partial"] & \hat{H}^{i+q+1}(G,M) \arrow[r] & \cdots
\end{tikzcd}$$

By the Five-Lemma $\cup u$ is an isomorphism for each i and for each $\mathbb{Z}G$-module M. So, u is a unit and G has periodic Farrell cohomology. □

Remark. In [15], an example is given of a group of the class $C_1(1)$ for which all the vertex groups are cyclic but it does not have periodic cohomology after 1-step and so it does not have periodic Farrell cohomology.

Corollary 6. (i) *A countable locally finite group has periodic Farrell cohomology of period q if and only if each finite subgroup has periodic cohomology of period q.*

(ii) *Let G be the fundamental group of a graph of groups $(\mathbf{G}, Y)$. Let A be an orientation of Y. Assume that $G_v = H$ for each $v \in \mathrm{ver}\, Y$, $\mathbf{cd}(H) < \infty$, $\mathbb{Z}$ admits a complete resolution over H, and the inclusions $i_{e,o} : G_e \to G_{o(e)}$ and $i_{e,t} : G_e \to G_{t(e)}$ are equal for each $e \in A$. Then G has periodic Farrell cohomology if and only if H has periodic Farrell cohomology.*

Proof. (i) This is [15, Theorem 3.7] together with Proposition 3 above.

(ii) Notice that G acts on a tree X with $G_v = H$ for each vertex v of X, and $\mathbf{cd}(H) < \infty$. By Lemma 4, we see that G is of class $C_\infty(1)$. In particular, we can define Farrell cohomology for G. Let $\{u_{o(E)}, u_{t(E)}\}$ be a collection of units $u_{o(E)}, u_{t(E)} \in \hat{H}^q(H, \mathbb{Z})$. If we choose $u_{o(E)} = u_{t(E)}$, then the collection is compatible because the two maps $i^*_{e,o}$ and $i^*_{e,t}$ are equal. By Proposition 5, G has periodic Farrell cohomology.

If G has periodic Farrell cohomology, then by Proposition 5, H has also periodic Farrell cohomology. □

Remark. A special case of Corollary 6 is the following:

Let H be a group with $\mathbf{cd}(G) < \infty$ and such that $\mathbb{Z}$ admits a complete resolution over H. Let S be a subgroup of H, and $G = H *_S H$ the amalgamated free product where the two monomorphisms from S to H are equal. Then G has periodic Farrell cohomology if and only if H does.

3. Free actions $\mathbb{R}^m \times S^{n-1}$

In this section, we will show that there are groups of class $C_\infty(1)$ which act freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$ and have infinite virtual cohomological dimension. We recall some terminology from [4]. For a group G of class C_∞, let $x \in H^n(G, \mathbb{Z})$ be such that $\iota(x) \in \hat{H}^n(G, \mathbb{Z})$ is a unit. By an x-polarized space we mean a free G-complex E and a class of homotopy equivalences $E \simeq S^{n-1}$ so that G acts trivially on $H^*(G, \mathbb{Z})$, E/G has the homotopy type of a countable finite dimensional complex, and the Euler class of the spherical fibration $EG \times_G E \to BG$ is x.

Remarks. (i) In [16], it was proven that, if G is a finite group, for each unit $u \in \hat{H}^n(G, \mathbb{Z})$, $n > 0$, there is a u-polarized space.

(ii) In [4], it was shown that if G is a group with $\mathrm{vcd}(G) < \infty$, and $x \in H^n(G, \mathbb{Z})$ be such that $\iota(x) \in \hat{H}^n(G, \mathbb{Z})$ is a unit, then there is a $(x^k + v)$-polarized space, for some $k \geq 1$ and $v \in H^*(G, \mathbb{Z})$ is nilpotent.

The first result is about groups of class $C_1(1)$.

Proposition 7. *Let G be a group of class $C_1(1)$ and X a tree which is an admissible G-complex. Let $x \in H^n(G, \mathbb{Z})$ be an element such that $\iota(x) \in \hat{H}^n(G, \mathbb{Z})$, $n > 2$, is a unit. Then there is a free G-complex E and a G-map $\pi : E \to X$ such that*:

(i) *$E \simeq S^{n-1}$ and G acts trivially on $H^*(G, \mathbb{Z})$.*

(ii) *π is a Hurewicz fibration.*

(iii) *The Euler class of the spherical fibration $E \to EG \times_G E \to BG$ is $x + v$, where $v \in H^n(G, \mathbb{Z})$ is nilpotent.*

Proof. The proof is similar to the proof given in [4, Proposition 2.4]. Set $u = \iota(x)$. Inductively, we construct a spherical fibration over the skeleta of the tree X. Let $v \in \mathrm{ver}\, X$ and $G_v = H$ its isotropy group. Since H is finite, $u_H = \mathrm{res}^G_H(u)$ is a unit in the Tate cohomology of H and by [16] there is a (u_H)-polarized space Y_H. Set $E|_v = Y_H$. We can do this for a complete set of orbit representatives of the action of G on $\mathrm{ver}\, X$ and then define for a vertex v with isotropy group H, $E|_{Gv} = G \times_H (E|_v)$ and $\pi : E|_{Gv} \to Gv$, $\pi(g, e) = gv$, if $(g, e) \in G \times_H (E|_v)$. This construction produces a free G-space $E|_{\mathrm{ver}\, X}$ and a G-map which is a spherical fibration $\pi : E|_{\mathrm{ver}\, X} \to \mathrm{ver}\, X$. The fiber over a vertex v with isotropy group H is a $(\mathrm{res}^G_H(u))$-polarized space.

This fibration can be extended over the edges as follows: Let e be an edge with isotropy group S. Then $E|_{\partial e}$ is an S-spherical fibration over the trivial S-space ∂e and the fiber is a $(\mathrm{res}^G_S(u))$-polarized space. There are no obstructions for extending S-equivariantly this fibration across e, see [4, Proposition 2.4]. Define $E|_{Ge} = E|_e \times_S G$.

This completes the construction of *a* the spherical fibration π. Notice that E is a free G-space and $E \simeq S^{n-1}$. For each finite subgroup H, the Euler class of the fibration

$$E \to EH \times_H E \to BH$$

is $\mathrm{res}_H^G(x)$. Hence the Euler class of the fibration

$$E \to EG \times_G E \to BG$$

is $x + v$, where $\mathrm{res}_H^G(v) = 0$ for all subgroups H which appear as isotropy groups of the G action on X. In particular, property (viii) of the spectral sequence (S) implies that $\iota(v)$ is nilpotent, and therefore v is nilpotent. □

For the next result, we make an assumption about the groups we are going to study. A group G satisfies the property (P) if:

(P) G is of class C_∞ and for each $x \in H^n(G, \mathbb{Z})$ such that $\iota(x) \in \hat{H}^n(G, \mathbb{Z})$ is a unit, there is an $(x^k + v)$-polarized space, where $v \in H^n(G, \mathbb{Z})$ is nilpotent and $k \geq 1$,

Remark. The remark above shows that finite groups and groups with finite virtual cohomological dimension satisfy property (P).

Let G be a group of class $C_\infty(1)$. Let X be a tree which is an admissible G-complex such that there is a group H such that G_v is isomorphic to H for each $v \in \mathrm{ver}\, X$. From the proof of Corollary 6(ii), there is a unit $u \in \hat{H}^n(G, \mathbb{Z})$, $n > 2$, such that $\mathrm{res}_{G_v}^G(u) = u_H$ for all vertices v of X.

Proposition 8. *Let G, X, H, u be as above. Assume that H satisfies property* (P). *Let $x \in H^n(G, \mathbb{Z})$ be an element such that $\iota(x) = u \in \hat{H}^n(G, \mathbb{Z})$, $n > 2$. Then there is a free G-complex E and a G-map $\pi : E \to X$ such that*:

(i) *$E \simeq S^{n-1}$ and G acts trivially on $H^*(G, \mathbb{Z})$.*

(ii) *π is a Hurewicz fibration.*

(iii) *The Euler class of the spherical fibration $E \to EG \times_G E \to BG$ is $x^k + v$, where $k \geq 1$ and $v \in H^n(G, \mathbb{Z})$ is nilpotent.*

Proof. The proof is similar to the proof of Proposition 7. Inductively, we construct a spherical fibration over the skeleta of the tree X. Let $v \in \mathrm{ver}\, X$ and $G_v = H$ its isotropy group. Then $u_H = \mathrm{res}_H^G(u)$ is a unit in the Farrell cohomology of H. Let Y_H be an $(\mathrm{res}_H^G(x^k) + v)$-polarized space, where $v \in H^n(G, \mathbb{Z})$ is nilpotent and $k \geq 1$. This way we can define E over the ver X as in Proposition 7. This construction produces a free G-space $E|_{\mathrm{ver}\, X}$ and a G-map which is a spherical fibration $\pi : E|_{\mathrm{ver}\, X} \to \mathrm{ver}\, X$ and the fiber over any vertex v is Y_H.

This fibration can be extended over the edges as follows: Let e be an edge with isotropy group S. Then $\partial e = \{v, w\}$, $E|_{\partial e}$ is an S-spherical fibration over the trivial S-space ∂e and the fiber is Y_H. Define $E|_e = Y_H \times [0, 1]$, where S acts on Y_H by the restriction of the action of H on Y_H, and trivially on $[0, 1]$. Extend $E|_e$ over Ge by $E|_{Ge} = E|_E \times_S G$.

As before, E is a free G-space and $E \simeq S^{n-1}$. For the subgroup H, the Euler class of the fibration

$$E \to EH \times_H E \to BH$$

is $\operatorname{res}_H^G(x^k) + v$. If y is the Euler class of the fibration

$$E \to EG \times_G E \to BG\,,$$

then $\operatorname{res}_H^G(y) = \operatorname{res}_H^G(x^k) + v \Rightarrow \operatorname{res}_H^G(y - x^k)$ is nilpotent $\Rightarrow \iota((y - x^k)^m)$ is the kernel of res_H^G for all subgroups H which appear as isotropy groups of vertices of X. Therefore, $\iota((y - x^k)^m)$ is nilpotent and $y - x^k$ is nilpotent. So $y = x^k + \mu$ where μ is nilpotent. □

Let G be as in Propositions 7 or 8. Assume further that G is countable and the tree X on which G acts is countable.

Lemma 9. *Let G be as above. Then the free complex E constructed in Propositions 7 and 8 can be chosen so that E/G has the homotopy type of a countable finite-dimensional simplicial complex.*

Proof. Let $p : E/G \to X/G$ be the map induced by π. By construction $p^{-1}(\operatorname{ver}(X/G))$ is a countable complex. Let e be an 1-simplex of X/G of orbit type H. Then $p^{-1}(e) \simeq Y/H$, where Y is an $(\operatorname{res}_H^G(x^k) + v)$-polarized space, with $v \in H^n(G, \mathbb{Z})$ nilpotent and $k \geq 1$. Also $p^{-1}(\partial e)$ is H-homotopy equivalent to $Y/H \times \partial e$. Then Y/H and $Y/H \times \partial e$ have the homotopy type of countable simplicial complexes of dimension less than or equal to kn, and

$$E/G \simeq (p^{-1}(\operatorname{ver} X/G)) \cup \Big(\bigcup_{e \in E(X/G)} (Y_e/G_e \times e)\Big)\,,$$

where Y_e is as above. The coproduct is taken over the 1-simplices of X/G and the attaching takes place along $Y_e/G_e \times \partial e$. So E/G is homotopy equivalent to a countable complex. The details appear in the proof of Lemma 2.5 in [4].

The Euler class of the fibration

$$E \to EG \times_G E \to BG$$

is $x^k + v$, where v is nilpotent. Therefore, $\iota(x^k + v) = \iota(x^k) + \iota(v)$ is a unit. So, in the Gysin cohomology sequence of the spherical fibration, the map given by cup product with $(x^k + v)$ is an isomorphism in high enough dimensions. Therefore, $H^i(E/G, B) = 0$ for i large and for each $\mathbb{Z}G$-module B.

The proof of the lemma is completed as in Lemma 2.6 in [4]. □

If a group G acts freely and simplicially on a simplicial complex so that the orbit space has the homotopy type of a countable finite-dimensional simplicial complex, then standard methods produce a free and properly discontinuous action of G on

$\mathbb{R}^m \times S^{kn-1}$ (see, for example, [4, Lemma 2.8 and Proof of Theorem A]). We summarize in the following theorem:

Theorem 10. *Let G be a countable group which satisfies the assumptions of Proposition 7 or 8. Assume, also, that the admissible G-complex can be chosen to be a countable tree. Then for each $x \in H^n(G, \mathbb{Z})$ such that $\iota(x) \in \hat{H}^n(G, \mathbb{Z})$ is a unit there is a free and properly discontinuous action of G on $\mathbb{R}^m \times S^{kn-1}$, for some m and n, so that $H_*(\mathbb{R}^m \times S^{kn-1}, \mathbb{Z})$ is a trivial $\mathbb{Z}G$-module and the Euler class of the spherical fibration*

$$\mathbb{R}^m \times S^{kn-1} \to EG \times_G \mathbb{R}^m \times S^{kn-1} \to BG$$

is $x^k + v$, where $v \in H^*(\mathbb{R}^m \times S^{kn-1}, \mathbb{Z})$ is nilpotent. □

Using Theorem 10, we construct actions of groups with infinite virtual cohomological dimension on $\mathbb{R}^m \times S^{kn-1}$.

(1) Let G be a countable locally finite group with period Farrell cohomology. In [15], it is proven that G is either locally cyclic or locally quartenionic group. In [7], a countable tree is constructed which is an admissible G-complex. So G is of class $C_1(1)$. Then Theorem 10 asserts that G acts freely and properly discontinuously on $\mathbb{R}^m \times S^{n-1}$ for some m and n. But G is a countable torsion group and $\mathrm{vcd}(G) = \infty$.

(2) The next example is constructed as follows [11, 3.1, 3.2]: Let A be the Higman group

$$A = \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle$$

and B an infinite cyclic subgroup of A. Define $C = A *_B A$ the amalgamated product, where the two monomorphisms from B to A are equal to the inclusion map. It turns out that A and C are duality groups, $H^2(C, \mathbb{Z}/k\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$ and $H^2(A, \mathbb{Z}/k\mathbb{Z}) = 0$ [11]. Let G be given by an extension

$$0 \to \mathbb{Z}/k\mathbb{Z} \to G \to C \to 0\,,$$

which represents a nonzero element in $H^2(C, \mathbb{Z}/k\mathbb{Z})$ and therefore the exact sequence does not split. Then G is not a virtually torsion free group [11, Theorem 1] and therefore $\mathrm{vcd}(G) = \infty$. If $p : G \to C$ is the epimorphism in the exact sequence above, then

$$G = H *_S H\,, \quad \text{where } H = p^{-1}(A)\,,\ S = p^{-1}(B)\,.$$

Notice that H and S of finite virtual cohomological dimension and have periodic Farrell cohomology since each finite subgroup is cyclic [2, Chapter X, Theorem

6.7]. Therefore, H and S satisfy property (P) [4, Theorem A]. By [13], G acts on a tree X with fundamental domain a single edge e with $G_{o(e)} = G_{t(e)} = H$ and $G_e = S$. In particular, X is countable, G is of class $C_\infty(1)$ and Corollary 6(ii) shows that G has periodic Farrell cohomology. So G satisfies the assumptions of Proposition 8 and acts on a countable tree. Therefore, Theorem 10 applies to G, and G acts freely and properly discontinuously on $\mathbb{R}^m \times S^{kn-1}$ for some m, n, and k.

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