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On the nil groups of Waldhausen nils ☆

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Abstract

We study the nil groups in the algebraic K-theory of the group ring of a virtually cyclic group. We prove these vanish in low degrees and mention some consequences in case the Farrell–Jones Isomorphism conjecture is valid.

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1. Introduction

A group is called *virtually cyclic* if it is either finite or has an infinite cyclic subgroup of finite index. The relevance of studying properties of virtually cyclic groups is that the so-called Isomorphism Conjecture (IC) suggests that they completely determine algebraic and topological invariants such as the algebraic K-theory of the group ring of any discrete group, see also [8] for other invariants. This conjecture has been verified in special cases.

Our main result is the following, this may be regarded as a natural consequence of the results in [11,12].

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Main Theorem. *Let Γ be a virtually cyclic group and R denote the ring of integers in an algebraic extension of the rationals. Then*

$$N^r K_i(R\Gamma) = 0, \quad \textit{for } r \geqslant 1,\ i \leqslant -1.$$

The case of finite groups and infinite virtually cyclic groups of the type $H \times T$ where H is a finite group and T the infinite cyclic group is in [3], hence our contribution is the case of infinite virtually cyclics.

Our analysis of infinite virtually cyclic groups is based on the following classification theorem due to P. Scott and T. Wall:

Proposition 1 [19, Theorem 5.12]. *Let Γ be an infinite virtually cyclic group. Then Γ is isomorphic to*

Type 1. *A semidirect product $H \rtimes T$ where H is a finite group and T is the infinite cyclic group, or*

Type 2. *A nontrivial amalgam of finite groups of the form*

$$G_0 \underset{H}{*} G_1,$$

where $|G_0 : H| = 2 = |G_1 : H|$.

Thus, the techniques are divided accordingly. For groups of type 1 we use a twisted version of Bass–Heller–Swan formula found by Farrell and Hsiang in [7] then extended in [5] and in [9]. Groups of type 2 require the analysis of Waldhausen's nil groups associated to a co-Cartesian square of rings. These techniques are essentially those of the paper by Connolly and Da Silva in [3] and then explored extensively by Connolly and Prassidis [5], Prassidis and Juan-Pineda [12], Farrell and Jones [9] and Juan-Pineda [11]. An important improvement in our techniques is that it allows us to work with group rings of the form $R\Gamma$ where R is any ring of integers in an algebraic extension of the rationals and not only the integers. This is relevant as more general Isomorphism Conjectures have been proposed, see, for example, [2].

This paper is organized as follows: we first recall the background material to study the algebraic K-theory of $R\Gamma$ when Γ is an infinite virtually cyclic group. Since there are two classes of infinite virtually cyclics we recall the machinery for each. Finally, we prove the main result in the last two sections. We thank the referee for many valuable suggestions to improve this presentation.

2. Statement of results

We set up our Main Theorem in this section, mention some of its consequences and relate these to other results. For any ring R, let $K_i(R)$ denote the algebraic K-groups of R for all $i \in \mathbb{Z}$. In the process of studying these K groups one is led to study the relation in algebraic K-theory of a ring with that of its polynomial (Laurent) extension, this was done by Bass–Heller–Swan [1] and leads to the discovery of the nil groups. These are defined as

$$NK_i(R) = \mathrm{Ker}\big[K_i\big(R[t]\big) \xrightarrow{\varepsilon_*} K_i(R)\big],$$

where ε_* is induced by the augmentation $\varepsilon : R[t] \to R$. The relation between $K_i(R)$ and $K_i(R[t])$ is given explicitly by Bass–Heller–Swan formula, [18, 3.2.22]:

$$K_i\big(R[T]\big) \cong K_i(R) \oplus K_{i-1}(R) \oplus 2NK_i(R) \quad \text{for } i \leqslant 1.$$

One instance in which there are no nil terms is the following

Theorem 2 (Bass–Heller–Swan). *Let R be a regular ring then*

$$K_i\big(R[T]\big) \cong K_i(R) \oplus K_{i-1}(R) \quad \textit{for all } i \leqslant 1.$$

Higher nils, N^r, are defined inductively by taking $R[t_1, \ldots, t_n]$. An alternative definition may be found in [3].

Main Theorem. *Let Γ be a virtually cyclic group and R denote the ring of integers in an algebraic extension of the rationals. Then*

$$N^r K_i(R\Gamma) = 0, \quad \textit{for } r \geqslant 1,\ i \leqslant -1.$$

To put this in context, we explain the motivation for looking at the nil groups associated to virtually cyclic groups. Let G be a discrete group and denote by $\mathcal{F}$ the family of all virtually cyclic subgroups of G, by $\mathcal{O}$ the orbit category of G restricted to $\mathcal{F}$, and by $\mathbb{K}R$ the Davis–Lück functor [6]:

$$\mathbb{K}R : \mathcal{O} \to \mathit{SPECTRA}.$$

This functor evaluated at each object G/H is weak homotopic to the nonconnective delooping [14] of the algebraic K theory spectrum of the ring RH; with this machinery Davis and Lück [6] constructed a generalized homology theory on the category of G-CW-complexes:

$$\mathbb{H}_*(--; \mathbb{K}R),$$

and associated to this we have a corresponding *assembly* map:

$$\mathcal{A}_{R,\mathcal{F}} : \mathbb{H}_*\big(\mathcal{E}/G; \{\mathbb{K}R\}\big) \to K_*(RG), \tag{1}$$

where $\mathcal{E}/G$ denotes a classifying space for the family $\mathcal{F}$. The Isomorphism Conjecture proposes:

Isomorphism Conjecture. *$\mathcal{A}_{R,\mathcal{F}}$ is an isomorphism for any discrete group G and any ring R.*

The Isomorphism Conjecture has been verified in some cases, for example, in [2], Bartels, Farrell, Jones and Reich prove the conjecture for any ring R, when G is the fundamental group of a compact Riemannian manifold with strictly negative sectional curvature and $* \leqslant 1$, see also [10].

Recall that the assembly maps are natural in the ring R and the group G, with this observation we have a split cofibration of spectra over the orbit category

$$\mathbb{K}R \to \mathbb{K}R[t] \to \mathcal{N}R$$

induced by the inclusion $R \to R[t]$ and where $\mathcal{N}R$ is by definition the cofiber and has homotopy groups the nil groups of RH at the object G/H.

From the long exact sequence in generalized homology associated to the above *split* cofibration and from the identity $R[t]G = RG[t]$, we have the following, compare with [2, Proposition 7.4].

Proposition 3. *Assume $\mathcal{A}_{R,\mathcal{F}}$ is an isomorphism. Then $\mathcal{A}_{R[t],\mathcal{F}}$ is an isomorphism if and only if*

$$\mathcal{N}\mathcal{A}_{R,\mathcal{F}}: \mathbb{H}_*(\mathcal{E}/G; \mathcal{N}R) \to NK_*(RG)$$

is an isomorphism as well, where $\mathcal{N}\mathcal{A}$ is the assembly map corresponding to the spectrum valued functor $\mathcal{N}R$.

We may rephrase the above proposition by saying that the nil groups of RG depend on regular nil groups, twisted nil groups and Waldhausen nil groups as well. The precise way in which all of these interact is built in the groups

$$\mathbb{H}_*(\mathcal{E}/G; \mathcal{N}R).$$

and in the space $\mathcal{E}/G$.

Corollary 4. *Assume the assembly map*

$$\mathcal{A}_{R,\mathcal{F}}: \mathbb{H}_*\big(\mathcal{E}/G; \{\mathbb{K}R\}\big) \to K_*(RG),$$

is an isomorphism for the ring of integers R in an algebraic extension of the rationals, its polynomial extension $R[t]$; and for $* \leqslant 1$. Then

$$NK_i(RG) = 0 \quad \text{for } i \leqslant -1.$$

Proof. From Proposition 3 we have that

$$\mathcal{N}\mathcal{A}_{R,\mathcal{F}}: \mathbb{H}_*(\mathcal{E}/G; \mathcal{N}R) \to NK_*(RG)$$

is an isomorphism for $* \leqslant 1$. There is an Atiyah–Hirzebruch–Quinn [15] spectral sequence computing $\mathbb{H}_*(\mathcal{E}/G; \mathcal{N}R)$ with E^2-term given by

$$E^2_{p,q} = H_p\big(\mathcal{E}/G; \big\{NK_q(RH)\big\}\big) \Longrightarrow \mathbb{H}_{p+q}(\mathcal{E}/G; \mathcal{N}R).$$

Where $H_p(\mathcal{E}/G; \{NK_q(RH)\})$ is the homology of the space $\mathcal{E}/G$ with local coefficients the $NK_q(RH)$-groups and H runs over the virtually cyclic subgroups of G. Now, in the range $p + q \leqslant -1$, we have by our Main Theorem that $NK_q(RH) = 0$, this gives that $E^2_{p,q} = 0$ and therefore $0 = \mathbb{H}_i(\mathcal{E}/G; \mathcal{N}R) \cong NK_i(RG)$ for $i \leqslant -1$. $\square$

3. Background material

In this section we outline the basic definitions and results to be needed. Given a finite group H of order n. Let R be the ring of integers in an algebraic extension L of the rationals. One of the main ideas is to find an appropriate R-order A with the property that $RH \subset A$ and $nA \subset RH$, the consequence is that we may place the ring RH in a Cartesian square where everything else is quasi-regular. Hence the Mayer Vietoris sequence associated to this square will give the vanishing of the $(K_i, N^r K_i)$-groups in a certain range. This has been observed by many authors, e.g., [16,17]. The extra ingredient is that we have to consider not only the ring RH but an automorphism $\alpha : RH \to RH$ as well, hence we need an α-invariant order in RH that is maximal with this property. If it were only maximal it would also be hereditary, hence the difficulty is to find this invariant under α, this was achieved in [9, Theorem 1.2] for the case $R = \mathbb{Z}$ and in the above generality in [11, Theorem 2], it has also been accomplished by Kuku and Tang in [13]. The precise statement is:

Proposition 5 [11, Theorem 2]. *Let H be a finite group of order n and $\alpha : H \to H$ an automorphism. Then there exists an R-order $A \subseteq LH$ with the following properties*

(1) $RH \subset A$;
(2) *A is α-invariant*;
(3) *A is a right hereditary ring, hence right regular*;
(4) $nA \subset RH$.

Corollary 6. *Given a finite group H of order n, a ring R and automorphism α as above, let A be an R order as in the above theorem. Then the following is a Cartesian square*

$$\begin{tikzcd} RH \arrow[r] \arrow[d] & A \arrow[d] \\ RH/nA \arrow[r] & A/nA \end{tikzcd}$$

Now, recall that given a Cartesian square of rings where at least one of the morphisms landing in S is surjective:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & S \end{tikzcd}$$

we have, for $r \geqslant 1$, a Mayer Vietoris exact sequence associated to this [1, Theorem 8.3, p. 677]

$$\begin{aligned} N^r K_1(R) &\to N^r K_1(A_0) \oplus N^r K_1(A_1) \to N^r K_1(S) \\ &\to N^r K_0(R) \to N^r K_0(A_0) \oplus N^r K_0(A_1) \to N^r K_0(S) \\ &\to N^r K_{-1}(R) \to \cdots . \end{aligned}$$

3.1. Waldhausen nils

We mentioned the Mayer Vietoris sequence associated in K theory when we have a Cartesian square of rings. The dual situation would be that of having a co-Cartesian square of rings (or a pushout). This time the relation in K-theory is more intricate and leads to Waldhausen nil groups, these were extensively studied in [20] and in some sense they measure the failure of the expected sequence to be exact. Here is the setup, the following material may be consulted in detail in [12,5].

Let τ be the following category: the objects are triples $\mathbf{R} = (R; B_0, B_1)$, where R is a ring and B_0, B_1 are R-bimodules. A morphism $(\phi, f_0, f_1) : (R; B_0, B_1) \to (S, C_0, C_1)$ where $\phi : R \to S$ is a ring homomorphism and $f_i : B_i \otimes_R S \to C_i$ $(i = 0, 1)$ are $(R-S)$-bimodule homomorphisms. The Waldhausen nil groups, $\widetilde{Nil}_i^{\mathcal{W}}$, are functors from τ to the category of Abelian groups, their definition is as follows: given an object $\mathbf{R}$ in τ, we define the category $Nil(\mathbf{R})$ whose objects are quadruples $(P, Q; p, q)$ where P and Q are finitely generated projective right R-modules and

$$p : P \to Q \otimes B_0, \qquad q : Q \to P \otimes B_1$$

are R-homomorphisms such that the following compositions vanish after finitely many steps

$$P \xrightarrow{p} Q \otimes B_0 \xrightarrow{q\otimes 1} P \otimes B_1 \otimes B_0 \to \cdots,$$
$$Q \xrightarrow{q} P \otimes B_1 \xrightarrow{p\otimes 1} Q \otimes B_0 \otimes B_1 \to \cdots.$$

Morphisms are homomorphisms on the modules compatible with the corresponding maps. Moreover, exact sequences are defined in the obvious way and observe that there is a forgetful functor

$$\mathcal{F} : Nil(\mathbf{R}) \to \mathcal{P}_R \times \mathcal{P}_R,$$

where $\mathcal{P}_R$ denotes the category of finitely generated projective R modules. The Waldhausen nil groups are defined as

$$\widetilde{Nil}_i^{\mathcal{W}}(\mathbf{R}) = \operatorname{Ker}\bigl(K_i\bigl(Nil(\mathbf{R})\bigr) \xrightarrow{\mathcal{F}_i} K_i(\mathcal{P}_R \times \mathcal{P}_R)\bigr), \quad \text{for } i \in \mathbb{Z}.$$

Our next task is to identify the Waldhausen nils with the reduced K-theory of an augmented ring, this is one of the main results in [4, Proposition 2.6 when $i = 1$] and [12, Remarks after Theorem 3.7 for $i < 1$]. We describe the augmented ring, given an R-bimodule M, let $T_R(M)$ denote the tensor algebra of M. Now, given a triple $\mathbf{R} = (R; B_0, B_1)$ we define the following matrix ring

$$R_\rho = \rho(\mathbf{R}) = \begin{pmatrix} T_R(B_1 \otimes_R B_0) & B_1 \otimes_R T_R(B_0 \otimes_R B_1) \\ B_0 \otimes_R T_R(B_1 \otimes_R B_0) & T_R(B_0 \otimes_R B_1) \end{pmatrix}$$

where multiplication is given by matrix multiplication and on each entry by concatenation. There is a natural augmentation map

$$\varepsilon : R_\rho \to \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

Recall that the NK groups of R_ρ are by definition the kernels of the induced by ε in the corresponding K theories. The relation between $NK(R_\rho)$ and $\widetilde{Nil}^{\mathcal{W}}(\mathbf{R})$ is given by the following proposition

Proposition 7 [4,12, loc. cit.]. *There are natural isomorphisms*

$$NK_i(R_\rho) \cong \widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R}), \quad \text{for } i \leqslant 1.$$

We now recall some vanishing results for Waldhausen nils

Proposition 8 [20]. *Let R be a regular ring, then for any triple $\mathbf{R} = (R, B_0, B_1)$, it follows that*

$$\widetilde{Nil}^{\mathcal{W}}_i(\mathbf{R}) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

The above may be improved, a ring R is *quasi-regular* if it has a two sided nilpotent ideal J such that R/J is regular.

Proposition 9 [5, Corollary 3.13]. *Let R be a quasi-regular regular ring, then for any triple $\mathbf{R} = (R, B_0, B_1)$, it follows that*

$$\widetilde{Nil}^{\mathcal{W}}_i(\mathbf{R}) = 0 \quad \text{for } i \leqslant -1.$$

Here is how triples arise from co-Cartesian square of rings. Let

co-CS 1.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & S \end{tikzcd}$$

be a co-Cartesian (a pushout) square in the category of rings. We call this an *admissible* square if the morphisms coming out of R are pure inclusions, i.e., they are inclusions and induce R-bimodule splittings

$$A_i = R \oplus B_i \quad \text{for } i = 0, 1.$$

These splittings define the triple

$$\mathbf{R} = (R; B_0, B_1).$$

Furthermore by [20], associated to a co-Cartesian square as above, there is a long exact sequence (for all $i \in \mathbb{Z}$)

$$\cdots \to K_i(A_0) \oplus K_i(A_1) \to K_i(S) \to K_{i-1}(R) \oplus \widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R}) \to \cdots.$$

We concentrate in a special kind of triples, introduced in [5], and defined as follows: let R be our ground ring and let $\alpha : R \to R$ be a ring isomorphism, we denote by R^α the free R-bimodule where the left R multiplication is the usual of R and the right multiplication is $rs = r\alpha(s)$.

Definition 10. An admissible triple is one of the form $\mathbf{R} = (R, R^\alpha, R^\beta)$ for some automorphisms α and β of R.

One of he main efforts of [5] is to analyze the Waldhausen nils of admissible triples. We recall one of the main results concerning admissible triples.

Proposition 11 [5, 3.2,3.8,3.13]. *There are natural isomorphisms*

$$\widetilde{Nil}_i^{\mathcal{W}}\big(R; R^\alpha, R^\beta\big) \cong \widetilde{Nil}_i^{\mathcal{W}}\big(R; R^{\alpha\beta}, R\big) \cong \widetilde{Nil}_i^{\mathcal{W}}\big(R; R, R^{\beta\alpha}\big).$$

We now specialize in triples arising from amalgams. Let $\Gamma = G_0 \underset{H}{*} G_1$ be an amalgam of finite groups where

$$|G_0 : H| = 2 = |G_1 : H|.$$

This is an infinite virtually cyclic group of type 2. The group ring $R\Gamma$ fits into a co-Cartesian square

co-CS 2.

$$\begin{tikzcd} RH \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & R\Gamma \end{tikzcd}$$

where $A_0 = RG_0, A_1 = RG_1$ and $B_0 = R[G_0 - H], B_1 = R[G_1 - H]$. Observe that each B_i is a free left RH module but the right RH-module structure is *twisted* by an automorphism of RH, induced by an automorphism of H. This gives the admissible triple

$$\mathbf{R} = \big(RH; RH^\alpha, RH^\beta\big). \tag{2}$$

Next, from an admissible diagram as co-CS 1, we get the following admissible diagram by taking polynomial extensions:

co-CS 3.

$$\begin{tikzcd} R[t] \arrow[r] \arrow[d] & A_0[t] \arrow[d] \\ A_1[t] \arrow[r] & S[t] \end{tikzcd}$$

and from this we get the triple $\mathbf{R}[t] = (R[t]; B_0[t], B_1[t])$. Now, there is an obvious inclusion from co-CS 2 into co-CS 3 whose induced cokernels define the Bass nils of all rings in the squares, this gives a natural transformation of long exact sequences, and for $i \leqslant 1$

$$\begin{tikzcd}
\cdots K_i(A_0) \oplus K_i(A_1) \arrow[r] \arrow[d] & K_i(S) \arrow[r] \arrow[d] & K_{i-1}(R) \oplus \widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R}) \to \cdots \arrow[d] \\
\cdots K_i(A_0[t]) \oplus K_i(A_1[t]) \arrow[r] & K_i(S[t]) \arrow[r] & K_{i-1}(R[t]) \oplus \widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R}[t]) \to \cdots
\end{tikzcd}$$

By taking cokernels of all vertical maps we have a long exact sequence. We may summarize the above discussion in the following:

Proposition 12. *Let Γ be an infinite virtually cyclic group of the form*

$$\Gamma = G_0 \underset{H}{*} G_1,$$

where $|G_0 : H| = 2 = |G_1 : H|$. Then, there is a long exact sequence for $i \leqslant 1$, $r \geqslant 0$

$$\begin{aligned}
&\cdots N^r K_i(RG_0) \oplus N^r K_i(RG_1) \to N^r K_i(R\Gamma) \\
&\quad \to N^r K_{i-1}(RH) \oplus N^r \widetilde{\mathrm{Nil}}^{\mathcal{W}}_{i-1}(\mathbf{R}) \to \cdots,
\end{aligned} \tag{3}$$

where $\mathbf{R}$ is as in (2)*, and $N^r(\,)$ are the Bass nils.*

This is a Bass–Connolly–Da Silva–Milnor type sequence associated to a pushout.

4. Twisted nils

Let $\Gamma = H \rtimes T$, where H is a finite group of order n and T denotes the infinite cyclic group. Let α denote the automorphism of H that defines the semidirect product structure of Γ. Then the group ring $R\Gamma$ is isomorphic to the twisted Laurent polynomial ring $RH_\alpha[T]$ see, for example [7]. By Proposition 5 there is an R-order A which is α-invariant and fits in the following Cartesian square

Cs 1.

$$\begin{tikzcd}
R[H \rtimes T] \arrow[r] \arrow[d] & A_\alpha[T] \arrow[d] \\
(RG/nA)_\alpha[T] \arrow[r] & (A/nA)_\alpha[T]
\end{tikzcd}$$

Now by Proposition 1.4 of [9], all the terms in the above square are quasi-regular, except $R[H \rtimes T]$. To simplify the writing, let us rename $B_\alpha = (RG/nA)_\alpha$ and $C_\alpha = (A/nA)_\alpha$, from the Mayer Vietoris sequence in nil groups associated to the above Cartesian square we have for $i \leqslant 1$ and $r \geqslant 0$:

$$\begin{aligned}
&\cdots N^r K_i(R\Gamma) \to N^r K_i\big(B_\alpha[T]\big) \oplus N^r K_i\big(A_\alpha[T]\big) \\
&\quad \to N^r K_i\big(C_\alpha[T]\big) \to N^r K_{i-1}(R\Gamma) \to \cdots.
\end{aligned}$$

But by quasi-regularity,

$$N^r K_i(S) = 0, \quad \text{for } r \geqslant 1,\ i \leqslant 0$$

when S is $B_\alpha[T]$, $A_\alpha[T]$ or $C_\alpha[T]$, hence it follows that

$$N^r K_i(R\Gamma) = 0 \quad \text{for } r \geqslant 1,\ i \leqslant -1.$$

5. $N(\widetilde{\mathit{Nil}}^{\mathcal{W}})$

Let $\Gamma = G_0 \underset{H}{*} G_1$ be our infinite cyclic group of the second type, where $|G_0 : H| = 2 = |G_1 : H|$, and H is a finite group.

By Proposition 7 we may identify $\widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R})$ with the NK_i groups of a suitable matrix ring R_ρ . Let A be a regular α-invariant R-order, contained in LH given by Proposition 5. This induces triples, that by Proposition 11 we may assume of the following form

$$\begin{aligned}
&\mathbf{A} = \big(A, A^\alpha, A\big)\\
&\mathbf{R}/n\mathbf{A} = \big(RH/nA, (RH/nA)^\alpha, RH/nA\big)\\
&\mathbf{A}/n\mathbf{A} = \big(A/nA, (A/nA)^\alpha, A/nA\big),
\end{aligned}$$

which in turn gives corresponding matrix rings A_ρ, $(RH/nA)_\rho$, and $(A/nA)_\rho$ which fit into a Cartesian square [5, Proposition 3.14]

Cs 2.

$$\begin{tikzcd}
RH_\rho \arrow[r] \arrow[d] & A_\rho \arrow[d] \\
(RH/nA)_\rho \arrow[r] & (A/nA)_\rho
\end{tikzcd}$$

Observe that the rings A_ρ, $(RH/nA)_\rho$, and $(A/nA)_\rho$ are quasi-regular, hence $NK_i() = 0$ for $i \leqslant 0$ and for any of these rings. The Mayer–Vietoris associated to the Cartesian square **CS** 2 gives for $i \leqslant -1$ exact sequences

$$0 \to NK_i(RH_\rho) \to 0.$$

Thus $\widetilde{Nil}^{\mathcal{W}}_{i-1}(\mathbf{R}) \cong NK_i(RH_\rho)$ vanishes for $i \leqslant -1$, and consequently $\widetilde{Nil}^{\mathcal{W}}_{i}(\mathbf{R})$ vanishes for $i \leqslant -2$.

Now, for $i \geqslant 1$, from the exact sequence

$$\begin{aligned}
&NK_{-i}(RG_0) \oplus NK_{-i}(RG_1) \to NK_{-i}(R\Gamma)\\
&\quad \to NK_{-i-1}(RH) \oplus N\widetilde{Nil}^{\mathcal{W}}_{-i-1}(\mathbf{R}) \to \cdots,
\end{aligned}$$

the fact that H, G_0, and G_1 are finite, and from [3, Proposition 4.4] it follows that

$$N^r K_i(R\Gamma) = 0 \quad \text{for } i \leqslant -1,\ r \geqslant 1.$$

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