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On the lower Nil-groups of Waldhausen

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Abstract. Let $\Gamma = \Gamma_0 *_G \Gamma_1$ be an amalgamated free product, where G is a finitely generated central subgroup of Γ_0 and Γ_1. We show that the negative Waldhausen Nil-groups that appear in the calculation of the K-theory of $\mathbb{Z}\Gamma$ vanish. If $G = H \times T^m$ is a decomposition of G with H a finite group and T the infinite cyclic group, we also show that the exponent of the NK_0-group depends on the order of H.

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1 Introduction

Let Γ be the amalgamated free product $\Gamma_0 *_G \Gamma_1$, where G is a subgroup of Γ_i, $i = 0, 1$. In calculating the K-theory of the integral group ring $\mathbb{Z}\Gamma$, we encounter the difficult task of calculating certain exotic Nil-groups ([19]).

In the sequel, we will outline certain methods for calculating the Nil-groups which appear in the computation of $K_i(\mathbb{Z}\Gamma)$ for $i \leq 0$ in the case that G is a central finitely generated subgroup of Γ_i ($i = 0, 1$). In particular, G is of the form $H \times T^m$ with H a finite abelian group and T an infinite cyclic group.

Main Theorem. *Let G be a finitely generated central subgroup of Γ_i, $i = 0, 1$. Then*

(1) $NK_j(\mathbb{Z}G; \mathbb{Z}[\Gamma_0 - G], \mathbb{Z}[\Gamma_1 - G]) = 0$, *for* $j \leq -1$.

(2) $NK_0(\mathbb{Z}G; \mathbb{Z}[\Gamma_0 - G], \mathbb{Z}[\Gamma_1 - G])$ *has exponent a power of the order of* H.

An immediate application of the Main Theorem is connected to a classical conjecture about the vanishing of the lower K-groups of integral group rings. More precisely,

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Vanishing Conjecture. *Let* Γ *be a discrete group. Then* $K_i(\mathbb{Z}\Gamma) = 0$ *for* $i \leq -2$.

The Vanishing Conjecture has been proved for all subgroups of cocompact discrete subgroups of Lie groups ([8], [9]) and for the groups of the form $\pi_1(K) \times \mathbb{Z}^n$ where K is a finite complex of nonpositive curvature ([12]). More examples of groups that satisfy the Vanishing Conjecture can be found in [4] and [1].

Using the conclusion of the Main Theorem, we can extend the class of groups for which the Vanishing Conjecture is true.

Theorem. *Let* Γ_i, $i = 0, 1$, *be two groups for which the Vanishing Conjecture is true, and let* G *be a finitely generated central subgroup of* Γ_i $(i = 0, 1)$. *Then the group* $\Gamma_0 *_G \Gamma_1$ *satisfies the Vanishing Conjecture.*

As a first application the theorem provides an algebraic proof for the vanishing of lower K-groups of an abelian virtually infinite cyclic group Γ that admits an epimorphism (with finite kernel) to the infinite dihedral group D_∞ ([9]).

Another application comes from combining the two classes of groups that were mentioned above. Let Γ_0 be a discrete subgroup of a discrete cocompact subgroup of a Lie group which is not torsion free and contains $\mathbb{Z}$ as a central subgroup. Let Γ_1 be a torsion free cocompact subgroup of $\mathrm{SL}_n(\mathbb{Q}_{(p)})$, where $\mathbb{Q}_{(p)}$ is the p-adic field. Then the Main Theorem implies that the group $\Gamma = \Gamma_0 *_{\mathbb{Z}} (\Gamma_1 \times \mathbb{Z})$ satisfies the Vanishing Conjecture and Γ does not belong to any of the two classes mentioned above. More complicated examples can be obtained by repeating the above construction.

The K-theory of amalgamated free products of groups has been studied in [17], [18], [19]. Amalgamated free products are pushouts in the category of groups. The failure of the Mayer-Vietoris sequence to be exact in this case is measured by certain Nil-groups. The Nil-groups in this case have been defined in [18], [19], and [7]. In [7], the Nil-groups were defined using twisted extensions of additive categories. We study the functorial properties of the Nil-groups using the description in [7]. Using the functorial properties of the Nil-groups we show that the lower Nil-groups vanish in the setting of the Main Theorem. The method used for proving the vanishing result is based in the methods used in [5] and [9] for proving the vanishing of the lower K-groups for finite groups and virtually infinite cyclic groups, respectively.

The calculation of the exponent of the NK_0-group uses a modification of the methods that were developed in [6].

The methods developed in this paper work also for HNN-extensions of groups. The relevant category theory could be developed as in [7] and a vanishing result will follow exactly as in the amalgamated free product case.

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2 Review of the twisted polynomial extensions of additive categories

All the rings have identity, unless it is mentioned otherwise, and the property that finitely generated free modules have well-defined rank. All ring homomorphisms preserve the identity.

We shall review certain basic constructions on rings and bimodules. Let R be a ring and B an $R-R$-bimodule. We write $T_R(B)$ for the tensor R-algebra defined by B. The algebra $T_R(B)$ is an augmented graded algebra which admits a decomposition as an R-bimodule

$$T_R(B) = R \oplus B \oplus (B \otimes_R B) \oplus \cdots.$$

The multiplication is given by concatenation.

For any ring R, $\mathscr{M}_R$ denotes the category of right R-modules, $\mathscr{P}_R$ the subcategory of finitely generated projective right modules, and $\mathscr{F}_R$ the subcategory of finitely generated right free R-modules. For $\mathscr{A} = \mathscr{M}, \mathscr{P}, \mathscr{F}$, $\mathscr{A}_R^n$ denotes the product category $\mathscr{A}_R \times \mathscr{A}_R \times \cdots \times \mathscr{A}_R$ (n times). Let $\mathscr{T}$ be the category with objects triples $\mathbf{R} = (R; B_0, B_1)$ where R is a ring with unit and B_i, $i = 0, 1$, are R-bimodules. A morphism $(f, \phi_0, \phi_1) : (R; B_0, B_1) \to (S; C_0, C_1)$ is a triple where $f : R \to S$ is a unit preserving ring homomorphism, and $\phi_i : B_i \otimes_R S \to C_i$ is an $R-S$-bimodule homomorphism for $i = 0, 1$ (the R-module structure on C_i is induced by the map f). Let

$$(R; B_0, B_1) \xrightarrow{(f, \phi_0, \phi_1)} (S; C_0, C_1) \xrightarrow{(g, \psi_0, \psi_1)} (T; D_0, D_1)$$

be two morphisms in $\mathscr{T}$. Their composition is the morphism

$$(R; B_0, B_1) \xrightarrow{(gf, \psi_0(\phi_0 \otimes 1_T), (\psi_1(\phi_1 \otimes 1_T))} (T; D_0, D_1).$$

Remark 2.1. Let $\mathbf{R} = (R; B_0, B_1)$ be an object in $\mathscr{T}$ and $f : R \to S$ be a unit preserving ring homomorphism. Then f induces a morphism in $\mathscr{T}$

$$[f] = (f, \phi_0, \phi_1) : (R; B_0, B_1) \to (S; S \otimes_R B_0 \otimes_R S, S \otimes_R B_1 \otimes_R S)$$

where, for $i = 0, 1$, $\phi_i : B_i \otimes_R S \to S \otimes_R B_i \otimes_R S$, is defined by $\phi_i(b \otimes s) = 1 \otimes b \otimes s$. The construction is natural. Let $\mathscr{R}$ be the category with objects (S, f) where S is a ring and $f : R \to S$ is a ring homomorphism and morphisms given by ring homomorphisms making the corresponding diagrams commutative. Then the above construction induces a functor $[\mathbf{R}] : \mathscr{R} \to \mathscr{T}$.

Main Construction. We shall define functors

$$\mathbb{F}_{\mathscr{A}} : \mathscr{T} \to \mathscr{A}dd$$

for $\mathscr{A} = \mathscr{P}$ or $\mathscr{F}$, where $\mathscr{A}dd$ is the category of additive categories. The construction is construction 2.1 of [7].

Description of $\mathbb{F}_{\mathscr{A}}$. Let $\mathbf{R} = (R; B_0, B_1)$ be an object of $\mathscr{T}$ as above. Then there is a functor $\alpha_R : \mathscr{M}_R^2 \to \mathscr{M}_R^2$ defined by

$$\alpha_R(M_0, M_1) = (M_1 \otimes_R B_0, M_0 \otimes_R B_1)$$

$$\alpha_R(f_0, f_1) = (f_1 \otimes 1, f_0 \otimes 1).$$

Then $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$ is the twisted polynomial extension construction on $\mathscr{A}_R^2$ ([7]). More precisely, the objects of $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$ are the same as the objects of $\mathscr{A}_R^2$ and

$$\mathbb{F}_{\mathscr{A}}(\mathbf{R})(u, v) = \sum_{i=0}^{\infty} \mathscr{M}_R^2(u, \alpha_R^i(v)) = \sum_{i=0}^{\infty} p_i t^i$$

where we write $p_i : u \to \alpha_R^i(v)$ for the i-th component of the morphism. Let $m = (f, \phi_0, \phi_1) : (R; B_0, B_1) \to (S; C_0, C_1)$ be a morphism in $\mathscr{T}$. We shall construct a functor $\mathbb{F}_{\mathscr{A}}(m) : \mathbb{F}_{\mathscr{A}}(\mathbf{R}) \to \mathbb{F}_{\mathscr{A}}(\mathbf{S})$ between additive categories:

If $u = (F_0, F_1)$ is an object in $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$ then

$$\mathbb{F}_{\mathscr{A}}(m)(F_0, F_1) = (F_0 \otimes_R S, F_1 \otimes_R S).$$

For the construction of $\mathbb{F}_{\mathscr{A}}$ on morphisms we first note that, for any object $v = (G_0, G_1)$ of $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$, we can define a morphism m_1 in $\mathscr{M}_S^2$ between $\mathbb{F}_{\mathscr{A}}(m)(\alpha_R(v))$ and $\alpha_S(\mathbb{F}_{\mathscr{A}}(m)(v))$, as follows.

$$\begin{aligned}\mathbb{F}_{\mathscr{A}}(m)(\alpha_R(v)) &= \mathbb{F}_{\mathscr{A}}(m)(G_1 \otimes_R B_0, G_0 \otimes_R B_1) \\ &= ((G_1 \otimes_R B_0) \otimes_R S, (G_0 \otimes_R B_1) \otimes_R S) \\ &\xrightarrow{\phi} (G_1 \otimes_R C_0, G_0 \otimes_R C_1) \\ &\simeq ((G_1 \otimes_R S) \otimes_S C_0, (G_0 \otimes_R S) \otimes_S C_0) \\ &= \alpha_S(\mathbb{F}_{\mathscr{A}}(m)(v)),\end{aligned}$$

where $\phi = (1_{F_1} \otimes \phi_0, 1_{F_0} \otimes \phi_1)$. Repeating the above process we construct a morphism m_i in $\mathscr{M}_S^2$ from $\mathbb{F}_{\mathscr{A}}(m)(\alpha_R^i(v))$ to $\alpha_S^i(\mathbb{F}_{\mathscr{A}}(m)(v))$, for all $i \geq 0$. For a morphism

$$\sum_{i=0}^{\infty} p_i t^i \in \mathbb{F}_{\mathscr{A}}(u, v),$$

where $p_i : u \to \alpha_R^i(v)$, define

$$\mathbb{F}_{\mathscr{A}}(m)\left(\sum_{i=0}^{\infty} p_i t^i\right) = \sum_{i=0}^{\infty} (m_i \circ \mathbb{F}_{\mathscr{A}}(m)(p_i)) t^i,$$

where $\mathbb{F}_{\mathscr{A}}(m)(p_i) : \mathbb{F}_{\mathscr{A}}(m)(u) \to \mathbb{F}_{\mathscr{A}}(m)(\alpha_R^i(v))$ is given by $p_i \otimes 1_S$.

Remark 2.2. There are some immediate observations arising from the definition.

1. The operation "$t = 0$" induces a forgetful natural transformation

$$\eta_{\mathscr{A}}(\mathbf{R}) : \mathbb{F}_{\mathscr{A}}(\mathbf{R}) \to \mathscr{A}_R^2.$$

   Equivalently, the functor $\eta_{\mathscr{A}}(\mathbf{R})$ is induced by the morphism $(R; B_0, B_1) \xrightarrow{(1;0,0)} (R; 0, 0)$ of objects of $\mathscr{T}$.

2. The different choices of $\mathscr{A}$ are connected by a forgetful natural transformation

$$\psi : \mathbb{F}_{\mathscr{F}} \to \mathbb{F}_{\mathscr{P}}.$$

The main natural examples of such triples arise from the study of the K-theory of pushout squares of rings ([18], [19]). Let

$$\begin{array}{ccc} R & \xrightarrow{\iota_0} & A_0 \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle j_0} \\ A_1 & \xrightarrow{j_1} & S \end{array}$$

be a pushout diagram of rings, where the homomorphisms ι_i, $i = 0, 1$, are assumed to be *pure inclusions* i.e. they are inclusions and there is a splitting $A_i = \iota_i(R) \oplus B_i$ as R-bimodules. In this case, the structure of S has been described in [17] and [19]. Notice that S contains the tensor algebras $T_R(B_0 \otimes_R B_1)$ and $T_R(B_1 \otimes_R B_0)$. The structure of a pushout diagram as above determines an object in $\mathscr{T}$, namely the triple $\mathbf{R} = (R; B_0, B_1)$. In this context, a functor is defined in [7]

$$r : \mathbb{F}_{\mathscr{A}}(\mathbf{R}) \to \mathscr{A}_S$$

connecting the two categories.

Let $\mathbf{R} = (R; B_0, B_1)$ be a triple in $\mathscr{T}$. Set $B_i = B_0$ for all even $i \geq 0$, $B_i = B_1$ for all odd $i > 0$, and put

$$B_i^{(j)} = B_i \otimes_R B_{i+1} \otimes_R \cdots \otimes_R B_{i+j-1}$$

for all $i, j \geq 0$. In particular, $B_i^{(0)} = R$, $B_i^{(1)} = B_i$. Similarly, if (Q_0, Q_1) is an object in $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$, we put $Q_i = Q_0$ for all even $i \geq 0$, and $Q_i = Q_1$ for all odd $i > 0$. With this notation

$$\alpha_R^i(Q_0, Q_1) = (Q_i \otimes_R B_{i+1}^{(i)}, Q_{i+1} \otimes_R B_i^{(i)}).$$

Thus, if $u = (P_0, P_1)$ and $v = (Q_0, Q_1)$ are objects in $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$, then

$$\begin{aligned}
&\mathbb{F}_{\mathscr{A}}(\mathbf{R})(u,v)\\
&\quad=\bigoplus_{i\geq 0}[\mathscr{M}_R(P_0,Q_i\otimes_R B_{i+1}^{(i)})\oplus\mathscr{M}_R(P_1,Q_{i+1}\otimes_R B_i^{(i)})]\\
&\quad=\left\{\sum_{i=0}^{\infty}(p_{(0,i)}\oplus p_{(1,i)})t^i : p_{(k,i)}\in\mathscr{M}_R(P_k,Q_{i+k}\otimes_R B_{i+k+1}^{(i)}),k=0,1\right\}.
\end{aligned}$$

The object $\rho=(R,R)$ is a basic object in $\mathbb{F}_{\mathscr{P}}(\mathbf{R})$, in the sense of Bass ([3], p. 197), i.e., each object u of $\mathbb{F}_{\mathscr{P}}(\mathbf{R})$ is isomorphic to a direct summand of $\rho^n=(R^n,R^n)$ for some integer n. We write $R_\rho=\mathrm{End}_{\mathbb{F}_{\mathscr{P}}(\mathbf{R})}(\rho)$ for the endomorphism ring of ρ. We shall give the structure of R_ρ in more detail. A morphism of degree i, $\phi=(\phi_{0,i},\phi_{1,i})t^i:\rho\to\alpha_R^i(\rho)$, can be identified with the element $(\phi_{0,i}(1),\phi_{1,i}(1))\in B_{i+1}^{(i)}\oplus B_i^{(i)}$. Multiplication in R_ρ, i.e. composition of endomorphisms, is then given by concatenation with the added convention that $B_iB_i=0$, $i=0,1$. Considering the degree mod 2 of components one obtains a natural splitting of R_ρ as an $R\times R$-bimodule

$$R_\rho=R_{\mathrm{even}}\oplus R_{\mathrm{odd}}.$$

The even component R_{even} is a subring of R_ρ which is isomorphic to the product of the tensor algebras $T_R(B_1\otimes_R B_0)\times T_R(B_0\otimes_R B_1)$, and R_{odd} is an R_{even}-bimodule. There is a split inclusion of rings $\imath:R\times R\to R_\rho$ by considering pairs of elements of R as endomorphisms of degree zero of ρ. The splitting ζ is given by the forgetful map to the zero degree component of any endomorphism.

We shall give a description of R_ρ as a "matrix ring". Define

$$R'_\rho=\begin{pmatrix} T_R(B_1\otimes_R B_0) & B_1\otimes_R T_R(B_0\otimes_R B_1)\\ B_0\otimes_R T_R(B_1\otimes_R B_0) & T_R(B_0\otimes_R B_1)\end{pmatrix},$$

with multiplication given as matrix multiplication and on each entry by concatenation. There is a split inclusion of rings $\imath':R\times R\to R'_\rho$ given by

$$\imath'(r_1,r_2)=\begin{pmatrix} r_1 & 0\\ 0 & r_2\end{pmatrix},$$

with splitting given by the natural projection ζ' to $R\times R$.

Proposition 2.3. *There is a natural ring isomorphism*

$$\kappa:R_\rho\to R'_\rho$$

such that $\kappa\circ\imath=\imath'$.

Proof. As an abelian group, $\mathrm{End}_{\mathbb{F}_{\mathscr{P}}(\mathbf{R})}(\rho)$ is generated by morphisms of degree i for all $i\geq 0$. We shall define the map κ on morphisms of degree i and extend linearly. A morphism $\phi_i \in \mathrm{End}_{\mathbb{F}_{\mathscr{P}}(\mathbf{R})}(\rho)$, of degree i is determined by a pair of elements $(b_{i+1}, b_i) \in B_{i+1}^{(i)} \times B_i^{(i)}$. Then κ is defined:

$$\kappa(\phi_i) = \begin{cases} \begin{pmatrix} b_{i+1} & 0 \\ 0 & b_i \end{pmatrix} & \text{if } i \text{ is even} \\ \begin{pmatrix} 0 & b_i \\ b_{i+1} & 0 \end{pmatrix} & \text{if } i \text{ is odd.} \end{cases}.$$

It is a straightforward calculation that κ is a ring isomorphism that commutes with the augmentation maps. □

In most calculations involving R_ρ from now on, we will represent elements of R_ρ as 2×2 matrices as in Proposition 2.3.

In [15] (§7, §8) one finds definitions of the polynomial extension and the finite Laurent extension of any additive category. The constructions are used for defining the lower K-groups of an additive category following the ideas in [2]. We shall review the basic definitions from [15]. We denote by $\mathbb{P}_0(\mathbf{A})$ the idempotent completion of the additive category $\mathbf{A}$. Objects of the new category are pairs (a, p) where p is a self-morphism of a such that $p^2 = p$. A morphism $f : (a, p) \to (b, q)$ is a morphism $f : a \to b$ such that $qfp = f$. There is an embedding $\iota : \mathbf{A} \to \mathbb{P}_0(\mathbf{A})$ that maps a to $(a, 1_a)$. It follows that ([15])

$$K_0(\mathbb{P}_0(\mathbf{A})) = \mathrm{Coker}(K_1(\mathbf{A}[z]) \oplus K_1(\mathbf{A}[z^{-1}]) \to K_1(\mathbf{A}[z, z^{-1}])).$$

We define the reduced K_0-group $\widetilde{K_0}(\mathbf{A})$ to be the cokernel of the map induced by ι on the K_0-group of the projective completion. Inductively, for an additive category $\mathbf{A}$,

$$K_{j-1}(\mathbf{A}) = \mathrm{Coker}(\widetilde{K_j}(\mathbf{A}[z]) \oplus \widetilde{K_j}(\mathbf{A}[z^{-1}]) \to \widetilde{K_j}(\mathbf{A}[z, z^{-1}])), \quad j \le 0,$$

where the map is induced by the natural inclusion of categories. Reduced and unreduced K_j groups are isomorphic for $j \le -1$.

Following [15], we define the reduced K_1-groups of the additive category $\mathbf{A}$ as follows ([15], §5): If L and M are two objects in $\mathbf{A}$, the sign

$$\varepsilon(L, M) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : L \oplus M \to M \oplus L.$$

The isomorphism $\varepsilon(L, M)$ determines an element in $K_1(\mathbf{R})$. Define

$$\widetilde{K_1}(\mathbf{A}) = \mathrm{Coker}(\varepsilon : K_0(\mathbf{A}) \otimes K_0(\mathbf{A}) \to K_1(\mathbf{A})).$$

The K_1-group of an additive category is isomorphic to the K_1 of its idempotent completion. The same is true for the reduced K_1-group.

We shall study the polynomial and Laurent extensions of an additive category of the form $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$. Let $\mathbf{R} = (R; B_0, B_1)$ as before. We write $\mathbf{R}[z, z^{-1}]$ ($\mathbf{R}[z]$, $\mathbf{R}[z^{-1}]$) for the objects of $\mathscr{T}$, $(R[z, z^{-1}]; B_0[z, z^{-1}], B_1[z, z^{-1}])$ $((R[z]; B_0[z], B_1[z])$, $(R[z^{-1}]; B_0[z^{-1}], B_1[z^{-1}])$ respectively). Here $B_i[z, z^{-1}] = B_i \otimes_R R[z, z^{-1}]$ ($B_i[z] = B_i \otimes_R R[z]$, $B_i[z^{-1}] = B_i \otimes_R R[z^{-1}]$ respectively), for $i = 0, 1$, and it is an $R[z, z^{-1}]$-bimodule with $z.b = bz$ for $b \in B_i$.

Lemma 2.4. *There are equivalences of categories*

$$f : \mathbb{F}_{\mathscr{A}}(\mathbf{R})[z, z^{-1}] \to \mathbb{F}_{\mathscr{A}}(\mathbf{R}[z, z^{-1}])$$

$$f_+ : \mathbb{F}_{\mathscr{A}}(\mathbf{R})[z] \to \mathbb{F}_{\mathscr{A}}(\mathbf{R}[z])$$

$$f_- : \mathbb{F}_{\mathscr{A}}(\mathbf{R})[z^{-1}] \to \mathbb{F}_{\mathscr{A}}(\mathbf{R}[z^{-1}]).$$

Proof. We shall give the proof for the finite Laurent extension. The other cases follow similarly. Let $u = (F_0, F_1) \in \mathscr{A}_R^2$ represent an object in $\mathbb{F}_{\mathscr{A}}(\mathbf{R})[z, z^{-1}]$. We define

$$f((F_0, F_1)[z, z^{-1}]) = (F_0[z, z^{-1}], F_1[z, z^{-1}]),$$

which is an equivalence on the set of objects ([15], Example 8.4). If $u = (F_0, F_1)$ is an object of $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$ we write $u[z, z^{-1}]$ for the object $(F_0[z, z^{-1}], F_1[z, z^{-1}])$ of $\mathbb{F}_{\mathscr{A}}(\mathbf{R})[z, z^{-1}]$. Then $f(u) = u[z, z^{-1}]$. For the definition of f on morphisms, we need the following general remark.

Claim. There is a natural isomorphism

$$\alpha_R^i(v)[z, z^{-1}] \cong \alpha_{R[z, z^{-1}]}^i(v[z, z^{-1}]),$$

for each object $v = (G_0, G_1)$ of $\mathbb{F}_{\mathscr{A}}(\mathbf{R})$.

Proof. We shall prove the claim for $i = 1$. The general case follows by repeating the argument.

$$\begin{aligned}(\alpha_R(v))[z, z^{-1}] &= ((G_1 \otimes_R B_0) \otimes_R R[z, z^{-1}], (G_0 \otimes_R B_1) \otimes_R R[z, z^{-1}]) \\ &\cong (G_1 \otimes_R (R[z, z^{-1}] \otimes_{R[z, z^{-1}]} B_0[z, z^{-1}]), \\ &\qquad G_0 \otimes_R (R[z, z^{-1}] \otimes_{R[z, z^{-1}]} B_1[z, z^{-1}])) \\ &= (G_1[z, z^{-1}] \otimes_{R[z, z^{-1}]} B_0[z, z^{-1}], G_0[z, z^{-1}] \otimes_{R[z, z^{-1}]} B_1[z, z^{-1}]) \\ &= \alpha_{R[z, z^{-1}]}(v[z, z^{-1}]).\end{aligned}$$

Using the Claim and the definitions we will show that f induces an equivalence on the morphisms.

$$\begin{aligned}\mathbb{F}_{\mathscr{A}}(\mathbf{R})[z,z^{-1}](u,v) &= (\mathbb{F}_{\mathscr{A}}(\mathbf{R})(u,v))[z,z^{-1}] \\ &= \left(\sum_{i=0}^{\infty}\mathscr{M}_R^2(u,\alpha_R^i(v))\right)[z,z^{-1}] \\ &= \sum_{i=0}^{\infty}\mathscr{M}_R^2(u,\alpha_R^i(v))[z,z^{-1}] \\ &= \sum_{i=0}^{\infty}\mathscr{M}_R^2[z,z^{-1}](u[z,z^{-1}],\alpha_R^i(v)[z,z^{-1}]) \\ &\cong \sum_{i=0}^{\infty}\mathscr{M}_{R[z,z^{-1}]}^2(u[z,z^{-1}],\alpha_{R[z,z^{-1}]}^i(v[z,z^{-1}])) \\ &= \mathbb{F}_{\mathscr{A}}(\mathbf{R}[z,z^{-1}])(u[z,z^{-1}],v[z,z^{-1}]).\end{aligned}$$

which implies that f is an equivalence of categories. □

The calculations in the proof of Lemma 2.4 imply the following result.

Corollary 2.5. *Let $\rho[z,z^{-1}]$ ($\rho[z]$, $\rho[z^{-1}]$) be the basic element of the category $\mathbb{F}_{\mathscr{F}}(\mathbf{R})[z,z^{-1}]$ ($\mathbb{F}_{\mathscr{F}}(\mathbf{R})[z]$, $\mathbb{F}_{\mathscr{F}}(\mathbf{R})[z^{-1}]$, respectively). Then $R_{\rho[z,z^{-1}]} \cong R_\rho[z,z^{-1}]$ ($R_{\rho[z]} \cong R_\rho[z]$, $R_{\rho[z^{-1}]} \cong R_\rho[z^{-1}]$ respectively).*

The next result studies the maps induced on K-groups by the forgetful natural transformation ψ defined in Remark 2.2, Part (2).

Lemma 2.6. *For each object $\mathbf{R} = (R; B_0, B_1)$ of $\mathscr{T}$ the natural transformation ψ induces an equivalence of categories*

$$\psi(\mathbf{R}) : \mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R})) \to \mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R})).$$

In particular, the map induced in K-groups is an isomorphism.

Proof. The proof is the same as the proof of the equivalence $\mathbb{P}_0(\mathscr{F}_R) \cong \mathbb{P}_0(\mathscr{P}_R)$. □

We shall compare the K-theory of $\mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R}))$ with the K-theory of the ring R_ρ. For this, notice that there is a functor $c : \mathscr{F}_{R_\rho} \to \mathbb{F}_{\mathscr{F}}(\mathbf{R})$ given by sending the free R_ρ-module of rank n to ρ^n. The functor c is full, faithful and cofinal.

Lemma 2.7. *The functor c induces an isomorphism*

$$c_j : \widetilde{K_j}(\mathbb{P}_0(\mathscr{F}_{R_\rho})) \to \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}))), \quad j \leq 1.$$

Proof. For $j = 1$, the result is classical ([11], Thm. 1.1). We shall prove the Lemma for $j = 0$. The other cases follow from Lemma 2.4, Corollary 2.5, and the definition of lower K-groups. Since the functor c is cofinal, the functor induced on the idempotent completions is also cofinal. Thus the map c_0 is a monomorphism. We shall show that c_0 is an epimorphism. The image of c_0 is generated by elements of the form (ρ^n, p), where p is a projection in $\mathbb{F}_{\mathscr{F}}(\mathbf{R})$. Let $((F, G), p)$ represent an element in $\widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R})))$. Then F and G are finitely generated projective R-modules and p is a projection in $\mathbb{F}_{\mathscr{P}}(\mathbf{R})$. There are finitely generated projective modules F' and G' such that $F \oplus F' \cong G \oplus G'$ and both modules are finitely generated free R-modules. Then, in $\widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R})))$,

$$[((F,G),p)] = [((F,G),p)] + [((F',G'),0)] = [((F \oplus F', G \oplus G'), p \oplus 0)],$$

which belongs to the image of c_0. □

Let $\mathbf{R} = (R; B_0, B_1)$ be an object of $\mathscr{T}$ and J be a two-sided ideal of R. Let $\chi : R \to R/J$ be the projection map. So $(R/J, \chi)$ determines an object of $\mathscr{R}$. By Remark 2.1, the projection χ induces a functor

$$[\chi] : \mathbf{R} \to [\mathbf{R}](R/J, \chi).$$

Thus it induces also a functor

$$\chi_* : \mathbb{F}_{\mathscr{F}}(\mathbf{R}) \to \mathbb{F}_{\mathscr{F}}([\mathbf{R}](R/J, \chi)) = \chi_*(\mathbb{F}_{\mathscr{F}}(\mathbf{R})).$$

Definition 2.8. The object $\mathbf{R} = (R; B_0, B_1)$ satisfies the condition (J^*), for a two-sided ideal J of R, if $JB_i = B_iJ$ for $i = 0, 1$.

Lemma 2.9. *Let* $\mathbf{R}$ *satisfy condition* (J^*) *for a two-sided ideal* J.

(i) *There is an isomorphism of* R/J*-bimodules*

$$R/J \otimes_R B_i \otimes_R R/J \cong R/J \otimes_R B_i, \quad i = 0, 1,$$

where the right action of R/J *on* $R/J \otimes_R B_i$ *is given by*

$$((r + J) \otimes b) \cdot (r' + J) = (r + J) \otimes (br').$$

(ii) *There is an isomorphism of* R/J*-bimodules* $\overline{\overline{B_i^{(j)}}} \cong R/J \otimes_R B_i^{(j)}$ *for all* i *and* j, *where*

$$\overline{\overline{B_i^{(j)}}} = (R/J \otimes_R B_i \otimes_R R/J) \otimes_{R/J} (R/J \otimes_R B_{i+1} \otimes_R R/J)$$
$$\otimes_{R/J} \cdots (R/J \otimes_R B_{i+j-1} \otimes_R R/J).$$

Proof. Condition (J^*) guarantees that the maps

$$R/J \otimes_R B_i \otimes_R R/J \to R/J \otimes_R B_i, \quad (r+J) \otimes b \otimes (r'+J) \mapsto (r+J) \otimes (br')$$

$$R/J \otimes_R B_i \to R/J \otimes_R B_i \otimes_R R/J, \quad (r+J) \otimes b \mapsto (r+J) \otimes b \otimes (1+J)$$

are inverse R/J-isomorphisms, proving Part (i).

Part (ii) follows by induction on j and Part (i). □

Let J be a two-sided ideal of R. We denote by $\mathbf{J}$ the two-sided ideal of R_ρ generated by $J \times J$.

Proposition 2.10. *Let* $\mathbf{R}$ *satisfy condition* (J^*) *for a two sided ideal* J *of* R*. Then there is a ring isomorphism*

$$\chi_J : R_\rho/\mathbf{J} \to (R/J)_{\rho/J},$$

where ρ/J *is the basic element of* $\chi_*(\mathbb{F}_{\mathscr{F}}(\mathbf{R}))$.

Proof. Using Lemma 2.9, we get that

$$T_{R/J}(\overline{\overline{B_0}} \otimes_{R/J} \overline{\overline{B_1}})) \cong R/J \otimes_R T_R(B_0 \otimes_R B_1)$$

$$\cong T_R(B_0 \otimes_R B_1)/JT_R(B_0 \otimes_R B_1),$$

as rings. Repeating the same argument to all the entries of the matrix representation of $(R/J)_{\rho/J}$, we get a ring isomorphism (we write BB' for $B \otimes_R B'$)

$$(R/J)_{\rho/J} \cong \begin{pmatrix} T_R(B_1B_0)/J \cdot T_R(B_1B_0) & B_1T_R(B_0B_1)/J \cdot B_1T_R(B_0B_1) \\ B_0T_R(B_1B_0)/J \cdot B_0T_R(B_1B_0) & T_R(B_0B_1)/J \cdot T_R(B_0B_1) \end{pmatrix}.$$

The assumption on the bimodules implies that the ideal $\mathbf{J}$ has the following matrix representation

$$\mathbf{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} T_R(B_1B_0) & B_1T_R(B_0B_1) \\ B_0T_R(B_1B_0) & T_R(B_0B_1) \end{pmatrix}$$

$$= \begin{pmatrix} J \cdot T_R(B_1B_0) & J \cdot B_1T_R(B_0B_1) \\ J \cdot B_0T_R(B_1B_0) & J \cdot T_R(B_0B_1) \end{pmatrix}.$$

Then it follows immediately that $R_\rho/\mathbf{J} \cong (R/J)_{\rho/J}$. □

Proposition 2.11. *Let* $\mathbf{R}$ *be a triple which satisfies condition* (J^*) *with* J *a two-sided nilpotent ideal of* R*. Then the map*

$$\chi_j : \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{T}}(\mathbf{R}))) \to \widetilde{K_j}(\mathbb{P}_0(\chi_*(\mathbb{F}_{\mathscr{T}}(\mathbf{R}))), \quad j \leq 0,$$

induced by χ is an isomorphism.

Proof. We shall give the proof for $j = 0$. Since J is nilpotent, condition (J^*) implies that the ideal $\mathbf{J}$ is a two-sided nilpotent ideal of R_ρ. Then we have a sequence of isomorphisms

$$\begin{aligned}
\widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{T}}(\mathbf{R}))) &\cong \widetilde{K_0}(R_\rho) && \text{by Lemma 2.7}\\
&\cong \widetilde{K_0}(R_\rho/\mathbf{J}) && \text{by [2], Ch. III, Proposition 2.12}\\
&\cong \widetilde{K_0}((R/J)_{\rho/J}) && \text{by Proposition 2.10}\\
&\cong \widetilde{K_0}(\mathbb{P}_0(\chi_*(\mathbb{F}_{\mathscr{T}}(\mathbf{R}))) && \text{by Lemma 2.7.}
\end{aligned}$$

□

3 Definition and properties of Nil-groups

Following [7], we can define the Nil-functor associated to $\mathbb{F}_{\mathscr{T}}$,

$$\mathbb{NF}_{\mathscr{T}} : \mathscr{T} \to \mathscr{A}dd.$$

The objects of the category $\mathbb{NF}_{\mathscr{T}}(\mathbf{R})$ are pairs (u, v) where u is an object of $\mathbb{F}_{\mathscr{T}}(\mathbf{R})$ and $v : u \to \alpha_R(u)$ is a degree one nilpotent morphism. Morphisms are given by commutative diagrams as in [7]. The action of $\mathbb{NF}_{\mathscr{T}}(\mathbf{R})$ on morphisms of $\mathscr{T}$ is defined as before. We are interested in the reduced version of the above functor. Notice that there is a functor $\mathscr{F}_R^2 \to \mathbb{NF}_{\mathscr{T}}(\mathbf{R})$ mapping an object u to the pair $(u, 0)$. We define

$$\widetilde{Nil}_0(\mathbf{R}, \alpha_R) = \operatorname{Coker}(K_0(\mathscr{F}_R^2) \to K_0(\mathbb{NF}_{\mathscr{T}}(\mathbf{R}))).$$

Following the ideas developed in the last section we define the lower $\widetilde{Nil}$-groups by

$$\widetilde{Nil}_j(\mathbf{R}, \alpha_R) = \operatorname{Coker}(K_j(\mathscr{F}_R^2) \to K_j(\mathbb{NF}_{\mathscr{T}}(\mathbf{R}))) \quad j \leq 0.$$

There is an alternative way for constructing the lower $\widetilde{Nil}$-groups using the methods of [15], i.e. as the cokernel of the inclusion induced map:

$$\widetilde{Nil}_{j+1}(\mathbf{R}[z], \alpha_{R[z]}) \oplus \widetilde{Nil}_{j+1}(\mathbf{R}[z^{-1}], \alpha_{R[z^{-1}]}) \to \widetilde{Nil}_{j+1}(\mathbf{R}[z, z^{-1}], \alpha_{R[z,z^{-1}]}).$$

We shall compare the two definitions. For this, we first need the analogue of Lemma 2.4 for the $\mathbb{NF}_{\mathscr{T}}$-functors.

Lemma 3.1. *There are equivalences of categories*

$$\mathbb{N}f : \mathbb{NF}_{\mathscr{F}}(\mathbf{R})[z, z^{-1}] \to \mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z, z^{-1}])$$

$$\mathbb{N}f_+ : \mathbb{NF}_{\mathscr{F}}(\mathbf{R})[z] \to \mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z])$$

$$\mathbb{N}f_- : \mathbb{NF}_{\mathscr{F}}(\mathbf{R})[z^{-1}] \to \mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z^{-1}]).$$

Proof. The proof is similar to the proof of Lemma 2.4. In the first case, the equivalence is defined by

$$\mathbb{N}f((u, v)[z, z^{-1}]) = (f(u), f(v))$$

where f is the equivalence defined in Lemma 2.4. The proof that $\mathbb{N}f$ is an equivalence follows as in Lemma 2.4. □

Lemma 3.2. *There is an isomorphism*

$$\widetilde{Nil}_{j-1}(\mathbf{R}, \alpha_R)$$
$$\cong \operatorname{Coker}(\widetilde{Nil}_j(\mathbf{R}[z], \alpha_{R[z]}) \oplus \widetilde{Nil}_j(\mathbf{R}[z^{-1}], \alpha_{R[z^{-1}]}) \to \widetilde{Nil}_j(\mathbf{R}[z, z^{-1}], \alpha_{R[z, z^{-1}]}))$$

for all $j \le 0$.

Proof. By definition, there is a diagram:

$$\begin{array}{ccc}
K_{j-1}(\mathscr{F}_R^2) & = & \operatorname{Coker}(K_j(\mathscr{F}^2_{R[z]}) \oplus K_j(\mathscr{F}^2_{R[z^{-1}]}) \to K_j(\mathscr{F}^2_{R[z,z^{-1}]})) \\
\downarrow & & \downarrow \\
K_{j-1}(\mathbb{NF}_{\mathscr{F}}(\mathbf{R})) & = & \operatorname{Coker}(K_j(\mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z])) \oplus K_j(\mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z^{-1}])) \to K_j(\mathbb{NF}_{\mathscr{F}}(\mathbf{R}[z, z^{-1}]))) \\
\downarrow & & \downarrow \\
\widetilde{Nil}_{j-1}(\mathbf{R}, \alpha_R) & & \operatorname{Coker}(\widetilde{Nil}_j(\mathbf{R}[z], \alpha_{R[z]}) \oplus \widetilde{Nil}_j(\mathbf{R}[z^{-1}], \alpha_{R[z^{-1}]}) \to \widetilde{Nil}_j(\mathbf{R}[z, z^{-1}], \alpha_{R[z,z^{-1}]}))
\end{array}$$

where the first equality is directly from the definition, the second equality follows from Lemma 3.1 and the bottom row consists of the cokernels of the vertical maps, by definition. The result follows. □

Let $NK_1(\mathbf{R})$ be the kernel of the map induced by the forgetful functor $\eta_{\mathscr{F}}(\mathbf{R})$ on the K_1-groups. As before, we define $NK_j(\mathbf{R})$ for all $j \le 1$.

Remark 3.3. The following are immediate from the definitions:

1. By construction, NK_j and the reduced NK_j are isomorphic for $j \le 1$.
2. The construction in [7] (Proposition 2.9 and Lemma 2.10) and Lemma 3.2 imply that there is an epimorphism $\sigma_j : \widetilde{Nil}_{j-1}(\mathbf{R}) \to NK_j(\mathbf{R})$, for all $j \le 1$.

3. There is a map $\psi'_j : NK_j(\mathbf{R}) \to \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}))$ which factors through $\widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R})))$ i.e. ψ'_j is the composition of two inclusions ($j \leq 1$):

$$\psi'_j : NK_j(\mathbf{R}) \longrightarrow \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R}))) \xrightarrow{\psi_j} \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}))).$$

4. Lemma 2.7 implies that

$$NK_j(\mathbf{R}) = \ker(\widetilde{K_j}(R_\rho) \to \widetilde{K_j}(R \times R)).$$

The following is an immediate consequence of Lemma 2.6.

Lemma 3.4. *There is a split exact sequence, for $j \leq 1$,*

$$0 \longrightarrow NK_j(\mathbf{R}) \xrightarrow{\psi'_j} \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}))) \xrightarrow{\theta_j} \widetilde{K_j}(\mathscr{P}_R^2) \longrightarrow 0.$$

$\theta_j = (\eta_{\mathscr{P}})_j$ for $j \leq 1$.

We shall study a Mayer-Vietoris type property of the functors NK_j. Let R be a commutative ring and B_i $(i = 0, 1)$ be two R-bimodules for which the left and the right actions of R coincide, for $i = 0, 1$. Thus the triple $(R; B_0, B_1)$ is an object of the category $\mathscr{T}$. Let

$$(*) \qquad \begin{array}{ccc} R & \xrightarrow{h_2} & R_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ R_1 & \xrightarrow{f_1} & R_0 \end{array}$$

be a pull-back diagram of rings such that either f_1 or f_2 is a ring epimorphism (usually such a diagram is called *Milnor square*). The rings R_j, $j = 0, 1, 2$, together with the ring homomorphisms from R, are objects of the category $\mathscr{R}$.

The above cartesian square is the reason we have introduced the category $\mathbb{F}_{\mathscr{P}}(\mathbf{R})$. The corresponding diagram of the categories of the free modules is not cartesian. So we derive the following exact sequence from the cartesian square $(*)$.

$$\widetilde{K_1}(\mathscr{P}_R^2) \to \widetilde{K_1}(\mathscr{P}_{R_1}^2) \oplus \widetilde{K_1}(\mathscr{P}_{R_2}^2) \to \widetilde{K_1}(\mathscr{P}_{R_0}^2) \to$$

$$\widetilde{K_0}(\mathscr{P}_R^2) \to \widetilde{K_0}(\mathscr{P}_{R_1}^2) \oplus \widetilde{K_0}(\mathscr{P}_{R_2}^2) \to \widetilde{K_0}(\mathscr{P}_{R_0}^2) \to$$

$$K_{-1}(\mathscr{P}_R^2) \to K_{-1}(\mathscr{P}_{R_1}^2) \oplus K_{-1}(\mathscr{P}_{R_2}^2) \to \cdots.$$

Also, we form the pull-back of the following diagram of categories

$$
(2)\qquad \begin{array}{ccc}
\mathbb{P} & \xrightarrow{h_2'} & \mathbb{F}_{\mathscr{P}}(\mathbf{R}_2) \\
{\scriptstyle h_1'}\big\downarrow & & \big\downarrow{\scriptstyle f_2'} \\
\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1) & \xrightarrow{f_1'} & \mathbb{F}_{\mathscr{P}}(\mathbf{R}_0)
\end{array} .
$$

Notice that $\mathbb{P}_0(\mathbb{P})$ is the pull-back of the projective completions.

Lemma 3.5. *The above pull-back diagram of categories induces a Mayer-Vietoris sequence in K-theory of the categories and their idempotent completions*

$$
\widetilde{K_1}(\mathbb{P}) \to \widetilde{K_1}(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1)) \oplus \widetilde{K_1}(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_2)) \to \widetilde{K_1}(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0)) \to \widetilde{K_0}(\mathbb{P}_0(\mathbb{P}))
$$

$$
\to \widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1))) \oplus \widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_2))) \to \widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0)))
$$

$$
\to K_{-1}(\mathbb{P}_0(\mathbb{P})) \to K_{-1}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1))) \oplus K_{-1}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_2)))
$$

$$
\to K_{-1}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0))).
$$

Proof. We shall use the terminology of [2], Ch. VII. It is obvious that the two functors f_1' and f_2' are cofinal. We shall show that, if f_1 is a surjective ring homomorphism, then f_1' is E-surjective in the sense of [2]. That means that for each object u of $\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0)$, after stabilization by an object u', there is an object v of $\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1)$ such that f_1' induces an epimorphism from the commutator subgroup of $\mathrm{Aut}(v)$ to the commutator subgroup of $\mathrm{Aut}(u \oplus u')$. So let u be an object of $\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0)$. We can stabilize u such that $u \oplus u' = (F_0, F_1)$ where F_0 and F_1 are free modules of the same rank. Set $r_i = (R_i, R_i)$, $i = 0, 1, 2$, for the object in the corresponding category, consisting of a pair of free modules of rank 1. As in the classical case, the commutator subgroup of $\mathrm{Aut}(u \oplus u')$ is generated by "elementary" matrices of the form $e_{ij}(x)$ which is a matrix with 1's in the diagonal and $x \in \mathrm{Mor}(r_0, r_0)$. A self-morphism of r_0 can be represented by a finite collection of pairs of elements in a tensor product of the bimodules. Since the map f_1 is surjective, it induces a surjective map on the elementary matrices. Thus, if v consists of a pair of free R_1-modules of the same rank with F_0 (or F_1), then f_1 induces an epimorphism on the corresponding commutator subgroups of the automorphism groups. Then, by [2], Ch. VII, §4, Ch. XII §8, there is a Mayer-Vietoris sequence in lower K-groups. The original argument in [2] gives an exact sequence for absolute K-groups. That sequence induces a sequence of the reduced K-groups. □

By the universal property of the pull-back diagrams of categories, there is a functor $g : \mathbb{F}_{\mathscr{P}}(\mathbf{R}) \to \mathbb{P}$ making the resulting diagrams commute up to natural equivalence.

Lemma 3.6. *The functor g induces a monomorphism for $j \leq 1$,*

$$
g_j : \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}))) \to \widetilde{K_j}(\mathbb{P}_0(\mathbb{P})).
$$

Proof. The ideas of the proof of Theorem 3.11 in [14] apply in our setting. The main ingredient of the proof of the theorem is that the two-sided reduction of coefficients is isomorphic to the one-sided reduction. That is obvious in our setting. The result in [14] implies that g is a full, faithfull and cofinal functor, which implies the result. □

The above properties of the K-theory associated to a Milnor square imply the following vanishing result for the lower NK-groups.

Theorem 3.7. *Let* $(*)$ *be a Milnor square as before and* $s \leq 0$. *If* $NK_j(\mathbf{R}_i) = 0$ *for all* $j \leq s$ *and* $i = 0, 1, 2$, *then* $NK_{j-1}(\mathbf{R}) = 0$ *for all* $j \leq s$. *Also, the boundary map* $\widetilde{K_{s+1}}(\mathbf{R}_0) \to \widetilde{K_s}(\mathbf{R})$ *induces an epimorphism.* $NK_{s+1}(\mathbf{R}_0) \to NK_s(\mathbf{R})$.

Proof. By the naturality of the exact sequences associated to the pull-back diagrams (1) and (2), we get a commutative diagram

$$\begin{array}{ccccc} \widetilde{K_j}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_0))) & \longrightarrow & \widetilde{K_{j-1}}(\mathbb{P}_0(\mathbb{P})) & \longrightarrow & \widetilde{K_{j-1}}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_1))) \oplus \widetilde{K_{j-1}}(\mathbb{P}_0(\mathbb{F}_{\mathscr{P}}(\mathbf{R}_2))) \\ \downarrow & & \downarrow{\scriptstyle \theta'_{j-1}} & & \downarrow \\ \widetilde{K_j}(\mathscr{P}^2_{R_0}) & \longrightarrow & \widetilde{K_{j-1}}(\mathscr{P}^2_R) & \longrightarrow & \widetilde{K_{j-1}}(\mathscr{P}^2_{R_1}) \oplus \widetilde{K_{j-1}}(\mathscr{P}^2_{R_2}). \end{array}$$

The vertical maps are induced by the maps θ_j of Lemma 3.4. The NK-groups are the kernels of the epimorphisms θ_j, $j \leq s$. Since they vanish, the maps θ_j are isomorphisms. The five-lemma implies that the map,

$$\theta'_{j-1} : K_{j-1}(\mathbb{P}_0(\mathbb{P})) \to K_{j-1}(\mathscr{P}^2_R)$$

is an isomorphism for $j \leq s$. But $(\theta_R)_{j-1} = \theta'_{j-1} g_{j-1}$. Lemma 3.6 implies that $(\theta_R)_{j-1}$ is a monomorphism and thus $NK_{j-1}(\mathbf{R}) = 0$ for $j \leq s$. A similar argument works for the second part of the theorem. □

There are two more functors of interest to our calculations. They are defined in [19].

$$\mathbb{N}il^W, \widetilde{\mathbb{N}il}^W : \mathscr{T} \to \mathscr{A}dd.$$

The objects of $\mathbb{N}il^W(\mathbf{R})$ ($\widetilde{\mathbb{N}il}^W(\mathbf{R})$) are quadruples (P, Q, p, q) where (P, Q) is an object of $\mathscr{P}^2_R$ ($\mathscr{F}^2_R$) and

$$p : P \to Q \otimes_R B_0, \quad q : Q \to P \otimes_R B_1$$

are R-homomorphisms such that there are filtrations

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = P, \quad 0 = Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$$

with the property that

$$p(P_{i+1}) \subset Q_i \otimes_R B_0, \quad q(Q_{i+1}) \subset P_i \otimes_R B_1.$$

Morphisms are given by commutative diagrams. On morphisms $\mathbb{N}il^W$ ($\widetilde{\mathbb{N}il}^W$) is defined as before. Then we define $Nil_0^W(\mathbf{R})$ ($\widetilde{Nil}_0^W(\mathbf{R})$) to be the $K_0(\mathbb{N}il^W(\mathbf{R}))$ ($K_0(\widetilde{\mathbb{N}il}^W(\mathbf{R}))$ respectively). Using the polynomial and Laurent extension categories we can define the lower Waldhausen's Nil-groups as in [15], §7 and §8. The lower Waldhausen's Nil-groups appear in the extension of the main exact sequence in [18] to the right. Notice that, for each object $\mathbf{R}$ of $\mathscr{T}$, there are functors ([19]):

$$\mathscr{P}_R^2 \xrightarrow{i(\mathbf{R})} \mathbb{N}il^W(\mathbf{R}) \xrightarrow{f(\mathbf{R})} \mathscr{P}_R^2.$$

The reduced Nil-groups are equal to the kernel of the map that $f(\mathbf{R})$ induces in K-theory. In [7], Proposition 2.6, a natural isomorphism is constructed

$$\Phi_0 : \widetilde{Nil}_0(\mathbf{R}, \alpha_R) \to \widetilde{Nil}_0^W(\mathbf{R})$$

in the case that B_i are free as left and right R-modules for $i = 0, 1$. Using Lemma 3.2, we see that such an isomorphism exists for all lower Nil-groups.

We shall use vanishing theorems for Waldhausen's Nil-groups to derive corresponding results for the twisted Nil-groups and the NK-groups. A ring R is called *coherent* if the category of finitely presented R-modules is abelian. The basic properties of coherent rings are summarized in [10]. In particular Noetherian rings are coherent. In [19], it was shown that if the ring R is coherent and has finite cohomological dimension then the Waldhausen Nil-groups vanish. We will prove the result for quasi-coherent rings (compare with the quasi-regular rings in [2], Proposition 10.1).

Definition 3.8. A ring R is called *quasi-coherent* if there is a two-sided nilpotent ideal of R such that R/J is coherent of finite cohomological dimension.

Proposition 3.9. *If R and all finite polynomial and Laurent extensions of R are commutative quasi-coherent rings then*

$$NK_j(\mathbf{R}) = \widetilde{Nil}_{j-1}(\mathbf{R}) = \widetilde{Nil}_{j-1}^W(\mathbf{R}) = 0, \quad j \leq 0.$$

Proof. We are going to show the result for $j = 0$. An application of Lemma 3.2 will imply the general case.

There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & NK_0(\mathbf{R}) & \longrightarrow & \widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R}))) & \xrightarrow{\eta_{\mathscr{F}}} & \widetilde{K_0}(\mathscr{F}_R^2) & \longrightarrow & 0 \\
 & & \downarrow \chi' & & \downarrow \chi_0 & & \downarrow \chi'' & & \\
0 & \longrightarrow & NK_0(\mathbf{R}/\mathbf{J}) & \longrightarrow & \widetilde{K_0}(\mathbb{P}_0(\mathbb{F}_{\mathscr{F}}(\mathbf{R}/\mathbf{J}))) & \xrightarrow{\eta_{\mathscr{F}}} & \widetilde{K_0}(\mathscr{F}_{R/J}^2) & \longrightarrow & 0
\end{array}$$

where the vertical maps are induced by the projection $\chi : R \to R/J$ of rings. By Proposition 2.11, the map χ_0 is a monomorphism which implies that χ' is a monomorphism. But the ring R/J is coherent and of finite cohomological dimension. Thus $\widetilde{Nil}_0^W(\mathbf{R}/\mathbf{J}) = 0$. Therefore $NK_0(\mathbf{R}/\mathbf{J}) = 0$ being the epimorphic image of $\widetilde{Nil}_0^W(\mathbf{R}/\mathbf{J})$ under $\sigma_0\Phi_0^{-1}$. Since χ' is a monomorphism, $NK_0(\mathbf{R}) = 0$. □

Remark 3.10. A commutative quasi-regular ring R satisfies the assumption of Proposition 3.9 ([2]). This is essentially Hilbert's basis theorem.

We now specialize to the case of amalgamated free products of rings as in [19]. Let $S = A_0 *_R A_1$ be the pushout in the category of rings ([19]). For simplicity we assume that all the maps are ring monomorphisms and we identify a ring with its image in the larger ring. We assume that $A_i = R \oplus B_i$ as R-bimodules, $i = 0, 1$. In this case, the epimorphism

$$\sigma_j : \widetilde{Nil}_{j-1}(\mathbf{R}, \alpha_R) \to NK_j(\mathbf{R})$$

is an isomorphism for all $j \leq 1$, and both of the groups are naturally isomorphic to Waldhausen's lower Nil-groups ([7]). Actually in this case there is a functor $r : \mathbb{F}_{\mathscr{F}}(\mathbf{R}) \to \mathscr{F}_S$, $r(F_0, F_1) = (F_0 \oplus F_1) \otimes_R S$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{Nil}_j(\mathbf{R}, \alpha_R) & \xrightarrow{\sigma_{j+1}} & K_{j+1}(\mathbb{F}_{\mathscr{F}}(\mathbf{R})) \\ \Phi_j \Big\downarrow & & \Big\downarrow r_* \\ \widetilde{Nil}_j^W(\mathbf{R}) & \xrightarrow{s_{j+1}} & K_{j+1}(S) \end{array}$$

where s_{j+1} is the split injection defined in [19]. The commutativity of the digram follows from Proposition 2.6 of [7] with the usual methods of extending results on K_0-groups of an additive category to lower K-groups.

Of course there is a more direct definition of the NK-groups as relative K-groups, using the result of Lemma 2.7. More specifically,

$$NK_j(\mathbf{R}) = \ker(\widetilde{K_j}(R_\rho) \to \widetilde{K_j}(R \times R)), \quad j \leq 1.$$

In other words, $NK_j(\mathbf{R}) \cong \widetilde{K_j}(R_\rho, \mathbf{I})$ where $\mathbf{I}$ is the augmentation ideal i.e. the kernel of the augmentation map. Actually this description allows us to define relative NK_j-groups. Let $\mathbf{J}$ be an ideal of R_ρ and J is its image under the augmentation map. Then J is an ideal of $R \times R$. We define

$$NK_j(R_\rho, \mathbf{J}) = \ker(\widetilde{K_j}(R_\rho, \mathbf{J}) \to \widetilde{K_j}(R \times R, J).$$

With this interpretation the following result becomes classical

Lemma 3.11. *With the above notation, there is a long exact sequence*

$$NK_1(R_\rho, \mathbf{J}) \to NK_1(R_\rho) \to NK_1(R_\rho/\mathbf{J}) \to NK_0(R_\rho, \mathbf{J}) \to \cdots.$$

Proof. Let $\mathbf{I}$ be the augmentation ideal as above. The exact sequence stated in the lemma is derived from the exact sequence of ideals

$$0 \to \mathbf{I} \cap \mathbf{J} \to \mathbf{J} \to \mathbf{J}/\mathbf{I} \cap \mathbf{J} \to 0.$$

The details appear in [13]. □

4 On the Nil-groups of commutative rings

Let R be a commutative Artinian ring and B_i $(i = 0, 1)$ be two R-bimodules such that the right and left R actions coincide. Let J be the Jacobson radical of R and $\mathbf{J}$ be the ideal of the endomorphism ring R_ρ generated by J. Then $\mathbf{J}$ is a nilpotent ideal. Let $\mathbf{I}$ be the augmentation ideal of $R_\rho \to R \times R$. Set $\bar{\mathbf{J}}$ for the ideal $\mathbf{J} \cap \mathbf{I}$.

Lemma 4.1. *With the above notation, $NK_1(R_\rho, \mathbf{J})$ is the image of the inclusion induced map*

$$K_1(R_\rho, \bar{\mathbf{J}}) \to K_1(R_\rho, \mathbf{J}).$$

Proof. By definition, the augmentation map induces isomorphisms

$$R_\rho/\mathbf{I} \cong R \times R, \quad \mathbf{J}/\bar{\mathbf{J}} \cong J \times J.$$

Thus the augmentation map induces an isomorphism of relative groups

$$K_1(R_\rho/\mathbf{I}, \mathbf{J}/\bar{\mathbf{J}}) \cong K_1(R \times R, J \times J).$$

With this observation, the exact sequence of the triple $\bar{\mathbf{J}} \subset \mathbf{J} \subset R_\rho$

$$K_1(R_\rho, \bar{\mathbf{J}}) \to K_1(R_\rho, \mathbf{J}) \to K_1(R_\rho/\mathbf{I}, \mathbf{J}/\bar{\mathbf{J}}),$$

is reduced to

$$K_1(R_\rho, \bar{\mathbf{J}}) \to K_1(R_\rho, \mathbf{J}) \to K_1(R \times R, J \times J).$$

The result follows from the definition of $NK_1(R_\rho, \mathbf{J})$ as the kernel of the augmentation map. □

Let $U(R_\rho, \bar{\mathbf{J}})$ be the units of R_ρ which map to the identity in $R_\rho/\bar{\mathbf{J}}$.

Lemma 4.2. *With the above assumptions,*

$$U(R_\rho, \bar{\mathbf{J}}) = 1 + \bar{\mathbf{J}}.$$

Proof. Since $\bar{\mathbf{J}}$ is nilpotent, $1 + \bar{\mathbf{J}} \subset U(R_\rho, \bar{\mathbf{J}})$. Also, if $u \in U(R_\rho, \bar{\mathbf{J}})$, then $u - 1$ belongs to $\bar{\mathbf{J}}$ which proves the other inclusion. □

Proposition 4.3. *The inclusion induced map*

$$1 + \bar{\mathbf{J}} \to K_1(R_\rho, \bar{\mathbf{J}})$$

is an epimorphism.

Proof. The result follows from Theorem 9.1, p. 266 of [2]. □

Combining Proposition 4.3 with Lemma 4.1, we derive the following

Corollary 4.4. *The image of the composition*

$$1 + \bar{\mathbf{J}} \to K_1(R_\rho, \bar{\mathbf{J}}) \to K_1(R_\rho, \mathbf{J})$$

is $NK_1(R_\rho, \mathbf{J})$.

Combining the above results with Lemma 3.11, we derive the following:

Corollary 4.5. *The composition*

$$1 + \bar{\mathbf{J}} \to NK_1(R_\rho, \mathbf{J}) \to NK_1(\mathbf{R})$$

is an epimorphism.

Proof. Since R is Artinian, the Jacobson radical is nilpotent and R/J is regular. By Proposition 3.9, $NK_1(\mathbf{R}/\mathbf{J})$ vanishes. Thus Lemma 3.11 implies that the second map is an epimorphism. The result follows from Corollary 4.4. □

The following lemma follows directly from the general form of the elements of the ring R_ρ.

Lemma 4.6. *The group* $1 + \bar{\mathbf{J}}$ *is generated by elements of the form*

$$\begin{pmatrix} 1 + j' & 0 \\ 0 & 1 + j'' \end{pmatrix}, \quad \begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ m'' & 1 \end{pmatrix},$$

where $j' \in JT_R(B_1 \otimes B_0)$, $j'' \in JT_R(B_0 \otimes B_1)$, $m' \in JB_1 \otimes T_R(B_0 \otimes B_1)$, $f'' \in JB_0 \otimes T_R(B_1 \otimes B_0)$.

Proof. A general element of $1 + \bar{\mathbf{J}}$ has the form

$$\begin{pmatrix} 1 + j' & m_1 \\ m_2 & 1 + j'' \end{pmatrix}$$

Set $m_1' = (1+j')^{-1}m_1$ and $m_2' = (1+j'')^{-1}m_2$. Then

$$\begin{pmatrix} 1+j' & m_1 \\ m_2 & 1+j'' \end{pmatrix} = \begin{pmatrix} 1+j' & 0 \\ 0 & 1+j'' \end{pmatrix}\begin{pmatrix} 1 & m_1' \\ m_2' & 1 \end{pmatrix}.$$

Now the last matrix can be written as a product

$$\begin{pmatrix} 1 & m_1' \\ m_2' & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-m_2'm_1' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (1-m_2'm_1')^{-1}m_2' & 1 \end{pmatrix}\begin{pmatrix} 1 & m_1' \\ 0 & 1 \end{pmatrix}.$$

The form of the decomposition completes the proof of the lemma. □

We further assume that B_i is isomorphic, as an R-bimodule, to a direct sum of copies of R. Let $\{x_{\lambda_i}^{(i)}\}_{\lambda_i \in \Lambda_i}$ be a basis for the module B_i, $i = 0, 1$. We write $F(i,i')$ for the free algebra on the set $\{x_{\lambda_i}^{(i)}.x_{\lambda_{i'}}^{(i')} : (\lambda_i, \lambda_{i'}) \in \Lambda_i \times \Lambda_{i'}\}$, where $i' \equiv (i-1) \bmod 2$. The algebra $F(i,i')$ is a subalgebra of the free algebra Ω on the union of the two sets of generators i.e. the set $\{x_{\lambda_i}^{(i)}, x_{\lambda_{i'}}^{(i')} : (\lambda_i, \lambda_{i'}) \in \Lambda_i \times \Lambda_{i'}\}$ We also write

$$M(i,i',i) = \bigoplus_{\lambda_i \in \Lambda_i} x_{\lambda_i}^{(i)} F(i',i) = \bigoplus_{\lambda_i \in \Lambda_i} F(i,i') x_{\lambda_i}^{(i)},$$

for the $F(i,i') - F(i',i)$-bimodule (again $i' \equiv (i-1) \bmod 2$). All the operations take place in the free algebra Ω.

Lemma 4.7. *With the above notation, there is a ring isomorphism*

$$R_\rho \cong \begin{pmatrix} F(1,0) & M(1,0,1) \\ M(0,1,0) & F(0,1) \end{pmatrix}.$$

Proof. When the bimodules are direct products of copies of the ring, then the tensor algebra is isomorphic to the free algebra on the set of generators. The result follows directly from this observation. □

Corollary 4.8. *Under the assumptions of Lemma 4.7, an element of* $1 + \bar{\mathbf{J}}$ *has the form*

$$\begin{pmatrix} 1+p_{1,0} & p_{1,0,1} \\ p_{0,1,0} & 1+p_{0,1} \end{pmatrix}.$$

where $p_{i,j} \in JF(i,j)$ *and* $p_{i,j,i} \in JM(i,j,i)$. *In particular, the entries of the matrix can be represented as polynomials on non-commuting variables with coefficients in* J.

Thus we have a relatively complete description of the generators of NK_1 in the case of interest.

Proposition 4.9. *Let R and J be as above. Then $NK_1(\mathbf{R})$ is generated by the images of elements of the form*

$$\begin{pmatrix} 1+p_{1,0} & 0 \\ 0 & 1+p_{0,1} \end{pmatrix}, \begin{pmatrix} 1 & p_{1,0,1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p_{0,1,0} & 1 \end{pmatrix},$$

under the composition

$$1+\bar{\mathbf{J}} \to K_1(R_p, \bar{\mathbf{J}}) \to NK_1(R_p, \mathbf{J}) \to NK_1(\mathbf{R}).$$

Proof. By Corollary 4.5 the composition is an epimorphism. But Lemma 4.6 and Corollary 4.8 imply that the three types of matrices generate $1+\bar{\mathbf{J}}$.

5 Calculation of the generating set

Now we specialize to the case of interest. Let G be a finitely generated abelian group, $G = H \times T^m$ with H a finite abelian group of order n and T the infinite cyclic group. Set $R = \mathbb{Z}H$ and $R' = \mathbb{Z}G$. Then R' is the Laurent ring of R in m commuting variables i.e. $R' = R[s_1, s_1^{-1}, \dots s_m, s_m^{-1}]$. Also, let $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ be the decomposition of n into prime factors. Choose integers $l_1, \dots, l_s$ such that $p_r^{l_r} \geq k_r n$ for all r and set $n' = p_1^{l_1} p_2^{l_2} \dots p_s^{l_s}$. Let B_i $(i = 0, 1)$ be two R'-bimodules such that the right and left R' actions coincide and $\mathbf{R}' = (R', B_0, B_1)$.

Theorem 5.1 (Main Theorem). *With the above notation*

(i) $NK_j(\mathbf{R}') = 0$ *for* $j \leq -1$,

(ii) $n' NK_0(\mathbf{R}') = 0$.

Proof. The proof is classical in this context. The basic ideas can be traced back to Bass ([2]) but we use more the methods of Connolly–daSilva ([6]). Let $\mathbb{Z}H \subset M \subset \mathbb{Q}H$ be a hereditary order. Then $nM \subset \mathbb{Z}H$ and we get the following cartesian square of ring homomorphisms

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & M[s_1, s_1^{-1}, \dots s_m, s_m^{-1}] \\ \downarrow & & \downarrow \\ (\mathbb{Z}G/nM)[s_1, s_1^{-1}, \dots s_m, s_m^{-1}] & \longrightarrow & (M/nM)[s_1, s_1^{-1}, \dots s_m, s_m^{-1}]. \end{array}$$

The ring M is regular and the rings R/nM and M/nM are quasi-regular because they are finite rings. Thus the Laurent rings satisfy the same properties. Therefore the result of Theorem 3.7 implies that

(i) $NK_j(\mathbf{R}') = 0$ for $j \leq -1$.

(ii) $NK_1(\mathbf{M/nM}) \to NK_0(\mathbf{R}')$ is an epimorphism

Combining part (ii) from above and Corollary 4.5 we have an epimorphism

$$1 + \bar{\mathbf{J}} \to NK_1(\mathbf{M}/n\mathbf{M}) \to NK_0(\mathbf{R}'),$$

where J is the Jacobson radical of M/nM, $\mathbf{J}$ is the ideal of the endomorphism ring of the basic object of $\mathbf{M/nM}$ generated by J, and $\bar{\mathbf{J}}$ is the intersection of $\mathbf{J}$ with the augmentation ideal. We shall show that each member of the generating set of $NK_0(\mathbf{R}')$ induced by the generators of $1 + \bar{\mathbf{J}}$ described in Proposition 4.9 has exponent n'. Let x_1 be the image of the generator of the form

$$x_1 = \partial\begin{pmatrix} 1 + p_{1,0} & 0 \\ 0 & 1 + p_{0,1} \end{pmatrix}.$$

The elements $p_{1,0}$ and $p_{0,1}$ are polynomials on non-commuting variables with coefficients in the ideal $J[s_1, s_1^{-1}, \dots s_m, s_m^{-1}]$. The calculations in the proof of the main theorem, part (b) in [6] show that

$$(1 + p_{1,0})^{n'} = (1 + p_{0,1})^{n'} = 1,$$

which implies that $n'x_1 = 0$. For the other generators, notice that if $a \in J[s_1, s_1^{-1}, \dots s_m, s_m^{-1}]$, $na = 0$ and thus $n'a = 0$ because $n|n'$. Therefore

$$\begin{pmatrix} 1 & p_{1,0,1} \\ 0 & 1 \end{pmatrix}^{n'} = \begin{pmatrix} 1 & n'p_{1,0,1} \\ 0 & 1 \end{pmatrix} = I.$$

That implies that n'-times the generator which is the image of the above matrix vanishes in $NK_0(\mathbf{R}')$. A similar argument applies for the other type of generators. □

Let Γ_i, $i = 0, 1$, be two groups that satisfy the Vanishing Conjecture. We can use Waldhausen's splitting theorem for amalgamated free products and the vanishing result of Theorem 5.1 to show that certain amalgamated products of Γ_i, $i = 0, 1$, satisfy the Vanishing Conjecture.

Theorem 5.2. *Let Γ_i, $i = 0, 1$, satisfy the vanishing condition and G a finitely generated central subgroup of their intersection. Let $\Gamma = \Gamma_0 *_G \Gamma_1$. Then*

$$K_j(\mathbb{Z}\Gamma) = 0, \quad j \leq -2.$$

Proof. By Theorem 5.1 the exotic Nil-groups vanish. Thus there is an exact sequence, for $j \leq -1$

$$(**) \qquad K_j(\mathbb{Z}\Gamma_0) \oplus K_j(\mathbb{Z}\Gamma_1) \to K_j(\mathbb{Z}\Gamma) \to K_{j-1}(\mathbb{Z}G).$$

But $K_j(\mathbb{Z}\Gamma_i) = K_j(\mathbb{Z}G) = 0$, for $i = 0, 1$, $j \leq -2$ by assumption. The vanishing result follows. □

For the discrete subgroups of the cocompact discrete subgroups of Lie groups, the K_{-1}-group is generated by the images of K_{-1}-groups of finite subgroups ([8], [9]). We generalize the result to certain amalgamated free products of such groups.

Corollary 5.3. *With the notation as in Theorem 5.2, assume further that* $K_{-1}(\mathbb{Z}\Gamma_i)$ *is generated by the images of* K_{-1} *of its finite subgroups,* $i = 0, 1$. *Then* $K_{-1}(\mathbb{Z}\Gamma)$ *is generated by the images of its finite subgroups.*

Proof. For $j = -1$, the exact sequence $(**)$ provides an epimorphism

$$K_{-1}(\mathbb{Z}\Gamma_0) \oplus K_{-1}(\mathbb{Z}\Gamma_1) \to K_{-1}(\mathbb{Z}\Gamma) \to 0.$$

By assumption, the groups $K_{-1}(\mathbb{Z}\Gamma_i)$, $i = 0, 1$, are generated by the images of the K_{-1}-groups of their finite subgroups. Thus $K_{-1}(\mathbb{Z}\Gamma)$ is generated by the images of the K_{-1}-groups of the finite subgroups of Γ_0 or Γ_1. By the Corollary in [16], p. 36, every finite subgroup of Γ is contained in a conjugate of Γ_0 or Γ_1. Since inner automorphisms induce the identity in K-theory, the images of $K_{-1}(\mathbb{Z}F')$, where F' is a finite subgroup of Γ_0 or Γ_1, and the images of $K_{-1}(\mathbb{Z}F)$ in $K_{-1}(\mathbb{Z}\Gamma)$ generate the same subgroup. The result follows. □

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