ON THE VANISHING OF CERTAIN K-THEORY NIL-GROUPS

HANS JØRGEN MUNKHOLM[1] AND STRATOS PRASSIDIS[1,2,3]

Abstract. Let Γ_i, $i = 0, 1$, be two groups containing C_p, the cyclic group of prime order p, as a subgroup of index 2. Let $\Gamma = \Gamma_0 *_{C_p} \Gamma_1$. We show that the Nil-groups appearing in Waldhausen's splitting theorem for computing $K_j(\mathbb{Z}\Gamma)$ $(j \leq 1)$ vanish. Thus, in low degrees, the K-theory of $\mathbb{Z}\Gamma$ can be computed by a Mayer-Vietoris type exact sequence involving the K-theory of the integral group rings of the groups Γ_0, Γ_1 and C_p.

1. Introduction

We prove the vanishing of Waldhausen's Nil-groups, in degrees less than or equal to 0, associated to certain amalgamated free products of groups ([12], [13]).

In more detail, let C_p denote the cyclic group of prime order p, and let Γ_0, Γ_1 be two groups, each containing C_p as a subgroup of index 2. Our main result concerns Waldhausen's Nil-groups associated to the amalgamated free product of groups $\Gamma = \Gamma_0 *_{C_p} \Gamma_1$. We write $B_i = \mathbb{Z}[\Gamma_i - C_p]$, $i = 0, 1$, for the $\mathbb{Z}C_p$-sub-bimodule generated by $\Gamma_i - C_p$.

Main Theorem. *With the above notation*

$$\widetilde{Nil}_j(\mathbb{Z}C_p; B_0, B_1) = 0, \quad j \leq 0.$$

Remark. *For $j \leq -1$, this is a special case of results obtained in* [10]. *The extension to the case $j = 0$ was prompted by a question put to the second author (by Jim Davis) in connection with the results appearing in* [3].

Using the Main Theorem and Waldhausen's splitting theorem, we can get information about the (lower) K-theory of Γ.

Corollary. *There are exact sequences*

$$K_1(\mathbb{Z}C_p) \to K_1(\mathbb{Z}\Gamma_0) \oplus K_1(\mathbb{Z}\Gamma_1) \to K_1(\mathbb{Z}\Gamma) \to K_0(\mathbb{Z}C_p) \to \cdots,$$

and

$$Wh(C_p) \to Wh(\Gamma_0) \oplus Wh(\Gamma_1) \to Wh(\Gamma) \to \widetilde{K}_0(\mathbb{Z}C_p) \to \cdots$$

Remark. *For each prime p, this covers precisely three different groups Γ. In fact, each Γ_i is cyclic of order $2p$ or dihedral of order $2p$.*

The proof involves an extension of the methods developed in [10]. There, the Nil-groups in question were shown to be related to the Nil-groups of certain additive categories given in [8]. And this fact was used to establish naturality properties and certain Mayer-Vietoris properties.

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For the present proof, we recall the classical Rim square associated to C_p, i.e., the cartesian square of rings

$$\begin{array}{ccc} \mathbb{Z}C_p & \longrightarrow & \mathbb{Z}[\zeta_p] \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

where ζ_p is a primitive p^{th} root of unity and $\mathbb{F}_p$ is the finite field of p elements. The methods of [10] can be extended to provide a long exact sequence of Nil-groups coming from this square. The three smaller rings in the diagram are Noetherian and have finite cohomological dimension (called regular in [1]). Hence, by Waldhausen's vanishing result, the Nil-groups associated to those rings vanish. Using the exact sequence, we can then derive vanishing results for the Nil-groups associated to the triple $(\mathbb{Z}C_p; B_0, B_1)$.

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2. Preliminaries

We assume that all rings considered have a unit which is preserved by all ring homomorphisms, and that finitely generated free modules have well-defined rank. For any ring R, $\mathcal{M}_R$ denotes the category of right R-modules, $\mathcal{P}_R$ the subcategory of finitely generated projective right R-modules, and $\mathcal{F}_R$ the subcategory of finitely generated right free R-modules. For $\mathcal{A} = \mathcal{M}$, $\mathcal{P}$, or $\mathcal{F}$, $\mathcal{A}^n_R$ denotes the category $\mathcal{A}_R \times \mathcal{A}_R \times \ldots \times \mathcal{A}_R$ (n times).

We will use the notation established in [10], and write $\mathbf{R} = (R; B_0, B_1)$ for a triple where R is a ring and B_i, $i = 0, 1$, are two R-bimodules. Moreover, $\mathbb{F}_{\mathcal{A}}(\mathbf{R})$ denotes the twisted polynomial extension category defined in [8] and [10], for $\mathcal{A} = \mathcal{P}, \mathcal{F}$. To recall its definition, from ([8]), we first note that the triple $\mathbf{R}$ gives rise to a functor $\alpha_R : \mathcal{M}^2_R \to \mathcal{M}^2_R$ defined by

$$\alpha_R(M_0, M_1) = (M_1 \otimes_R B_0, M_0 \otimes_R B_1), \quad \alpha_R(f_0, f_1) = (f_1 \otimes 1, f_0 \otimes 1).$$

Now, the objects of $\mathbb{F}_{\mathcal{A}}(\mathbf{R})$ are simply those of $\mathcal{A}^2_R$, and

$$\mathbb{F}_{\mathcal{A}}(\mathbf{R})(u, v) = \bigoplus\nolimits_{i=0}^{\infty} \mathcal{M}^2_R(u, \alpha^i_R(v)) = \left\{ \sum\nolimits_{i=0}^{\infty} p_i t^i : p_i \in \mathcal{M}^2_R(u, \alpha^i_R(v)) \right\}$$

where we write $p_i : u \to \alpha^i_R(v)$ for the i^{th} component of the morphism. Thus the morphism sets are graded abelian groups, and the powers of the formal variable t are there simply to keep track of degrees. In order to give a different description of these morphism sets, we set $B_i = B_0$ for all even $i \geq 0$, $B_i = B_1$ for all odd $i > 0$, and put

$$B^{(j)}_i = B_i \otimes_R B_{i+1} \otimes_R \ldots \otimes_R B_{i+j-1}$$

for all $i, j \geq 0$. In particular, $B^{(0)}_i = R$, $B^{(1)}_i = B_i$. Similarly, if (Q_0, Q_1) is an object in $\mathbb{F}_{\mathcal{A}}(\mathbf{R})$, we put $Q_i = Q_0$ for all even $i \geq 0$, and $Q_i = Q_1$ for all odd $i > 0$. With this notation

$$\alpha^i_R(Q_0, Q_1) = (Q_i \otimes_R B^{(i)}_{i+1}, Q_{i+1} \otimes_R B^{(i)}_i)$$

Thus, if $u=(P_0,P_1)$ and $v=(Q_0,Q_1)$ are objects in $\mathbb{F}_{\mathcal{A}}(\mathbf{R})$, then

$$\begin{aligned}\mathbb{F}_{\mathcal{A}}(\mathbf{R})(u,v) &= \bigoplus\nolimits_{i\geq 0}\big[\mathcal{M}_R(P_0,Q_i\otimes_R B_{i+1}^{(i)})\oplus\mathcal{M}_R(P_1,Q_{i+1}\otimes_R B_i^{(i)})\big]\\ &= \Big\{\sum\nolimits_{i=0}^{\infty}(p_{(0,i)}\oplus p_{(1,i)})t^i : p_{(k,i)}\in\mathcal{M}_R(P_k,Q_{i+k}\otimes_R B_{i+k+1}^{(i)}),\ k=0,1\Big\},\end{aligned}$$

and there is a forgetful functor ("evaluation at $t=0$")

$$\eta_{\mathcal{A}}:\mathbb{F}_{\mathcal{A}}(\mathbf{R})\ \to\ \mathcal{A}_R^2,\ \mathcal{A}=\mathcal{P},\mathcal{F}.$$

In [10], it was shown that $\mathbb{F}_{\mathcal{A}}$ is a functor on a category $\mathcal{T}$ of triples $\mathbf{R}=(R;B_0,B_1)$ as above with suitable, rather obvious, morphisms. In particular, if $h:R\to S$ is a ring homomorphism, there is a functor

$$h_*:\mathbb{F}_{\mathcal{A}}(\mathbf{R})\ \to\ \mathbb{F}_{\mathcal{A}}(\mathbf{S})$$

where $\mathbf{S}=(S;\overline{\overline{B_0}},\overline{\overline{B_1}})$ with $\overline{\overline{B_i}}=S\otimes_R B_i\otimes_R S$ $(i=0,1)$ given by two-sided reduction of scalars along h.

Triples of the form $\mathbf{R}$ arise naturally from certain co-cartesian diagrams ([12], [13]). To wit, let

$$\begin{array}{ccc} R & \xrightarrow{\alpha_0} & A_0\\ {\scriptstyle\alpha_1}\big\downarrow & & \big\downarrow{\scriptstyle\beta_0}\\ A_1 & \xrightarrow{\beta_1} & \Lambda\end{array}$$

be a co-cartesian diagram of rings and assume further that the maps α_i, $i=0,1$, are pure inclusions, i.e., they are inclusions and they induce R-bimodule splittings

$$A_i = R\oplus B_i,\quad i=0,1$$

where we have identified R with its image under α_i. There result a triple

$$\mathbf{R}=(R;B_0,B_1)\in\mathcal{T};$$

a splitting of Λ as an R-bimodule

$$\Lambda = R\oplus B_0\oplus B_1\oplus(B_0\otimes_R B_1)\oplus(B_1\otimes_R B_0)\oplus(B_0\otimes_R B_1\otimes_R B_0)\oplus\ldots;$$

and an induced filtration of Λ as a ring

$$\begin{aligned}&F_0\Lambda = R,\\ &F_1\Lambda = R\oplus B_0\oplus B_1,\\ &F_2\Lambda = R\oplus B_0\oplus B_1\oplus(B_0\otimes_R B_1)\oplus(B_1\otimes_R B_0),\\ &F_3\Lambda = R\oplus B_0\oplus B_1\oplus(B_0\otimes_R B_1)\oplus(B_1\otimes_R B_0)\oplus(B_0\otimes_R B_1\otimes_R B_0)\oplus(B_1\otimes_R B_0\otimes_R B_1),\\ &\ \ldots\ldots\end{aligned}$$

Moreover, by [13], there is an exact sequence (for $j\in\mathbb{N}$)

$$\cdots\to K_j(A_0)\oplus K_j(A_1)\ \to\ K_j(\Lambda)\ \to\ K_{j-1}(R)\oplus\widetilde{Nil}_{j-1}^{W}(R;B_0,B_1)\ \to\ \cdots\ (*)$$

In other words, the Nil-groups measure the failure of exactness of a K-theory Mayer-Vietoris sequence associated to a co-cartesian diagram of rings with the extra purity assumption.

For any amalgamated free product of groups, $\Gamma = \Gamma_0 *_G \Gamma_1$, the integral group ring $\mathbb{Z}\Gamma$ fits into such a co-cartesian diagram of rings

$$\begin{array}{ccc} R = \mathbb{Z}G & \longrightarrow & A_0 = \mathbb{Z}\Gamma_0 \\ \downarrow & & \downarrow \\ A_1 = \mathbb{Z}\Gamma_1 & \longrightarrow & \Lambda = \mathbb{Z}\Gamma \end{array}$$

with $B_i = \mathbb{Z}[\Gamma_i - G]$, $i = 0, 1$.

In this case, each B_i is free both as a left and a right R module, but we shall start more generally by considering a triple $\mathbf{R}$ which is associated to a co-cartesian diagram of rings for which the bimodules B_i are only assumed to be flat as left R-modules. We set

$$NK_j(\mathbf{R}) = \mathrm{Ker}((\eta_{\mathcal{F}})_j : K_j(\mathbb{F}_{\mathcal{F}}(\mathbf{R})) \to K_j(\mathcal{F}_R^2))$$

(for $j \leq 0$, the K_j-group of an additive category is understood as the K_j-group of its idempotent completion). Then, for $j \leq 1$, there is a natural isomorphism

$$NK_j(\mathbf{R}) \to \widetilde{Nil}^W_{j-1}(R; B_0, B_1)$$

([8], Theorem 2.11, for $j = 1$; [10], Proposition 13, for the lower K-groups) identifying the kernel of the "augmentation" induced map for $\mathbb{F}_{\mathcal{F}}(\mathbf{R})$ with Waldhausen's Nil-groups of one degree less. Since $\mathbb{F}_{\mathcal{F}}(\mathbf{R})$ is cofinal in $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$, and $\mathcal{F}_R^2$ is cofinal in $\mathcal{P}_R^2$, one also has the identification

$$NK_j(\mathbf{R}) = \ker((\eta_{\mathcal{P}})_j).$$

The main purpose of the comparison between the kernel of $(\eta_{\mathcal{P}})_j$ and Waldhausen's Nil-groups is that we can use vanishing results for the former to derive similar results for the latter. Thus, the next result follows immediately from [12], [13].

Lemma 2.1. *Let R be a regular Noetherian ring and $\mathbf{R} = (R; B_0, B_1)$ be a triple associated to a co-cartesian diagram of rings such that B_i is flat as a left R-module for $i = 0, 1$. Then*

$$NK_j(R; B_0, B_1) = 0, \quad j \leq 1.$$

Proof. In fact, by Theorem 4, p. 138, of [13], $\widetilde{Nil}^W_{j-1}(R; B_0, B_1)$ is zero for $j \leq 1$. □

Remark. *The assumption of the Lemma can be weakened to coherent regular rings but we will not use the stronger version in this paper.*

The main result of the present section is Proposition 2.4 which extends the vanishing result of Lemma 2.1 to $j \geq 2$ in case $B_0 \cong B_1 \cong R$. The case $j = 2$ is the one we actually need (in the proof of Theorem 3.15).

We start by establishing the appropriate terminology. Let $\mathbf{R} = (R; B_0, B_1)$ be a triple in $\mathcal{T}$. Then $\rho = (R, R)$ is a basic object in $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$, in the sense of Bass ([2], p. 197), i.e., each object u of $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$ is isomorphic to a direct summand of $\rho^n = (R^n, R^n)$ for some integer n. We write $R_\rho = \mathrm{End}_{\mathbb{F}_{\mathcal{P}}(\mathbf{R})}(\rho)$ for the endomorphism ring of ρ. There is a split inclusion of rings $\iota : R \times R \to R_\rho$ by considering pairs of elements of R as endomorphisms of degree zero of ρ. The splitting σ is given by the forgetful map to the zero degree component of any endomorphism. A morphism of degree i, $\phi = (\phi_0, \phi_1)t^i : \rho \to \alpha_R^i(\rho)$, can be identified with the element $(\phi_0(1), \phi_1(1)) \in B_{i+1}^{(i)} \oplus B_i^{(i)}$. Multiplication in R_ρ, i.e., composition of endomorphisms, is then given by concatenation with the added convention that

$B_iB_i = 0$, $i = 0, 1$. Considering the degree mod 2 of components one obtains a natural splitting of R_ρ as an $R\times R$-bimodule

$$R_\rho = R_{even} \oplus R_{odd}.$$

The component R_{even} is a subring of R_ρ, and R_{odd} is an R_{even}-bimodule.

The ring R_ρ is also $\mathbb{N}$-graded. The abelian group of degree i is $R_{\rho,i} = B_{i+1}^{(i)} \oplus B_i^{(i)}$, which also has a natural diagonal $R\times R$-bimodule structure. In case the triple $\mathbf{R}$ is associated to a co-cartesian diagram of rings, then R_ρ is the associated grading of the filtration of $\Lambda\times R$. Another grading of the ring Λ is given in [11].

Lemma 2.2. *With the above notation, there is an isomorphism $F_j : K_j(R_\rho) \to K_j(\mathbb{F}_\mathcal{P}(\mathbf{R}))$ making the diagram*

$$\begin{array}{ccc} K_j(R_\rho) & \xrightarrow{F_j} & K_j(\mathbb{F}_\mathcal{P}(\mathbf{R})) \\ {\scriptstyle K_j(\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle (\eta_\mathcal{P})_j} \\ K_j(R\times R) & \longrightarrow & K_j(\mathcal{P}_R^2) \end{array}$$

commute for $j \geq 1$. The horizontal map at the bottom is the natural isomorphism.

Proof. Since ρ is a basic object in $\mathbb{F}_\mathcal{P}(\mathbf{R})$, the functor

$$F : \mathcal{F}_{R_\rho} \to \mathbb{F}_\mathcal{P}(\mathbf{R}), \quad F(R_\rho^n) = \rho^n$$

is a full, faithful and cofinal functor. Thus, it induces an isomorphism on K_j-groups for $j \geq 1$ ([7], Thm 1.1; also [5], Proposition 1.1; [6], p. 225).

The bottom arrow is an isomorphism, in degrees $j \geq 1$, because $R \times R$ can be thought of as the endomorphism ring of the basic object (R, R) in the category $\mathcal{F}_R^2$ which is cofinal in $\mathcal{P}_R^2$.

Commutativity of the diagram is clear. □

We now restrict our attention to the case where $B_i \cong R$, $i = 0, 1$, as R-bimodules. In this case, the ring R_ρ has a description as a matrix ring. In fact, let S be the subring of $M_2(R[x])$ given by

$$S = \begin{pmatrix} R[x^2] & xR[x^2] \\ xR[x^2] & R[x^2] \end{pmatrix}$$

and let $\epsilon : S \to R\times R$ be the natural augmentation map

$$\begin{pmatrix} a(x^2) & xb(x^2) \\ xc(x^2) & d(x^2) \end{pmatrix} \longmapsto (a(0), d(0))$$

Proposition 2.3. *Let $B_0 \cong B_1 \cong R$ as R-bimodules. Then there is a ring isomorphism*

$$\kappa : R_\rho \to S$$

which commutes with the augmentation maps, i.e., $\epsilon\circ\kappa = \sigma$.

Proof. Because of the assumption on B_i, the degree i component $R_{\rho,i}$, is isomorphic to $R\times R$ as an $R\times R$-bimodule with the degree i endomorphism $(id_R, id_R)t^i$

corresponding to the element $(1,1)$. We define $R{\times}R$-bimodule maps

$$\kappa|R_{\rho,2i} : R_{\rho,2i} \;\to\; \begin{pmatrix} Rx^{2i} & 0 \\ 0 & Rx^{2i} \end{pmatrix}, \qquad (1,1) \mapsto \begin{pmatrix} x^{2i} & 0 \\ 0 & x^{2i} \end{pmatrix}$$

$$\kappa|R_{\rho,2i+1} : R_{\rho,2i+1} \;\to\; \begin{pmatrix} 0 & Rx^{2i+1} \\ Rx^{2i+1} & 0 \end{pmatrix}, \qquad (1,1) \mapsto \begin{pmatrix} 0 & x^{2i+1} \\ x^{2i+1} & 0 \end{pmatrix}$$

The resulting map κ is the required ring isomorphism. By construction, it commutes with the augmentation homomorphisms. □

The above result reduces the problem of computing the NK-groups to a problem in the K-theory of certain matrix rings.

Proposition 2.4. *Let R be a regular Noetherian ring and assume that $B_0 \cong B_1 \cong R$ as R-bimodules. Then for all $j{\in}\mathbb{Z}$, $NK_j(\mathbf{R}) = 0$.*

Proof. For $j \leq 0$ the result follows from [10]. Let $j \geq 1$. By Lemma 2.2, it is enough to prove the vanishing of the kernel of the map induced on the K-groups by the augmentation σ. If R is regular Noetherian, then $R[x]$ is regular Noetherian by Hilbert's Basis and Syzygy Theorems. Then $M_2(R[x])$ is Noetherian (because is finitely generated as an $R[x]$-module) and it has finite cohomological dimension. Also, $M_2(R[x])$ is a free S-module with basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Then S is a regular Noetherian ring ([9], p. 96, Proposition 2.30) and the same is true for R_ρ. Since R_ρ is a graded ring with zero grading $R{\times}R$, the augmentation induced map

$$\sigma_j : K_j(R_\rho) \;\to\; K_j(R{\times}R)$$

is an isomorphism ([9], Theorem 2.37, p. 98). Therefore $NK_j(\mathbf{R})$, being the kernel of σ_j, vanishes. □

Corollary 2.5. *Let $\mathbf{F} = (\mathbb{F}_p; \mathbb{F}_p, \mathbb{F}_p)$ where $\mathbb{F}_p$ is the field of p elements with p prime. Then*

$$NK_j(\mathbf{F}) = 0, \quad j{\in}\mathbb{N}.$$

In particular, $(\eta_{\mathcal{P}})_j : K_j(\mathbb{F}_{\mathcal{P}}(\mathbf{R})) \;\to\; K_j(\mathcal{P}^2_R))$ is an isomorphism.

3. Mayer-Vietoris Sequences

Let $h : R \;\to\; S$ be a ring homomorphism. Then h induces a functor

$$h^* : \mathcal{M}_S \;\to\; \mathcal{M}_R$$

which maps an S-module M to the R-module with underlying abelian group M and R-structure induced by h. We are interested in the image of the functor h^*.

Definition 3.1. Let $h : R \;\to\; S$ be a ring epimorphism. A right R-module M is called h-extended, if there is a right S-module structure on M such that $mr = mh(r)$ for all $m{\in}M$, $r{\in}\, R$. In other words, M is h-extended if M is in the image of h^*.

The main technical property of such extended modules is expressed in the following Lemma.

Lemma 3.2. *Let $h : R \to S$ be a ring epimorphism and M an h-extended right R-module. Then there is a natural right S-module isomorphism*

$$k : M \otimes_R S \to M.$$

Proof. It is easy to check that there is a well defined homomorphism given by $k(m \otimes s) = ms$, and that $\ell(m) = m \otimes 1_S$, defines an inverse. □

For an R-bimodule B we write $\overline{B} = S \otimes_R B$ for the $S - R$ bimodule obtained by left-sided reduction of scalars. Also, recall that $\overline{\overline{B}} = S \otimes_R B \otimes_R S$. We immediately get the following result.

Corollary 3.3. *Let B_i, $i = 0, 1$, be R-bimodules. If the $S - R$-bimodules $\overline{B_0}$ and $\overline{B_1}$ are h-extended as right R-modules, then*

$$\overline{\overline{B_0}} \otimes_S \overline{\overline{B_1}} = (S \otimes_R B_0 \otimes_R S) \otimes_S (S \otimes_R B_1 \otimes_R S) \cong S \otimes_R B_0 \otimes_R B_1 = \overline{B_0 \otimes_R B_1}$$

as S-bimodules. In particular,

$$h^*(\overline{\overline{B_i^{(j)}}}) \cong \overline{B_i^{(j)}}$$

as S-bimodules.

We will study the extension properties of a pull-back diagram of rings. We start with a cartesian diagram of rings

$$\begin{array}{ccc} R & \xrightarrow{h_2} & R_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ R_1 & \xrightarrow{f_1} & R_0 \end{array}$$

where we assume that h_1 and h_2 are epimorphisms. The diagram induces a pull-back diagram of categories ([1], Ch. IX, Thm. 5.1)

$$\begin{array}{ccc} \mathcal{P}_R & \longrightarrow & \mathcal{P}_{R_2} \\ \downarrow & & \downarrow \\ \mathcal{P}_{R_1} & \longrightarrow & \mathcal{P}_{R_0} \end{array}$$

Notice that the diagram induces an exact sequence of R-bimodules

$$0 \to R \xrightarrow{(\,h_1\ h_2\,)} R_1 \oplus R_2 \xrightarrow{\binom{f_1}{-f_2}} R_0 \to 0 \qquad (E)$$

where the action of R on R_j, $j = 0, 1, 2$, is induced by the maps in the cartesian square.

First we recall a routine algebraic lemma which uses the following notation. Let $h : R \to S$ be a ring homomorphism, Q and P right R-modules, and B an R-bimodule. Then there is a right R-module homomorphism

$$Q \otimes_R B \to Q \otimes_R S \otimes_R B, \quad q \otimes b \mapsto q \otimes 1_s \otimes b$$

and an induced abelian group homomorphism

$$\overline{h} : Hom_R(P, Q \otimes_R B) \to Hom_R(P, Q \otimes_R S \otimes_R B)$$

Lemma 3.4. *If P and Q are projective, and B is left flat, then the sequence*

$$0 \to Hom_R(P, Q \otimes_R B) \xrightarrow{(\overline{h_1}\ \overline{h_2})} Hom_R(P, Q \otimes_R R_1 \otimes_R B) \oplus Hom_R(P, Q \otimes_R R_2 \otimes_R B)$$
$$\xrightarrow{\begin{pmatrix} \overline{f_1} \\ -\overline{f_2} \end{pmatrix}} Hom_R(P, Q \otimes_R R_0 \otimes_R B) \to 0$$

is exact.

Proof. The assumptions on Q and B show that the induced sequence

$$0 \to Q \otimes_R B \to Q \otimes_R R_1 \otimes_R B \oplus Q \otimes_R R_2 \otimes_R B \to Q \otimes_R R_0 \otimes_R B \to 0$$

is exact. The result follows because P is projective. □

Corollary 3.5. *Let P and Q be projective right R-modules and B an R-bimodule which is flat as a left R-module. Assume further that $R_j \otimes_R B$ is h_j-extended as a right R-module ($j = 1, 2$), and that $R_0 \otimes_R B$ is $f_1 h_1$-extended (= $f_2 h_2$-extended) as a right R-module. Then the exact sequence (E) induces an exact sequence of abelian groups*

$$0 \to Hom_R(P, Q \otimes_R B) \xrightarrow{(\overline{h_1}\ \overline{h_2})}$$
$$Hom_{R_1}(P \otimes_R R_1, Q \otimes_R R_1 \otimes_R B) \oplus Hom_{R_2}(P \otimes_R R_2, Q \otimes_R R_2 \otimes_R B) \xrightarrow{\begin{pmatrix} \overline{f_1} \\ -\overline{f_2} \end{pmatrix}}$$
$$Hom_{R_0}(P \otimes_R R_0, Q \otimes_R R_0 \otimes_R B) \longrightarrow 0.$$

Proof. This follows from Lemma 3.4 using the adjointness isomorphisms

$$Hom_R(P, Q \otimes_R R_i \otimes_R B) \cong Hom_{R_i}(P \otimes_R R_i, Q \otimes_R R_i \otimes_R B)$$

($i = 0, 1, 2$). □

We now consider a triple $\mathbf{R} = (R; B_0, B_1)$ such that $R_j \otimes_R B_i$ is h_j-extended as a right R-module ($j = 1, 2$ and $i = 0, 1$) and $R_0 \otimes_R B_i$ is $f_1 h_1$-extended as a right R-module ($i = 0, 1$). It follows that $R_j \otimes_R B_i$ is f_j-extended as a right R_j-module ($i = 0, 1$, $j = 1, 2$). We further assume that the modules B_i are flat as left R-modules ($i = 0, 1$). Then we get corresponding objects in $\mathcal{T}$,

$$\mathbf{R}_j = (R_j; R_j \otimes_R B_0, R_j \otimes_R B_1), \quad j = 0, 1, 2;$$

a pull-back of additive categories (defining $\mathbb{P}$)

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{h_2'} & \mathbb{F}_{\mathcal{P}}(\mathbf{R}_2) \\ {\scriptstyle h_1'}\downarrow & & \downarrow{\scriptstyle f_2'} \\ \mathbb{F}_{\mathcal{P}}(\mathbf{R}_1) & \xrightarrow{f_1'} & \mathbb{F}_{\mathcal{P}}(\mathbf{R}_0) \end{array}$$

where f_j' is induced by f_j ($j = 1, 2$); and a functor

$$\phi : \mathbb{F}_{\mathcal{P}}(\mathbf{R}) \to \mathbb{P}$$

induced by the universal properties of the pull-back.

In [10], it has been shown that if a ring homomorphism is surjective, then the map induced in the twisted polynomial extension categories is E-surjective in the sense of [1] (Def. 2.4, p. 356). Thus by [2], A.13, p. 151, the above square induces a commutative diagram of exact sequences in K-theory

$$\begin{array}{ccccccc} K_2(\mathbb{F}_{\mathcal{P}}(\mathbf{R}_1))\oplus K_2(\mathbb{F}_{\mathcal{P}}(\mathbf{R}_2)) & \to & K_2(\mathbb{F}_{\mathcal{P}}(\mathbf{R}_0)) & \to & K_1(\mathbb{P}) & \to & K_1(\mathbb{F}_{\mathcal{P}}(\mathbf{R}_1))\oplus K_1(\mathbb{F}_{\mathcal{P}}(\mathbf{R}_2)) \\ \downarrow & & \downarrow & & \kappa\downarrow & & \downarrow \\ K_2(\mathcal{P}^2_{R_1})\oplus K_2(\mathcal{P}^2_{R_2}) & \to & K_2(\mathcal{P}^2_{R_0}) & \to & K_1(\mathcal{P}^2_R) & \to & K_1(\mathcal{P}^2_{R_1})\oplus K_1(\mathcal{P}^2_{R_2}) \end{array}$$

The vertical maps are induced by the obvious functors between two pull-back diagrams.

Lemma 3.6. *The functor ϕ is full and faithful.*

Proof. Let $u = (P_0, P_1)$, $v = (Q_0, Q_1)$ be two objects in $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$. We must show that the map induced by ϕ

$$\phi' : \mathbb{F}_{\mathcal{P}}(\mathbf{R})(u, v) \to \mathbb{P}(\phi(u), \phi(v))$$

is a group isomorphism, and start by setting the notation. For $k = 0, 1$, and $i \geq 0$,

$$\overline{h^{(i)}_{j,k}} : \mathcal{P}_R(P_k, Q_{i+k}\otimes_R B^{(i)}_{i+k+1}) \to \mathcal{P}_R(P_k\otimes_R R_j, Q_{i+k}\otimes_R R_j\otimes_R B^{(i)}_{i+k+1})$$

and

$$\overline{f^{(i)}_{j,k}} : \mathcal{P}_R(P_k\otimes_R R_j, Q_{i+k}\otimes_R R_j\otimes_R B^{(i)}_{i+k+1}) \to \mathcal{P}_R(P_k\otimes_R R_0, Q_{i+k}\otimes_R R_0\otimes_R B^{(i)}_{i+k+1})$$

are the maps induced by h_j, f_j, $j = 1, 2$.

<u>ϕ' is a monomorphism</u>. An element β in the morphism set $\mathbb{F}_{\mathcal{P}}(\mathbf{R})(u, v)$ can be written

$$\beta = \sum\nolimits_{i\geq 0} (p_{(0,i)}\oplus p_{(1,i)})t^i$$

and its image under ϕ' has the form

$$\phi'(\beta) = \sum\nolimits_{i\geq 0} \big((\overline{h^{(i)}_{1,0}}(p_{(0,i)}) \oplus \overline{h^{(i)}_{2,0}}(p_{(0,i)})\big) \oplus \big((\overline{h^{(i)}_{1,1}}(p_{(1,i)}) \oplus \overline{h^{(i)}_{2,1}}(p_{(1,i)})\big)t^i$$

If $\phi'(\beta) = 0$, then for all $k = 0, 1$ and $i \geq 0$,

$$\overline{h^{(i)}_{1,k}}(p_{(k,i)}) \oplus \overline{h^{(i)}_{2,k}}(p_{(k,i)}) = 0$$

Using Corollary 3.5, we see that each $p_{(k,i)} = 0$. Thus $\beta = 0$.

<u>ϕ' is an epimorphism</u>. Let $\gamma\in\mathbb{P}(\phi(u), \phi(v))$. Then $\gamma = \gamma_1\oplus\gamma_2$ where

$$\gamma_j\in\mathbb{F}_{\mathcal{P}}(\mathbf{R}_j)((h_j)_*(u), (h_j)_*(v)), \quad j = 1, 2$$

Since γ is a morphism in the pull-back category

$$(f_1)_*(\gamma_1) = (f_2)_*(\gamma_2) \text{ in } \mathbb{F}_{\mathcal{P}}(\mathbf{R}_0)((f_1h_1)_*(u), (f_1h_1)_*(v)) \qquad (1)$$

As before, each γ_j can be written as a direct sum of homomorphisms

$$\gamma_j = \sum\nolimits_{i\geq 0} (p_{(0,i)j}\oplus p_{(1,i)j})t^i$$

and condition (1) implies that, for $k = 0, 1$, $i \geq 0$,

$$\overline{f^{(i)}_{1,k}}(p_{(k,i)1}) = \overline{f^{(i)}_{2,k}}(p_{(k,i)2}), \quad \text{i.e.,} \quad (p_{(k,i)1}, p_{(k,i)2}) \in \operatorname{Ker}\big(\overline{f^{(i)}_{1,k}} - \overline{f^{(i)}_{2,k}}\big)$$

By Corollary 3.5, there is $p_{(k,i)} \in Hom_R(P_k, Q_{i+k} \otimes_R B^{(i)}_{i+k+1})$ such that $\overline{h^{(i)}_{j,k}}(p_{(k,i)}) = p_{(k,i)j}$. Set

$$\beta = \sum_{i \geq 0} (p_{(0,i)} \oplus p_{(1,i)}) t^i$$

Then $\phi'(\beta) = \gamma$. □

We recall the definition of an elementary morphism in an additive category. Let $\mathbf{A}$ be an additive category, u an object of $\mathbf{A}$. An automorphism a of u is called elementary if there is a decomposition $u = u_0 \oplus u_1$ such that a takes the form

$$a = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some $b : u_1 \to u_0$. Also, $K_1(\mathbf{A})$ can be defined as the group generated by all pairs (u, a), where u is an object of $\mathbf{A}$ and a an automorphism of u, divided by the subgroup generated by pairs (v, e) with e elementary.

We will prove the analogue of E-surjectivity for functors induced by ring epimorphism on the twisted polynomial extension category of finitely generated projective modules ([1], p. 449). Let $\mathbf{R} = (R; B_0, B_1)$ be a triple. We start with an observation on the morphism sets of objects in $\mathbb{F}_{\mathcal{F}}(\mathbf{R})$. Let $u = (F_0, F_1)$ and $v = (G_0, G_1)$ be two objects in $\mathbb{F}_{\mathcal{F}}(\mathbf{R})$ of ranks (m_0, m_1) and (n_0, n_1). As before, we write $G_i = G_0$ for all even $i \geq 0$, $G_i = G_1$ for all odd $i > 0$ and we write n_i for the rank of G_i. For any R-bimodule B, we write $M_{m \times n}(B)$ for the abelian group of $m \times n$ matrices with entries in B.

Lemma 3.7. *With the above notation, a choice of bases of the free modules involved induces an isomorphism of abelian groups*

$$\mathbb{F}_{\mathcal{F}}(\mathbf{R})(u, v) \cong \bigoplus_{i \geq 0} \left[M_{m_0 \times n_i}(B^{(i)}_{i+1}) \oplus M_{m_1 \times n_{i+1}}(B^{(i)}_{i}) \right]$$

Proof. This is standard matrix calculation. □

Let $h : R \to S$ be a ring epimorphism and $\mathbf{R} = (R; B_0, B_1)$. Let $\mathbf{S} = (S; \overline{\overline{B_0}}, \overline{\overline{B_1}})$. The map h induces a functor

$$h_* : \mathbb{F}_{\mathcal{F}}(\mathbf{R}) \to \mathbb{F}_{\mathcal{F}}(\mathbf{S})$$

Let $\overline{u_j}$, $j = 1, 2$, be objects in $\mathbb{F}_{\mathcal{F}}(\mathbf{S})$. Then there are objects u_j in $\mathbb{F}_{\mathcal{F}}(\mathbf{R})$ such that $h_*(u_j) \cong \overline{u_j}$, $j = 1, 2$. A choice of isomorphisms induces an abelian group homomorphism

$$h_* : \mathbb{F}_{\mathcal{F}}(\mathbf{R})(u_1, u_2) \to \mathbb{F}_{\mathcal{F}}(\mathbf{S})(\overline{u_1}, \overline{u_2})$$

The next result is an easy corollary of Lemma 3.7.

Corollary 3.8. *With the above notation, h_* is an epimorphism.*

Proof. Let B be any R-bimodule. Since h is a ring epimorphism, the map

$$h' : B \to \overline{\overline{B}}, b \mapsto 1_S \otimes b \otimes 1_S$$

is an abelian group epimorphism. Thus for any $m, n > 0$, the induced map on the matrix group

$$h'_{m \times n} : M_{m \times n}(B) \to M_{m \times n}(\overline{\overline{B}})$$

is an epimorphism. The result follows from the identifications proved in Lemma 3.7. □

The next Lemma is on the E-surjectivity of the functor h_*.

Lemma 3.9. *Let $\overline{u}$ be an object of $\mathbb{F}_{\mathcal{P}}(\mathbf{S})$ and g an elementary automorphism of $\overline{u}$. Then there is an object $\overline{v}$ in $\mathbb{F}_{\mathcal{P}}(\mathbf{S})$, an object w in $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$ and an elementary automorphism f of w such that $h_*(w) \cong \overline{u} \oplus \overline{v}$ and under the isomorphism $h_*(f)$ is conjugate to $g \oplus 1_{\overline{v}}$.*

Proof. Since g is elementary, there is a splitting $\overline{u} = \overline{u_1} \oplus \overline{u_2}$ such that

$$g = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

where $\gamma \in \mathbb{F}_{\mathcal{P}}(\mathbf{S})(\overline{u_1}, \overline{u_2})$. Choose objects $\overline{v_j}$, $j = 1, 2$, in $\mathbb{F}_{\mathcal{P}}(\mathbf{S})$ such that $\overline{u_j} \oplus \overline{v_j}$ is in $\mathbb{F}_{\mathcal{F}}(\mathbf{S})$. Set $\overline{v} = \overline{v_1} \oplus \overline{v_2}$. Then $\overline{u} \oplus \overline{v}$ is an object in $\mathbb{F}_{\mathcal{F}}(\mathbf{S})$ and under the isomorphism

$$\overline{u} \oplus \overline{v} \cong (\overline{u_1} \oplus \overline{v_1}) \oplus (\overline{u_2} \oplus \overline{v_2})$$

$g \oplus 1_{\overline{v}}$ corresponds to

$$g \oplus 1_{\overline{v}} = \begin{pmatrix} 1 & \gamma' \\ 0 & 1 \end{pmatrix}$$

where

$$\gamma' : \overline{u_1} \oplus \overline{v_1} \xrightarrow{\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 0 \end{smallmatrix}\right)} \overline{u_2} \oplus \overline{v_2}$$

Since $\overline{u_j} \oplus \overline{v_j}$, $j = 1, 2$, are objects in $\mathbb{F}_{\mathcal{F}}(\mathbf{S})$, the result follows from Corollary 3.8. □

Using the above basic results, we will show that the functor ϕ is cofinal.

Lemma 3.10. *The functor $\phi : \mathbb{F}_{\mathcal{P}}(\mathbf{R}) \to \mathbb{P}$ is cofinal.*

Proof. Let $u = (u_1, u_2, g)$ be an object in $\mathbb{P}$. We can add an object $v = (v_1, v_2, g')$, with g' an isomorphism of degree zero, to u such that $u_j \oplus v_j$ is an object of $\mathbb{F}_{\mathcal{F}}(\mathbf{R}_j)$, $j = 1, 2$. Thus we can assume that u has the property that u_j is an object in $\mathbb{F}_{\mathcal{F}}(\mathbf{R}_j)$, $j = 1, 2$. Then g is an isomorphism between $f_1'(u_1)$ and $f_2'(u_2)$ in $\mathbb{F}_{\mathcal{F}}(\mathbf{R}_0)$. By choosing bases for the free modules involved, we can assume that g is an automorphism. First we will show that (u_1, u_2, g) is in the image of ϕ (up to equivalence) if g is an elementary automorphism in $\mathbb{F}_{\mathcal{P}}(\mathbf{R}_0)$. In this case, after more stabilization, $g = f_1'(g_1)$ for some automorphism g_1 of u_1 (Lemma 3.9) because f_1 is onto. Then the pair $(g_1, 1)$ induces an isomorphism between $(u_1, u_2, 1)$ and (u_1, u_2, g). But $(u_1, u_2, 1)$ is in the image of ϕ. The general case follows because the morphism in the object $(u_1 \oplus u_1, u_2 \oplus u_2, g \oplus g^{-1})$ can be written as a composition of elementary matrices. □

The following theorem is the main technical result of this paper. From it, the truly main result, Theorem 3.15, follows essentially by manipulation of definitions.

Theorem 3.11. *The functor ϕ induces an isomorphism*

$$\phi_j : K_j(\mathbb{F}_{\mathcal{P}}(\mathbf{R})) \to K_j(\mathbb{P}), \quad j \geq 1$$

Proof. The functor ϕ is full and faithful and cofinal. Thus $\mathbb{F}_{\mathcal{P}}(\mathbf{R})$ can be identified with a full cofinal subcategory of $\mathbb{P}$. The result follows from [7], Thm 1.1. □

Corollary 3.12. *Let $NK_1(\mathbf{R}_j) = 0$ for $j = 1, 2$ and $NK_2(\mathbf{R}_0) = 0$. Then $NK_1(\mathbf{R}) = 0$.*

Proof. The NK_1-group associated to $\mathbf{R}$ is given as the kernel of the composition

$$K_1(\mathbb{F}_{\mathcal{P}}(\mathbf{R})) \xrightarrow{\phi_1} K_1(\mathbb{P}) \xrightarrow{\kappa} K_1(\mathcal{P}^2_R)$$

If we use ϕ_1 from Theorem 3.11 to identify $K_1(\mathbb{F}_{\mathcal{P}}(\mathbf{R}))$ with $K_1(\mathbb{P})$, then $NK_1(\mathbf{R})$ is identified as the kernel of κ in the diagram preceding Lemma 3.6. The vanishing assumptions guarantee that the immediate neighbors of κ are monomorphims (actually isomorphisms). Also, the leftmost vertical map is a split epimorphism (in fact, an obvious splitting exists at the level of categories). Thus, by the five lemma, κ is a monomorphism. □

We finally specialize to the case of interest. Let $\Gamma = \Gamma_0 *_G \Gamma_1$ where G is a finite normal subgroup of Γ_i, $i = 0, 1$. Let $B_i = \mathbb{Z}[\Gamma_i - G]$, $i = 0, 1$, be the two $\mathbb{Z}G$-bimodules which appear in the definition of Waldhausen's Nil-groups in this case. Let N be the norm element in $\mathbb{Z}G$ i.e. N is the sum of all the group elements, and $\langle N \rangle$ the ideal generated by N. Let $n = |G|$. Notice that $\mathbb{Z}$ is isomorphic to the quotient of $\mathbb{Z}G$ by the ideal generated by the elements of the form $g-1$, $g{\in}G$ and $\mathbb{Z}/n\mathbb{Z}$ is the quotient of $\mathbb{Z}G$ by the ideal generated by the elements N and $g-1$, $g \in G$. Then we have a cartesian square

$$\begin{array}{ccc} \mathbb{Z}G & \xrightarrow{p_1} & \mathbb{Z}G/\langle N \rangle \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle q_1} \\ \mathbb{Z} & \xrightarrow{q_2} & \mathbb{Z}/n\mathbb{Z} \end{array}$$

Lemma 3.13. *With the above notation, the right module $\mathbb{Z}G/\langle N \rangle \otimes_{\mathbb{Z}G} B_i$ (respectively $\mathbb{Z} \otimes_{\mathbb{Z}G} B_i$) is p_1 (respectively p_2) extendable, $i = 0, 1$. That implies that B_i is $q_1 p_1$-extendable.*

Proof. Notice that if $\gamma{\in}\Gamma_i$ then $\gamma N = N\gamma$ because G is normal in Γ_i. That implies that the ideal generated by N acts trivially, from the right, on $\mathbb{Z}G/\langle N \rangle \otimes_{\mathbb{Z}G} B_i$. For the other ring, notice that all the elements of $\mathbb{Z}G$ of the form $g - g'$, $g, g'{\in}G$ act trivially on the right on $\mathbb{Z} \otimes_{\mathbb{Z}G} B_i$. □

In the special case that $n = p$ is a prime then $\mathbb{Z}G/\langle N \rangle \cong \mathbb{Z}[\zeta_p]$, $\mathbb{Z}$ with a primitive p-th root of unity attached. This ring is regular Noetherian. The rings $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ are also regular Noetherian rings and therefore the corresponding NK_j - groups vanish for $j \leq 1$, cf. Lemma 2.1.

Let C_p have index 2 in a group G. We will describe the $\mathbb{F}_p$-bimodule structure of $\mathbb{F}_p \otimes_{\mathbb{Z}C_p} \mathbb{Z}[G - C_p]$. The action of $\mathbb{Z}C_p$ on $\mathbb{F}_p$ is given via the epimorphism:

$$\mathbb{Z}C_p \to \mathbb{Z}C_p/\langle N, g-1, g{\in}C_p \rangle \cong \mathbb{F}_p$$

Lemma 3.14. *With the above notation, there is an $\mathbb{F}_p$-bimodule isomorphism*

$$\alpha : \mathbb{F}_p \otimes_{\mathbb{Z}C_p} \mathbb{Z}[G - C_p] \to \mathbb{F}_p$$

Proof. The $\mathbb{Z}C_p$-bimodule $\mathbb{Z}[G - C_p]$ has a decomposition

$$\mathbb{Z}[G - C_p] = \bigoplus_{h \in G - C_p} \mathbb{Z}h$$

as an abelian group. The action of C_p is given by permuting the summands according to the action of C_p on $G - C_p$. Let h, h' be two elements in $G - C_p$. Then there is $g{\in}C_p$ such that $gh = h'$. Therefore, in $\mathbb{F}_p \otimes_{\mathbb{Z}C_p} \mathbb{Z}[G - C_p]$,

$$x \otimes h' = x \otimes (gh) = x \otimes h, \quad \text{for all } x{\in}\mathbb{F}_p.$$

Define α on $\mathbb{Z}h$ by setting $\alpha(x \otimes h) = x$ and extending linearly. Then α is the required isomorphism. □

The next theorem is a combination of the above observations and Theorem 3.11.

Theorem 3.15. *In the above notation, if $n = p$ is a prime, then*

$$\widetilde{Nil}_0^W(\mathbb{Z}C_p; B_0, B_1) = NK_1(\mathbb{Z}C_p; B_0, B_1) = 0.$$

Proof. In this case the pull-back diagram above becomes

$$\begin{array}{ccc} \mathbb{Z}C_p & \longrightarrow & \mathbb{Z}[\zeta_p] \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

The rings $\mathbb{Z}$, $\mathbb{Z}[\zeta_p]$, and $\mathbb{F}_p$ are regular. By Lemma 2.1 the NK_1-groups of the induced triples vanish for the three rings above. The bimodules B_i, $i = 0, 1$, are extendable over $\mathbb{F}_p$ (Lemma 3.13). Thus Lemma 3.14 applies and Corollary 2.5 implies that

$$NK_2(\mathbb{F}_p; \mathbb{F}_p \otimes_{\mathbb{Z}C_p} B_0, \mathbb{F}_p \otimes_{\mathbb{Z}C_p} B_1) = 0$$

Then the result follows from Corollary 3.12. □

Combining with the results in [10] and Waldhausen's exact sequence (*) of Section 2, we have

Corollary 3.16. *With the above assumptions, there are exact sequences*

$$K_1(\mathbb{Z}C_p) \to K_1(\mathbb{Z}\Gamma_0) \oplus K_1(\mathbb{Z}\Gamma_1) \to K_1(\mathbb{Z}\Gamma) \to K_0(\mathbb{Z}C_p) \to \cdots,$$

and

$$Wh(C_p) \to Wh(\Gamma_0) \oplus Wh(\Gamma_1) \to Wh(\Gamma) \to \widetilde{K}_0(\mathbb{Z}C_p) \to \cdots$$

As a particular application we have the following result which was used in the calculations in [3].

Corollary 3.17. $Wh(S_3 *_{\mathbb{Z}/3\mathbb{Z}} S_3) = 0$ *where S_3 is the symmetric group on 3 letters.*

Proof. From Theorem 3.15, we know that Waldhausen's Nil group vanish. We also know that the lower K-theory of S_3 and $\mathbb{Z}/3\mathbb{Z}$ vanish. The result follows from Corollary 3.16. □

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IMADA, Odense Universitet, Campusvej 55, DK 5230 Odense, Denmark

Department of Mathematics, Vanderbilt University, Nashville TN 37240, U.S.A.